EARL E. ECKSTROM

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STRUCTURAL ENGINEERING

STRENGTH OF MATERIALS

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STRUCTURAL ENGINEERING

STRENGTH OF MATERIALS

BY

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DEDICATED

AT THE SUGGESTION OF MY WIFE,
TO MY LONG-SUFFERING FAMILY

PREFACE

The present volume is the first of a series treating of the theory and design of structures, embodying the course which the writer has been giving for many years, with amplifications which, it is hoped, will make it a fairly complete treatise for engineers.

Considering the extent of the field, some subjects are omitted, and others touched upon briefly. Foundations, for instance, though extremely important for the structural engineer, are not discussed. There are several satisfactory treatises on that subject, and it is more largely a matter of judgment, experience, and practical detail, and less dependent on fundamental principles, than the topics here included.

It was originally intended that this work should comprise two volumes only; but the theory of structures is so intimately connected with that of the strength of materials, of which it is, strictly speaking, a continuation, that it was decided to limit this first volume to the latter subject. There are many excellent works on the strength of materials, but it was necessary for the writer, in order to present the theory of structures in the way that he prefers, to cover a good part of what is usually included in strength of materials; and if the writer's views and methods are of value, it is hoped that this volume will prove attractive and useful to other than structural engineers.

In connection with the strength of materials, it is desirable that the engineer should be familiar with the fundamental properties of the materials themselves, including their principal physical characteristics, and the most important matters connected with their strength. There are several excellent and exhaustive works on the subject of materials, but it has been thought desirable to prepare a briefer one particularly designed to accompany this volume on strength of materials.

Another volume will complete the theory and the design of simple structures of wood, metal, and masonry; and a final volume will treat of more complicated and statically indeterminate structures.

The first two volumes, namely the present volume on strength of materials, and the one on the fundamental properties of materials, are completed; the last two are well on the way to completion.

The aim of the present volume is to give a complete, clear, and simple discussion of the fundamental principles of the strength of materials applicable in the design of structures of various kinds, not merely of framed structures but of those of masonry and concrete also.

It is believed to contain all that is needed for machine design also, and so is suited for any student of engineering.

Some books on this subject are so brief that they do not serve as a foundation for further study, while others are so encyclopedic that the principles are likely to be obscured by a mass of detail. The writer believes that every student

of engineering should be required to study a treatise on the strength of materials, and that such treatise should go beyond the limits of the course which his curriculum provides, thus giving him material for further study. By reference to original sources and other works, it should also put him in a position to pursue the subject as far as our present knowledge permits. To accomplish this, has been the aim of the present volume. It is recognized that the curriculum of most engineering schools will not permit the student to go through the whole of this volume. Indeed, the importance of the subject and the necessity of getting thorough and accurate fundamental notions are not sufficiently appreciated. The teacher will be able to omit such parts of the present work as he considers least important for his own purposes, or which his time does permit him to cover.

A brief statement follows suggesting the portions of his work which may be so omitted.

In the present volume there is perhaps little that cannot be found in other works on the subject, but it is believed that the method of presentation is somewhat novel, and that there are original points of view and methods in many places. The material will, it is hoped, be found to be thoroughly digested and presented in such a way and with such abundant references that it may serve as a guide for more extensive study. The writer is acquainted with most of the published works on the subject and is indebted to many of them. Acknowledgements are made in the proper places.

There is some repetition, but the writer does not consider it objectionable. The student of any subject must continually repeat and review if he wishes to impress on his mind.

The student of this book is assumed to possess a copy of the Pocket Companion issued by the Carnegie Steel Company, and he should familiarize himself with the various shapes used in steel construction and should check some of the results and tables in the Companion as he learns the principles involved. It is believed that the work will be found appropriate for students in Mechanical and Electrical Engineering as well as for those in Civil Engineering. Students of any branch of engineering should, in the writer's view, be thoroughly grounded in the strength of materials, but some parts of the present work may be omitted by students who do not intend to pursue the subject of structures.

The writer will be grateful to any reader who will point out any errors, typographical or others, which may be found in this book, and he will welcome suggestions of possible improvements.

The writer is especially indebted to his former assistant, Albert Haertlein, for reading the text and proof, and in other ways too numerous to mention; also to his assistant, M. S. Bromwell, for suggestions and corrections; and to many others for permission to use cuts or tables.

GEORGE FILLMORE SWAIN.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.
September, 1924.

SUGGESTION FOR A SHORTER COURSE

Subject to the discretion of the teacher, the following suggestions are made of parts of this book which may be omitted in case time is not available to cover the entire work:

Chapter II may be omitted if the students are thoroughly familiar with statics. Arts. 8 and 10 may be omitted if forces not in a plane are not considered.

In Chapter III, Arts. 1 to 11 may be omitted or used for reference. Arts. 20 to 24 may be omitted.

Chapter IV. Omit Arts. 13, 14, 15 inclusive.

Chapter V. Omit Arts. 9 and 10, and 20 to 25 inclusive. Arts. 12 to 19 may be omitted by those whose time is short, although they contain matter which is important and interesting.

Chapter VI. Omit Arts. 5, 6, 7.

Chapter VII. Omit Art. 5.

Chapter VIII. Omit Arts. 13 to 19 inclusive.

Chapter IX. Omit Arts. 12, 13, 14, 16, 18, 19, 20. This chapter may be taken up after Chapter X if the teacher considers advisable. It treats of what is often termed the "general theory of flexure."

Chapter X. Omit 10, 11, 12, part of 17, 17*a*, 18 to 25 inclusive.

Chapter XI. Most of the contents of this chapter is not included in books on the Strength of Materials, but is left for works on Structures. Being concerned with Shears and Moments, it properly belongs in this volume. Some parts of it should be included in any course, but if the Theory of Structures is considered relatively unimportant by the teacher, he may omit Arts. 13 to 17*a* inclusive, 23 to 26 inclusive, and 34.

Chapter XII. This chapter consists of examples of the application of principles demonstrated in Chapter X. The teacher may omit such cases as he desires, though all are important. Thus he may omit 6, 7, 11, 14 to 19, 24, 27, 29 to 32.

Chapter XIII. Omit 4, 5, 6, 15, 16.

Chapter XIV. Omit nothing.

Chapter XV. Omit nothing.

Chapter XVI. Omit 16, 32 to 35.

Chapter XVII. Omit 3, 15, 17, 18, 23, 24.

Chapter XVIII. Omit 14, 15, 16.

Chapter XIX. May be omitted entirely if time is short.

Chapter XX. May be omitted entirely if time is short, or may omit Arts. 4 to 11 inclusive.

Chapter XXI. Omit nothing or all, the latter if the subject is adequately treated elsewhere in the curriculum.

Chapter XXII. Omit nothing.

Chapter XXIII. Omit nothing or all.

Chapter XXIV. Omit nothing or all.

CONTENTS

		Page
Preface		vii
Suggestion for a Shorter Course		ix
Chapter		
1.	Introduction	1
2.	Mechanics	4
3.	Geometrical Relations	24
4.	Behavior Under Stress and Strain	48
5.	Relations between Stresses at a Point	87
6.	Pure Tension or Compression	116
7.	Shearing	131
8.	Torsion	136
9.	Distribution of a Normal Force over a Section	150
10.	Flexure	178
11.	Shears and Moments on Beams. Influence Lines	248
12.	Slope and Deflection of Beams in Pure Flexure	294
13.	Combined Stresses	335
14.	Beams Over More Than Two Supports	358
15.	Reinforced Concrete Beams	368
16.	Riveted Joints	377
17.	Columns	411
18.	Hollow Cylinders and Spheres	461
19.	Initial Stresses	470
20.	Flexure of Curved Bars	476
21.	Tests of Materials	485
22.	Repetition or Reversal of Stress. Fatigue	508
23.	Theories of Failure	536
24.	Mathematics, Testing and Common Sense	557
Index		563

"Read not to contradict and confute, nor to believe and take for granted, nor to find talk and discourse, but to weigh and consider."

—Bacon.

STRUCTURAL ENGINEERING

THE STRENGTH OF MATERIALS

CHAPTER I

INTRODUCTION

1. The constructions of the engineer, broadly speaking, may be divided into two classes, *structures* and *machines;* the distinction between these being that structures are at rest, while machines are either in motion as a whole (as in the case of a locomotive), or have certain parts which are in motion (as in the case of a stationary engine or a loom). Machines, as thus defined, may be of three kinds: *Motors*, which generate power, like a stationary engine; *transmission machinery*, which transmits power from the motor to the place of utilization, like shafting, gearing, or belting; and *machines for utilizing power*, like looms, lathes, or punches. Sometimes the term machine is limited to the last named class. Coming down to a more specific definition, we may say:

A *structure* is a part or assemblage of parts forming an engineering construction, subjected to loads, which are known, and which the structure is required to sustain or hold in equilibrium.

A *machine* is a part or assemblage of parts forming an engineering construction, subjected to loads, which are known, and designed to produce certain desired effects accompanied by certain determined motions.

The design of structures or of machines requires a thorough knowledge of the principles of mechanics. In the case of a machine, the essence of its action consists in the proper performance of its motions, or of the motions of its different parts with relation to each other. The study of these motions constitutes what is known as *kinematics*, which may be regarded as a branch of geometry rather than of mechanics. But in either structures or machines, the design and proportioning of the different parts, and of the construction as a whole, requires a knowledge of the principles of statics and dynamics (or kinetics) which constitute properly mechanics. Structures, being at rest, and the loads being assumed to be at rest also, require for their design, in general, only a knowledge of statics; but since the loads are often in motion, as in the case of a railroad train upon a bridge, a knowledge of dynamics (kinetics) is often necessary. Machines, whose fundamental peculiarity consists in motion, require for their design a knowledge of statics and dynamics. The loads acting upon a structure or machine are assumed as stationary, and if they are actually in motion, d'Alembert's principle makes it possible to reduce them to an equivalent system of stationary loads in equilibrium. The parts of a moving body are thus treated as stationary structures; and some-

times they are primarily structures rather than parts of movable systems, like the body of a railroad car.

2. The forces or loads acting upon a body produce internal *stresses* between the particles of which the body is composed; and these produce deformations, or *strains*. The problem of design is to provide that at no point shall these stresses and strains be large enough to cause rupture or to exceed a safe limit. The subject of this volume, the *Strength of Materials*, or, as it is sometimes called *Resistance of Materials*, or *Mechanics of Materials*, or *Elasticity and Strength* (Elasticität und Festigkeit), deals with the various kinds of stresses and their relation to each other and to the applied loads, the various kinds of strains and their relations to each other, and the relations between stresses and strains. It investigates the stresses and strains produced in the parts of the structure or machine, due to the given loads. It also deals with the strength of the various engineering materials under the various kinds of stresses, as determined necessarily by experiment, and the effects of variations of material upon that strength. These variations of material may be in chemical composition, physical characteristics, or methods of handling and treatment. In this latter part of the subject, the strength of materials touches upon what is often (and properly) looked upon as a distinct subject, that of *Materials of Engineering*, which deals with the occurrence of these materials in nature, their chemical and physical characteristics, their methods of manufacture or preparation for use, their strength and other properties. It is a large and very important subject, with which the engineer should be quite familiar if he is to use materials intelligently in his structures and machines.

3. The subjects of *Mechanics*, *Materials* and *Strength of Materials* should therefore be mastered by the engineer. He may then proceed to the *Theory of Structures or of Machines*, and to *Structural or Machine Design*.

The subject of Strength of Materials deals with single and comparatively simple bodies, each by itself, such as beams, ties, columns, shafts, and tubes. In structures or machines these simple bodies are combined, sometimes in very complicated ways. The Theory of Structures or of Machines deals with the computation of the stresses in these different parts, and when this has been done, a knowledge of the Strength of Materials makes possible the proportioning of each part. The Theory of Structures is thus bounded on one side by the Strength of Materials—of which it may perhaps be regarded as an extension—and on the other by the subject of Design, which teaches not only how to design each part, but how to combine the parts economically and properly into the structure or machine. Design is the highest and culminating branch, involving the conception and design of a structure or machine which will best fulfil certain conditions, carry certain loads, or produce certain results, safely and economically.

The various branches or stages referred to—Mechanics, Materials, Strength of Materials, Theory of Structures or Machines, and Structural or Machine Design—shade imperceptibly into one another. No sharp lines of demarcation can be drawn, except between Mechanics and the others. Mechanics may be separated and considered as an abstract subject by itself; but its illustrations, of which there should be many, will be largely drawn from the others.

The present work presupposes a thorough knowledge of Mechanics, but a chapter is devoted to a brief summary of the most essential principles required by

the engineer. The subject of Materials is touched upon only so far as necessary, and suggestions given for further study. The Strength of Materials is treated as fully as the writer deems necessary. Later volumes will treat of the Theory and Design of Structures.

4. Forces.—In studying the body which is considered in the subject of Strength of Materials, or the structure or machine which is considered in the Theory of Structures or of Machines, two kinds of forces must be carefully distinguished, namely, the *Outer Forces* and the *Inner Forces* or *Stresses*. The outer forces are the applied loads, and the reactions or resisting forces which balance them. The loads are the active forces, and it is the function of the structure to sustain them; they first cause the reactions to be brought into action, and both cause the stresses. The loads must first be determined, and sometimes they are very uncertain, and much investigation is necessary to determine what are reasonable. In the Strength of Materials, the outer forces are always supposed to be given, and the stresses are to be determined; or sometimes it is required to find what load a given piece, such as a column, will carry with safety.

The loads are always of two kinds, namely the weight of the piece itself (the *dead load*), and the outwardly applied loads (*live loads*), applied to some surface of the body. The latter are supposed known; the weight can only be known after the piece is designed. In the Strength of Materials the weight is also generally supposed known, for the subject does not deal with design.

The reactions or supporting forces are exerted at the places where the piece or structure is supported, as at the ends of a beam. They are called into action by the loads, and are determined by the condition that loads and reactions are in equilibrium. So far as the mechanics of the problem is concerned, loads and reactions are indistinguishable in character, but the reactions are always determined after the loads have been assumed, and are such that they must balance those loads, subject to any physical conditions imposed by the construction of the supports, or any other physical conditions. For instance, a reaction acting through a frictionless roller must act perpendicular to the surface on which the roller rests.

The forces considered are therefore:

$$\text{Forces} \begin{cases} \text{Outer} \begin{cases} \text{Loads} \begin{cases} \text{Dead.} \\ \text{Live.} \end{cases} \\ \text{Reactions} \end{cases} \\ \text{Inner, or stresses} \end{cases}$$

CHAPTER II[1]

MECHANICS

A SHORT SUMMARY OF THE ESSENTIAL PRINCIPLES OF MECHANICS

1. Force: Moment.—A force is completely determined when a point in its line of action is given, and also the magnitudes of its two components along any two given rectangular axes. If, as is usual, these axes are taken horizontal and vertical, and if horizontal forces are taken as positive when they act toward the right and negative when they act toward the left, and vertical forces positive when they act upward and negative when they act downward, then the determination of the magnitudes of these components determines also, by their signs, the directions in which they act. It is sometimes stated that, to determine a force, its magnitude (in pounds), its direction, and a point in its line of action must be fixed. In this case, by direction is meant, not simply the inclination of its line of action, but the *sense* in which the force acts along that line, whether upward or downward, to the right or to the left. In any case, three independent factors determine a force. The two axes along which the components are found need not be horizontal and vertical, but are frequently or generally so taken, and the components designated as H and V; the axes need not even be at right angles, but are rarely taken otherwise.

The two meanings of the word "direction" should be carefully noticed. Sometimes it means merely the angle which a line makes with some fixed direction, and this is its meaning when we speak of the direction of a line; but a force or a motion is not simply along a line, but may have either of two *senses* along that line, that is, it may be directed in one way or the other, and when we speak of the direction of a force or of a motion we mean also its sense. If, as we shall always do in this book, we define a force by giving its two rectangular components and a point in its line of action, having adopted conventions as to the signs of the components, we define not only direction but sense.

When a force is distributed over an area, its *intensity* at any point is the magnitude of the force per unit of area at that point, *i.e.*, the force on an infinitesimal area surrounding the point, divided by the area. The distinction between a force and the intensity of a force must be constantly born in mind.

The moment of a force about a point in a plane in which it acts, is defined as the force multiplied by the perpendicular distance from the point to the line of action of the force. Such a moment is *right-handed* when the direction of the force with reference to the point as a center is the direction in which the hand of a clock moves, as seen from the point from which we suppose ourselves looking

[1] This chapter is not intended as in any sense a treatise on mechanics. It is simply inserted here in order that the student may have before him in one book a summary of the essential principles required by the structural engineer, to aid him in systematizing them in his own mind.

at the plane. The direction will be different according as we look at the plane from one side or from the other. *Right-handed* is therefore sometimes termed *clockwise; left-handed* is *anti-clockwise.*

2. To Find the Resultant of a Set of Forces in a Plane (*composition of forces*). The resultant of two forces in a plane is found by prolonging their lines of action till they meet, and constructing at the point of intersection the parallelogram of forces. The resultant of any number of forces in a plane may be obtained by finding the resultant of two, as just described, combining this resultant with a third, and continuing the process till the resultant of all is found. Thus (Fig. 1) to find the resultant R_{1-3} of three forces, F_1, F_2, F_3 acting along the lines shown, prolong the lines of action of F_1 and F_2 till they meet at A, lay off AC representing the magnitude and direction of F_1 and AB representing the magnitude

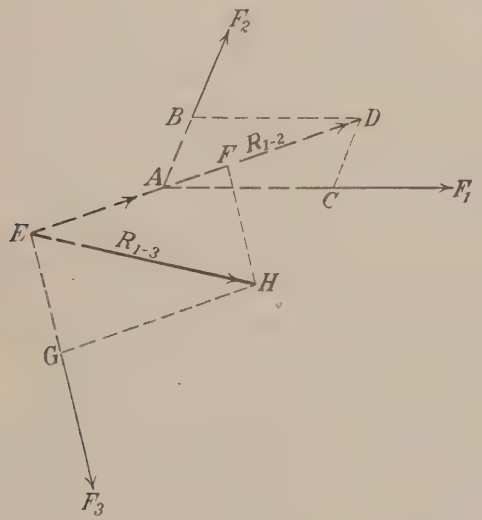


Fig. 1.

and direction of F_2, and the resultant of F_1 and F_2 (which we call R_{1-2}) will be a force represented by AD in magnitude and direction, and acting along the line AD from A toward D. Prolong AD till it meets the line of action of F_3 at E, lay off $EF = AD$, and EG representing F_3 in magnitude and direction, and R_{1-3} will be a force, represented by EH in magnitude and direction, and acting along the line EH from E toward H. The forces F_1, F_2 and F_3 are *components* of R_{1-3}.

If the resultant force acting on a body is a single force acting through the center of gravity of the body, it will cause a motion of translation in the direction of the force. If it acts at any other point, it may be resolved into an equal force acting at the center of gravity and causing a motion of translation, and a couple causing rotation.

Another method will be explained in the chapter on Graphical Statics, which is more generally applicable; for if two forces, whose resultant is desired, meet

outside the limits of the paper, the above method involves the problem of drawing through this inaccessible intersection a line in a given direction.

3. Resultant of Parallel Forces in a Plane.—If two forces are parallel, they intersect at an infinite distance, and their resultant will be parallel to either. If the two parallel forces act in the same sense, the resultant will equal in magnitude the sum of the two, and will act in the same direction and sense, and *between them;* if the two forces, though parallel, act in opposite senses, the resultant will equal in magnitude the difference of the two, and will act in the sense of the larger, and not between them, but on the side of the larger. Stated generally, the resultant of two parallel forces is parallel to them, and its magnitude is the algebraic sum of the two. The line of action of the resultant is found by the principle that the *resultant of a set of forces is the simplest system causing the same external effect upon a body to which it is applied:* in other words *the resultant of a set of forces must have the same components along any two rectangular axes, and the same moment about any point, as the algebraic sum of the components or moments of the given set of forces.* If, therefore, two parallel forces F_1 and F_2 act along the lines

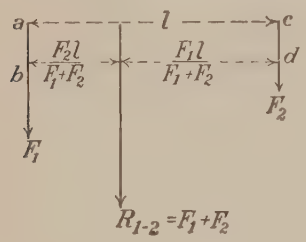


Fig. 2.

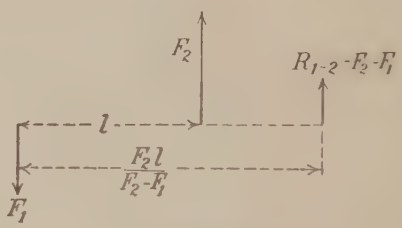


Fig. 3.

ab and *cd* (Fig. 2) the resultant R_{1-2} will equal the sum of F_1 and F_2 in magnitude and will be parallel to them in direction, and will act at a distance to the right of F_1 equal to $\frac{F_2 l}{F_1 + F_2}$. Hence if the two parallel forces act in the same direction the resultant will act *between them.* If the parallel forces F_1 and F_2 act in opposite directions (Fig. 3) and if F_2 is greater than F_1, the resultant R_{1-2} acts in the direction of F_2 and at a distance to the right of F_1 equal to $\frac{F_2 l}{F_2 - F_1}$, that is, it acts beyond or outside of F_2, or *farther from F_1 than F_2 is.*

If two parallel forces are *equal in magnitude,* but act in opposite senses, their resultant is zero in magnitude. Such a pair of equal forces is called a *couple,* and tends simply to rotate the body to which it is applied about an axis perpendicular to the plane containing the couple, not to move it longitudinally. The precise effect of a couple in rotating a rigid body is treated in books on mechanics and need not be further pursued here. The *moment of a couple* is the magnitude of either force multiplied by the perpendicular distance between their lines of action. The direction of a couple is termed right-handed or left-handed according as it tends to rotate the body in a clockwise direction or the reverse. A couple cannot be reduced to any simpler equivalent system; all couples in a plane or in parallel planes, in the same direction and having the same moment, are equiva-

lent, and may be substituted one for the other so far as the equilibrium of the body is concerned, but not so far as concerns its deformation.[1]

4. To find analytically the resultant of any number of forces in a plane, each force is resolved, *at any convenient point in its line of action*, into two rectangular components, H and V, parallel to axes in the plane at right angles to each other, assumed in any convenient direction and intersecting at any convenient point O; then with reference to the point O, each component of each force has a moment M. The resultant of all the forces will have as its components parallel to the same axes

$$H_r = \Sigma H;\ V_r = \Sigma V \text{ (taken algebraically)}$$

and its line of action will be determined by the condition that its moment must be in the same direction, and must have the same magnitude, as the algebraic sum of the moments of all the components. That is, if d = distance from O perpendicular to resultant,

$$d\sqrt{H_r^2 + V_r^2} = \Sigma M.$$

Generally it is unnecessary to make use of this last equation, for all that is desired in most engineering problems is the magnitude of the resultant (or its two components) and the point where it intersects some given line h.[2] For this purpose resolve each separate force into two components, one H parallel to the given line h, and the other V at right angles to it, and find the algebraic sum of the moments M of all the components about some convenient point O on the line. Then the resultant force will have a component parallel to h equal to ΣH, and a component at right angles to h equal to ΣV, and it will meet h at a distance x from O equal to $\frac{\Sigma M}{\Sigma V}$; for if the resultant is resolved at this point into $H_r = \Sigma H$ and $V_r = \Sigma V$, only the latter has a moment about O, and $\Sigma M = V_r x$. It will also meet a line through O at right angles to h at a distance from O equal to $\frac{\Sigma M}{\Sigma H}$. By observing signs carefully, the direction in which these distances are to be measured will be evident. The above simple principle should be thoroughly mastered by the student, and he should be prepared to use it without hesitation.

5. Resolution of a Force into Components in a Plane.—The separation or resolution of a force into components is the converse of the composition of forces. The following principles should be thoroughly mastered.

A single force may be resolved into an infinite number of sets of components in the same plane. The analytical conditions which are *necessary and sufficient* are

$$H_r = \Sigma H;\ V_r = \Sigma V;\ M_r = \Sigma M.$$

[1] While the resultant of a set of forces causes the same *external* effect (that is, the same motion) upon a body to which it is applied, as the set of forces of which it is the resultant, it may cause entirely different *internal* effects, such as stress or deformation. If the body is at rest, the resultant force acting upon it is zero, or the resultant applied load is equal and opposite to the resultant of the reactions or supporting forces, yet the loads may be such that the body may or may not safely carry them; and though it may safely carry a given set of loads, it might be fractured by another set of applied loads having the same resultant, or by that resultant itself if applied alone, and *vice versa*. The student should reflect on this.

[2] Remember that a given force may be resolved into components at any point in its line of action, and that, no matter where resolved, the algebraic sum of the moments of the components about any given point equals the moment of the resultant about the same point, and therefore equals the algebraic sum of the moments of any other set of components of the given force.

If a force is resolved into two components, all three of the forces must lie in one plane, the lines of action of the components must meet upon the line of action of the resultant, and the lines representing the magnitudes and directions of the three must form a triangle. If AD (Fig. 1) represent the magnitude and direction of the force to be resolved, the magnitude and direction of the components may be represented by AC and CD (not CA or DC).[1] The triangle of forces has only to do with magnitude and direction (including sense): It has nothing to do with the line of action, though it may be placed so that the line representing a force may coincide with its line of action.

A given force may thus be resolved into any number of sets of two components. The problem may present itself in various ways; thus there may be given

(*a*) lines parallel to the line of action of each component; or

(*b*) the magnitude of one component and a line parallel to the line of action of the other; or

(*c*) the magnitude of each component.

The student must clearly see how to attack the problem in each case and any limitations affecting the solution.[2]

A given force cannot be resolved into two components along lines which are parallel to each other, but not parallel to the given force.

A given force may be resolved into two components along lines both of which are parallel to the line of action of the given force. If both these lines of action are on the same side of the given force, the two components will be opposite in direction, and the larger will be nearer to the given force and in the same sense. If one is on one side and one on the other side of the given force, the two components will both be in the sense of the given force, and the one lying nearer to the given force will be the larger. The determination of the components should be made by the principle of moments.

A given force may be resolved into three components in an infinite number of ways; and in this case the four forces need not lie in one plane; but one component and the resultant of the other two must equal the given force. We are only considering, however, the case in which all are in a plane. The analytical conditions have been already stated. The graphical conditions will be more fully studied in a later chapter, but one of them is that if the components are laid off in order of magnitude, direction, and sense, from A to N, the resultant must be represented in magnitude and sense by AN (polygon of forces). A similar proposition holds true when the forces are not in a plane.

This problem may also present itself in many forms. For instance, the force to be resolved being completely given, there may be given the lines of action of all three components, provided these three lines do not meet in a point which is *not* on the resultant. This problem may be solved as follows (Fig. 4): Let R_{1-3} be the force to be resolved, acting along the line ad, and let AD (Fig. 4*a*) represent its magnitude and direction. Let cb, ef, and gh be respectively the lines

[1] When the line AB is said to represent the magnitude and direction of a force, the force acts in the sense from A toward B. The force AB is therefore very different from the force BA: they act in opposite senses. There is the same difference between AB and BA as there is between debit and credit.

[2] *Questions:*

1. Under (*a*) is more than one solution possible?
2. If so, what more must be given to make the solution definite?
3. Answer the same questions with respect to (*b*) and (*c*).

of action of the three components F_1, F_2, F_3. First, resolve R_{1-3} into two components, one of which shall be F_1 and the other the resultant of F_2 and F_3. These two components must meet at k, and the second must act along the line kl since it must pass through l, in order to be the resultant of F_2 and F_3; hence drawing lines in the force polygon, from A parallel to F_1 and from D parallel to kl, we see that $AB = F_1$, $BD = R_{2-3}$; then resolving BD into F_2 and F_3 by drawing BC parallel to F_2 and DC parallel to F_3 we find $BC = F_2$ and $CD = F_3$. The figure should be drawn, as indicated, in such a way that the components will come in numerical order from A to D, although this is not necessary. Note that we speak of drawing BC (not CB); in other words, we mention first the letter whose position is known (B). The student should form and adhere to this habit of thinking.

The problem of resolving a given force into three components may also present itself in other forms, and the student should endeavor to formulate them, together with the solutions.[1]

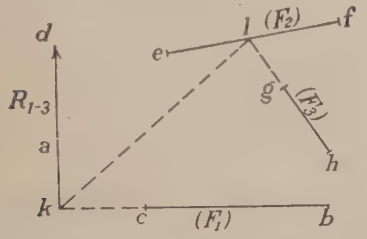


FIG. 4.

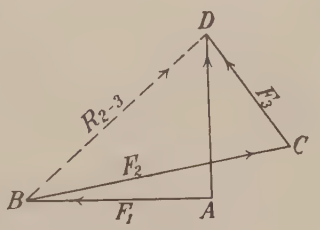


FIG. 4a.

6. Equilibrium of Forces in a Plane.—The conditions of equilibrium of forces follow from the preceding principles, since in this case the resultant of the forces is zero. The following theorems are fundamental:

(1) If a set of forces are in equilibrium, their resultant is zero, and any of the single forces is equal in magnitude, and opposite in sense, to the resultant of all the others, and acts in the same line as that resultant.

(2) If a set of forces in equilibrium be divided into any two portions, the resultant of one portion is equal and opposite to the resultant of the other portion, and acts in the same line.

(3) If the forces in equilibrium be laid off successively in magnitude, direction and sense from any starting point, *in any order*, they will form a closed figure (whether the forces are in a plane or not).

(4) If a set of forces in a plane are in equilibrium, the algebraic sum of the moments of all the forces, about any point whatsoever in the plane, will be zero.

[1] *Questions:*

1. Can a force be resolved into three components in the same plane with it, if the following elements are given; and is more than one solution possible?

(*a*) The lines of action of two components and the magnitude of the third.

(*b*) The magnitude of each component.

(*c*) The line of action of one component and the magnitudes of the other two.

(*d*) The lines of action of all three components and the magnitude of one.

If any of the above cases admit of more than one solution, what other elements may be given to make the solution definite?

(5) A couple can only be balanced by another couple (in the same plane or in a parallel plane) having an opposite direction of rotation and an equal moment.

(6) The analytical conditions of equilibrium of forces *in a plane and at a point* are *two*, namely

$$\Sigma H = 0;\ \Sigma V = 0.$$

Graphically, these two conditions are represented by one, namely that the force polygon must close.

(7) The analytical conditions of equilibrium of forces *in a plane and not at a point* are *three*, namely

$$\Sigma H = 0;\ \Sigma V = 0;\ \Sigma M = 0.$$

Graphically these three conditions reduce to two, namely that the force polygon must close (corresponding to $\Sigma H = 0;\ \Sigma V = 0$), and that the line of action of the resultant of all but one of the forces shall fall in the line of action of the remaining force; in other words, the resultant might be a couple if only $\Sigma H = 0$ and $\Sigma V = 0$.

7. The resultant of a couple and a single force in the plane of the couple must frequently be found. To do so we must observe that a couple may be revolved or transferred laterally in its plane or in planes parallel thereto to any extent, or the magnitude of its forces may be altered, provided the moment and direction of rotation remain the same.

Now to find the resultant of a force and a couple, reduce the couple to an equivalent couple having forces equal and parallel to the given force, and place this couple in such a position that one of its forces is in the same line with the given force but acts in the opposite sense. The given force will then balance this force of the couple, leaving the other force, which is unbalanced, as the resultant of the couple and the single force. In other words, *the resultant of a force and a couple is a force equal to the given force, acting parallel to it and in the same sense, but at a distance from it equal to the moment of the couple divided by the force.* Or, in still other words, the effect of adding a couple M to a single force F is to transfer the single force, parallel to itself, a distance equal to $\frac{M}{F}$. The direction in which the shifting takes place depends upon the direction of the couple, and can be easily seen in any special case.

Similarly, *a single force may be resolved into an equal and parallel force acting in the same sense, anywhere in space, together with a couple (in the plane of the two forces) whose moment equals the given force multiplied by the distance between it and the equal and parallel force.* This is obvious when we recollect that the *motion* of a body is not affected (although its condition of internal stress would be) when at any point two equal and opposite forces are applied acting in the same line, or when along any line two equal and opposite forces are added.[1]

[1] *Questions* (Forces in a plane):

1. What is the resultant of a vertical downward force F and a left-handed couple M, and where does it act?

2. Answer the same question for a horizontal force acting to the left, and a right-handed couple.

3. Resolve a vertical force of 100 pounds acting upward into an equal force acting 10 feet to the right, and a couple.

4. If a couple acting on a body in equilibrium be replaced by another couple having the same moment, but lying in a parallel plane, will the equilibrium of the body be affected?

It is to be observed that a couple may be considered to act anywhere in its plane or in any parallel plane, since its only statical effect is to cause rotation about an axis perpendicular to its plane. A single force, however, can be considered as acting at any point in its line of action; to transfer it to any parallel line, the addition of a couple is necessary.

8. Forces Not in a Plane.—Forces which do not lie in a plane may or may not all act at a point. The laws of composition, resolution, and equilibrium are similar to those for forces in a plane. Such forces, however, are not usually treated by graphical methods.

The resultant of such a set of forces is found by resolving each force, at any convenient point in its line of action, into components parallel to any three rectangular axes, OX, OY, OZ. The components parallel to OX are the only ones which tend to cause motion in this direction, and similarly for the components parallel to the other axes. Also, the components in the direction of OX and OZ are the only ones which tend to cause rotation around the axis OY, since the other components act parallel to that axis; and similarly for the other axes. The resultant of the given forces, therefore, must be such that its components and their moments about the axes, taken algebraically, satisfy the six equations

$$X_r = \Sigma X;\ Y_r = \Sigma Y;\ Z_r = \Sigma Z;$$

$$M_{rx} = \Sigma M_x;\ M_{ry} = \Sigma M_y;\ M_{rz} = \Sigma M_z.$$

The conditions of equilibrium of the set of forces are fixed by the condition that the resultant is zero, that is, there must be no tendency to change of motion or of rotation, and hence they are formulated by making the left-hand term, in each of the above equations, equal to zero. Hence there are six conditions of equilibrium of forces not in a plane nor at a point.

If all the forces act at a point, there is no tendency to rotation about axes passing through that point, and therefore the resultant goes through the same point, and is determined by the first three of the above equations.

9. Composition of Couples.—Couples (like forces), may be compounded, the resultant of any number of couples in the same plane being simply a couple in that plane whose moment equals the algebraic sum of the moments of the several couples. If two couples are in different planes, not parallel, imagine one of them replaced by a couple of equal moment, whose arm equals that of the other couple, and let these two couples be placed so that the common arms are coincident, and lie in the line of intersection of the two planes; then each couple will have each of its forces applied at the same point at which a force of the other couple acts, and these two pairs of forces can be compounded. The resultant of two couples is therefore another couple whose moment is the resultant of the two forces multiplied by the common lever arm, or

$$M_r = \sqrt{M_1^2 + M_2^2}.$$

This resultant couple will act in the plane of the resultant of the forces. Expressed otherwise, if we imagine each couple represented by a straight line drawn perpendicular to its plane from a common point O in the line of intersection of the two planes, in such a direction that looking from the end of this line to the point O, the moment of the couple in question is right-handed, the length of the line

representing to some scale the moment of the couple, and if we complete the parallelogram of which these two lines are adjacent sides, the diagonal of this parallelogram will represent the moment of the couple which is the resultant of the two given couples, the plane of the resultant couple will be perpendicular to this diagonal, and the resultant couple, looked at from the end of this diagonal toward O, will be right-handed.

This is illustrated by Fig. 5. The two couples have the forces F_1 and F_2, with the common arm ad, and lie in the planes ac and af respectively. The resultant couple has the resultant force F_{1-2} and arm ad, and lies in the plane of ah.

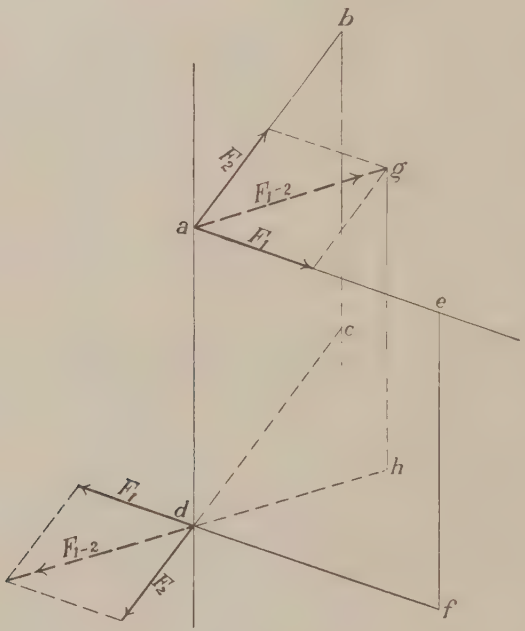


Fig. 5.

Since the couple which is the resultant of two given couples may be found, so, by combining this with a third couple, may the couple which is the resultant of any number of couples in different planes be found.

10. Summary Statement.—It is evident from what has preceded that any number of forces in a plane may always be combined into a single force or a couple in that plane. The procedure for finding this resultant analytically may be summarized thus: Take any two rectangular axes in the plane of the forces, and resolve each of the given forces into components parallel to these axes; find the moment of each component about the origin, or intersection of the axes; take the algebraic sum of the components parallel to each axis; the resultant will have the ame components, hence its magnitude will be the square root of the sum of the quares of these resultant components; divide the resultant moment about he origin by the resultant component along the axis X, and the quotient will

be the distance along the axis Y to the point where the resultant intersects this axis; similarly, the distance along the axis X to the point where the resultant intersects it may be found; hence the resultant acts through the two points so found. If the sums of the components of the given forces along each axis is zero, but the sum of the moments is M, then the resultant is a couple having this moment.

This procedure should be thoroughly mastered by the student.

In a similar way, if the given forces do not lie in a plane, they may always be reduced to (or have a resultant equal to) a single determinate force cutting any assumed plane in a determinate point and a couple in that plane; or, instead of this, a determinate couple composed of two forces *perpendicular* to the assumed plane, together with a determinate couple *in* the assumed plane. The procedure for determining the force, its point of application in the assumed plane, or the couples, is as follows (Fig. 6): Let the assumed plane be called A; assume any three rectangular axes, OX, OY, and OZ, OX and OY being in the plane A, and OZ perpendicular to it; resolve each of the given forces, at the point (x, y) where its line of action cuts the plane A, into three rectangular components X, Y and Z, parallel to the axes. Let the algebraic sums of the components parallel to the axes be $\Sigma X = X_r$; $\Sigma Y = Y_r$; $\Sigma Z = Z_r$. Then the only components which cause moments about X and Y are the components parallel to Z, while the only components causing moments about Z are those parallel to X and Y, and the resultant moments about the axes (positive when right-handed if observed from the positive side of O looking toward O) will be $M_x = -\Sigma Zy$; $M_y = \Sigma Zx$; $M_z = \Sigma(Xy - Yx)$.[1] The resultant of all the forces will therefore be (Fig. 6):

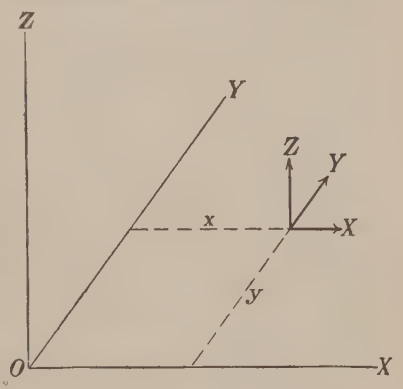


Fig. 6.

(1) A single force R whose magnitude is $\sqrt{X_r^2 + Y_r^2 + Z_r^2}$ and whose sense is found by combining the components X_r, Y_r and Z_r; it is applied at a point in the plane A whose coordinates are x_o and y_o, such that

$$Z_r y_o = \Sigma Zy = -M_x,$$
$$Z_r x_o = \Sigma Zx = M_y.$$

If this point (x_o, y_o) so determined, is such that $X_r y_o - Y_r x_o = M_z$ then the resultant of all the given forces is the single force R. If not, then there is also

(2) A couple in the given plane, whose moment is

$$M = M_z - X_r y_o + Y_r x_o.$$

If $R = \sqrt{X_r^2 + Y_r^2 + Z_r^2} = o$, and at the same time M_x or M_y, or both, are not equal to zero, the resultant is the couple M and also another couple M_1

[1] The signs will depend upon the direction assumed positive along the axes.

composed of forces *perpendicular* to the plane XY. If M_x is zero and M_y is not, the couple M_1 is composed of two forces in a plane parallel to the plane XZ; if M_y is zero and M_x is not, the couple M_1 is composed of two forces in a plane parallel to YZ; if neither M_x nor M_y are zero, the couple M_1 is composed of two forces (perpendicular to the plane XY) and acting in a plane which intersects the plane XY in a line making the angle α with OX such that $\tan \alpha = -\frac{M_x}{M_y}$.

We see, therefore, that in the case of forces in space the resultant is either

(1) A force R making, in general, an angle with the assumed plane, and a couple M in that plane or

(2) Two couples, one M in the assumed plane, and the other M_1 in a plane at right angles to it.

In the second case the two couples may be combined into a single couple in a plane oblique to the assumed plane. Generally, however, it is more convenient not to combine the two couples.

In the first case, the resultant force R may be resolved into two components, one ($\sqrt{X_r^2 + Y_r^2}$) in the assumed plane, the other Z_r perpendicular to it, and the first component may be combined with the couple M into a single force in the plane. Or, the couple M may be resolved into two couples, one in a plane P perpendicular to the force R, and the other in a plane P_1 containing R, and therefore perpendicular to the plane P. The couple in the plane P_1 may be combined with the force R to form a single force. The above results may, therefore, be restated as follows:

The resultant of any system of forces in space is either

1*a*. A single force (Z_r) perpendicular to any assumed plane, acting at a point ($x_o y_o$) such that

$$Z_r y_o = \Sigma Z y;\ Z_r x_o = \Sigma Z x,$$

together with a force $\sqrt{X_r^2 + Y_r^2}$ in the assumed plane, in such a position that its moment about $OZ = \Sigma(Xy - Yx)$; 1*b*. A single force $R = \sqrt{X_r^2 + Y_r^2 + Z_r^2}$, and a couple in a plane perpendicular to R or, (2) A single couple, the resultant of two couples, one in the assumed plane having a moment equal to $\Sigma(Xy - Yx)$ and the other in a plane at right angles to the assumed plane.

This whole matter may be stated in still another manner, thus: The resultant of any set of forces in space is a single determinate force acting at any assumed point O, together with three couples in any three assumed rectangular planes through O. The resultant force at O will have components along OX, OY and OZ of respectively ΣX, ΣY and ΣZ. The three couples will be the following:

$$\text{in the plane of } YZ,\ M_x = \Sigma Z y$$
$$\text{in the plane of } XZ,\ M_y = \Sigma Z x$$
$$\text{in the plane of } XY,\ M_z = \Sigma(Xy - Yx).$$

(The signs in the above equations are based on the assumption that OX and OY lie in the plane of the paper and OZ perpendicular to it; x positive to the right, y positive upward, and z positive toward the reader; components of forces positive when acting from O in the positive direction of the axes; moments positive when right-handed if viewed from the positive side of the axes toward O. The student should visualize these relations without the aid of a figure.)

This shows that if we wish to find the resultant of any set of forces, we may choose any plane we like, and the resultant will then be a couple lying in that plane together with a force cutting that plane, generally at an angle; or else it will be a single couple lying in a plane that may be determined. If we do not care to choose any plane, the resultant may be found as a force in a direction that may be determined, acting through a point that may be determined, together with a couple in a plane perpendicular to that force; or else it will be a single couple.

11. d'Alembert's Principle.—This is a principle by which any material system, or body, in motion, may be reduced to a condition in which it may be considered to be at rest, by the adding or introduction of certain known forces.

If every particle of such a system or body is moving with a uniform velocity in a straight line, the forces on each particle are balanced, and the particle is in equilibrium. But if the particle, whose weight is dw, is at any moment moving with an acceleration a in a given direction, there is a force acting on it in that direction equal to ma, if m is the mass of the particle or $\frac{dw}{g}$. If, in addition to the actual forces acting on the particle, a force is added *equal and opposite* to the force producing the acceleration, that is, equal and opposite to the force $\frac{a \cdot dw}{g}$, the particle may be considered as in equilibrium. Hence, *any material body, acted on by any external forces, may be considered as in a state of equilibrium, if to each particle of weight dw, having an actual acceleration a, there is added a force equal and opposite to the force* $\frac{a \cdot dw}{g}$, *where g is the acceleration of gravity.*

This is d'Alembert's principle.

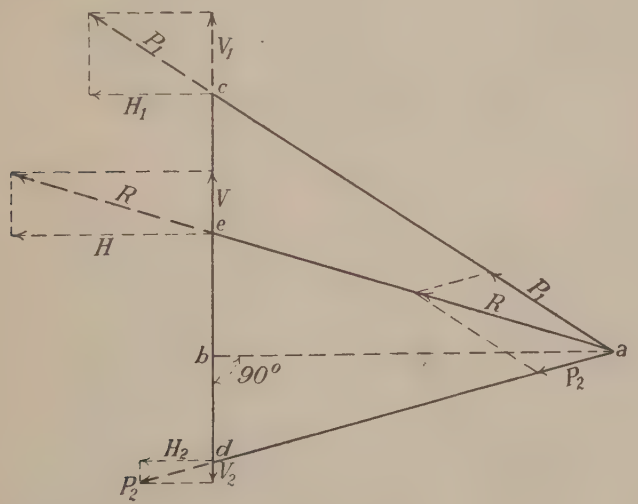


Fig. 7.

12. Analytical Methods of Resolving Forces.—If the preceding fundamental principles of equilibrium are kept clearly in mind, concrete problems occurring in actual practice may generally be solved with ease and rapidity, without the use of any but the simplest arithmetical formulas, and without trigonometric

functions. For instance, suppose that a given force R acting along a given line, ae (Fig. 7) is to be resolved into two components acting along given lines ad and ac. Graphically, the force R may be laid off as shown, and by drawing the triangle of forces the components P_1 and P_2 may be found. But the result may generally be found more quickly analytically. Usually the lines along which the components act are given by the dimensions of the structure; for instance, the distances ab, bc, bd, ce, and ed, may be directly given, or may be easily found. Let P_1, P_2, and R be resolved into components at c, d, and e respectively, these components being parallel and perpendicular to ab, as shown.

Taking moments about c, the principle that the moment of the resultant equals the sum of the moments of the components gives us at once

$$H_2 = H\frac{ce}{cd}$$

and similarly

$$H_1 = H\frac{ed}{cd}$$

Then clearly $V_1 = H_1\frac{cb}{ab}$ and $V_2 = H_2\frac{db}{ab}$.

This gives us at once the components of P_1 and P_2 along rectangular axes.

Analytical methods are applicable in cases where the figure of the structure has not been drawn to scale or where draughting instruments are not at hand. If the student will accustom himself to such methods, he will find them generally superior to graphical methods in ease as well as in rapidity.

13. Stress.—The term stress, as generally used, and as always used in this book, means a force distributed over a surface. When the student thinks of *stress*, therefore, he must have in his mind the complete conception of a force, including its sense, and he must also have in his mind the surface upon which this force is acting and over which it is distributed. These two elements, force and surface, are essential to the conception. Unless otherwise stated, the surface is a plane.

The resultant inner force, or stress, acting upon a plane surface, may be resolved into two components, one (a tension or a compression) acting *normal* to the surface, and one (a shear) acting *in* the plane of the surface; the latter is frequently resolved into two shears along rectangular axes in the plane of the surface. It is thus obvious that there are only two kinds of internal stress, *direct* and *tangential*. There can be no others. The direct stress, however, is different in character according as it acts away from the surface, tending to pull the particles apart, a *tension*, or as it acts toward the surface, a *compression*. There is no essential difference between a tangential force acting in one direction or sense, and one acting in the opposite sense, *i.e.*, either to the right or to the left, as when we try to slide a book along a table to the right or toward the left, although the resistance to the tangential tendency or stress may be greater in one way than in the other. A tangential stress is called a *shear*. Hence we may say that there are *three*, and *only three*, *distinct kinds of stress*, *viz. tension*, *compression*, and *shear*. The distribution of these forces over the plane surface on which they act is a very important matter, and determines the intensity of

the stress (which is the force per square unit) at any given point. If the stress is uniformly distributed over the surface, its resultant acts through the center of gravity of the surface, and its intensity is constant and equal to the total force divided by the area of the surface. If the stress is distributed not uniformly over the surface, the intensity at any point is obtained by finding the total stress upon an infinitesimal area da including the point, and dividing this by the area da. The distribution of a normal force N upon a plane surface, A, is generally represented by drawing lines at each point of the surface perpendicular to it, the length of each line being proportional to the *intensity* of the *stress* at that point (Fig. 8). The other ends of these lines will then lie in a surface A_1, A_2, etc., plane or curved. If this surface is a plane, the stress is either *uniformly distributed* (A_1), or it is *uniformly varying* (A_2) from the line of intersection of the two surfaces A and A_2, *i.e.*, if the intensity of stress at a perpendicular distance of one unit from that line be p, then at a perpendicular distance d the intensity of the stress will be pd. If the surface A is curved (A_3), the stress is distributed in a more complex manner, represented clearly by the shape of A_3. The volume included between A and A_3 may be said to represent the total force upon A, and it is at once evident that the resultant force passes through the center of gravity of this volume.

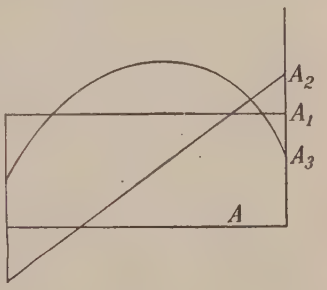


FIG. 8.

The resultant force acting upon a curved surface is similarly distributed over that surface, and the force acting upon any infinitesimal area may be resolved into *normal* and *tangential components* as in the case of a plane surface; but these components will, of course, vary in direction at each point of the curved surface.

The principles relating to distribution of stress are fully treated in Chap. IX, and it is very important that they should be mastered.

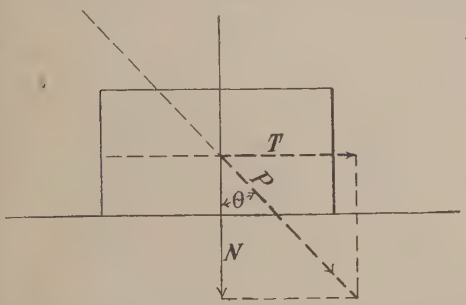


FIG. 9.

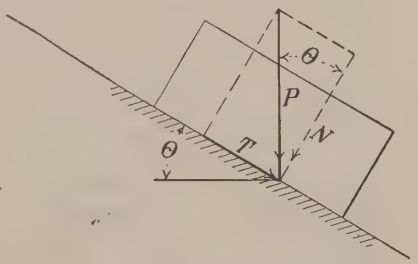


FIG. 10.

14. Friction.—If a body rests upon a plane surface (Fig. 9), the resultant pressure P which the body exerts upon the surface may be resolved into a normal component N and a tangential component T. The latter tends to slide the body along the surface, and when the angle θ becomes large enough, *i.e.*, equal to the angle of repose φ, the body will slide. When this condition exists $T = T_1$ and this is the friction which the surface is capable of exerting to resist the tendency to slide.

If the body is on an inclined plane, and acted upon only by its weight P (Fig 10) there is a normal and a tangential component as before, and when the doby is just on the point of sliding, the angle that the plane makes with the horizontal is the angle of repose. Values of this angle are given in books on mechanics. The value of the tangent of this limiting angle

$$\tan \varphi = \frac{T_1}{N_1} = f$$

is called the *coefficient of friction.*

The coefficient of friction varies with many circumstances, such as the roughness and character of the surface, the velocity (if the body is moving), the lubrication, the pressure, the temperature; but roughly speaking, and within reasonable limits of accuracy so far as the theory of structures is concerned, for dry surfaces, and for low speeds and pressures, the law of friction between solid bodies may be stated as follows:

(1) The coefficient of friction between two surfaces depends upon the character of the surfaces, varying with the materials, the smoothness, and the lubrication. The roughness alone does not determine the coefficient. Two materials having surfaces of equal roughness, one hard, the other soft, will have different coefficients. Clearly, also, the adhesiveness, the grain or fibre, and other qualities, will affect the coefficient.

(2) The coefficient is independent of the normal pressure and of the area of contact, and therefore of the intensity of pressure. That is to say, it takes, approximately the same tangential force to slide a brick whether it rests on its largest side or its smallest, since the total pressure is the same in either case.

This law is not strictly true, especially for very small or for very great intensities. In the former case, a considerable part of the resistance to sliding may be due to adhesion, and in the latter case, an indentation or other change in the character of the surfaces may result.

(3) The cœfficient of friction between solids, when one of the bodies is in motion, is generally considered approximately independent of the velocity. This again is not strictly true, especially for very low or for very high speeds.[1]

Strictly, the coefficient is a maximum just as motion is about to start; the surfaces are interlocked, as it were. When motion occurs, the coefficient decreases rapidly at first, as the velocity increases, then at a slower rate as the velocity continues to increase. Thus in tests of the friction between cast-iron brake shoes applied to the steel wheels of railroad cars, the coefficient was found to be as follows:

Velocity of train in miles per hour	17	21	27	31	37	51
Coefficient of friction	0.16	0.15	0.13	0.11	0.10	0.08

[1] It is interesting to note that the laws of fluid friction, *i.e.*, of a fluid flowing in a channel, are found to be, in general, the opposite of the above That is, for *solid friction* we have, referring to the tangential force necessary to move a body upon another, in general and within usual limits,

Solid friction (not the *coefficient* of friction) { varies directly with pressure; is independent of area of contact; is independent of velocity.

while for *fluid friction* we have

Fluid friction { is independent of pressure; varies directly with area; varies with some power of the velocity.

It will be evident upon reflection, also —and experience shows it to be true— that the coefficient gradually diminishes as the time increases, owing to the gradual smoothing out of irregularities of the surfaces.

Lubrication, and the character of the lubricant, modify the above principle. In such cases, the coefficient may be almost independent of the pressure, and for low pressures will vary almost as the speed, and will depend more upon temperature (which affects the viscosity of the lubricant) than upon any other condition. It will also vary less with the character of the surfaces than when they are dry, and may be almost independent of that element. In this case, also, the friction may *increase* with the time.

The friction of rest is in any case greater than the friction during motion.

It follows from the above, that if the body rests upon a surface (Fig. 11) and we lay off on either side of the normal to that surface the angle φ, the resultant force upon the body must lie within a cone of friction whose apex angle equals 2φ.

If a surface is perfectly smooth ($f = o$), then the pressure exerted upon it by another body must be normal to that surface, if equilibrium exists.

If a body, instead of resting directly upon a surface as shown in Figs. 9 to 11, rests upon rollers, or a single roller, supported upon the surface (Fig. 12), the

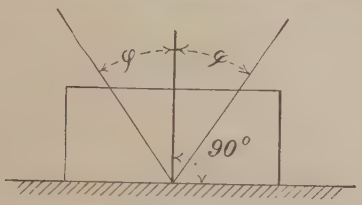


FIG. 11.

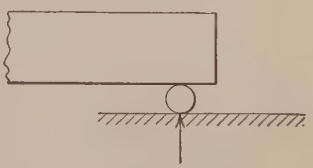


FIG. 12.

frictional resistance is very much diminished, and for practical purposes is taken as zero. That is to say, if one body rests upon another, with a roller between them, whether or not it is in equilibrium, the force exerted at the point of contact through the roller, must be normal to the surface, if the frictional resistance of the roller, as assumed, is zero.

The student should be thoroughly familiar with these elementary principles of friction, and should have mastered the subject as treated, for instance, in the chapter on Friction in Maurer's "Technical Mechanics," or the chapter on Friction and Lubrication in Slocum's "Theory and Practice of Mechanics."

15. Values of the Coefficient of Static Friction between Unlubricated Solids at Rest.—The structural engineer is not often concerned, except in the case of movable bridges, with friction during motion, or with thoroughly lubricated surfaces. He is generally concerned with the frictional force which may be developed between dry surfaces before motion will occur, termed *static friction.*

This coefficient is found either by measuring the angle of repose, or by measuring the force which is necessary to slide a given weight upon a surface. It may also be measured by first observing the velocity with which a body is sliding or rolling upon another, and then the time or distance required for it to come to a stop, when no force is applied (although this method involves friction during motion). The following table gives the average values of some coefficients of static friction:

Metals on metals	0.15 to 0.25
Metals on wood	0.20 to 0.60
Metals on stone	0.30 to 0.70
Wood on wood	0.20 to 0.50
Wood on stone	0.40 to 0.60
Stone on stone	0.60 to 0.75
Brick on brick	0.50 to 0.75
Brick on stone	0.60 to 0.75
Concrete on concrete (blocks)	about 0.65
Masonry on gravel	about 0.60
Masonry on dry clay	about 0.50
Masonry on sand	about 0.40
Gravel on moist clay	about 0.33
Earth on earth	0.25 to 1.00

The wide variation in these coefficients is to be observed. The point ot remember is that much depends upon the character of the surfaces in any given case.

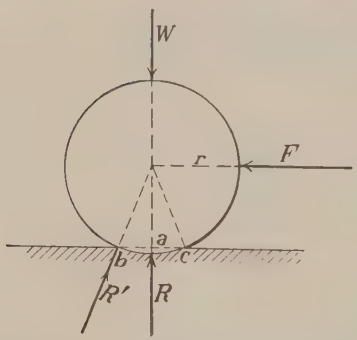


FIG. 13.

16. Rolling Friction (Fig. 13).—The force F necessary to cause a cylinder or wheel whose weight or load is W to roll upon a plane surface, involves a different principle from that involved in the friction between two surfaces. If the wheel touches the surface at a line only, the coefficient will be zero, since any force F, however small, will disturb the equilibrium between W and R. As a matter of fact, the weight W causes the wheel to indent the surface slightly, distributing that weight over an uncertain surface from b to c. In that case the value of F which will just cause the wheel to roll, will be that which causes the resultant of W and F to go through b, since if it should fall to the left of that point, motion would result. Hence, the limiting value of F is reached when very nearly

$$Fr = W.ab$$

$$F = W \cdot \frac{ab}{r}$$

The coefficient of friction in this case $\frac{F}{W}$, is the ratio $\frac{ab}{r}$. In any case, it is very small, and may be considered zero between metallic bodies.

17. Hinge: Ball Bearing.—If one body is connected to another body by a frictionless hinge, *i.e.*, a cylinder about which either body is free to turn without restraint, then the resultant force acting upon either body (omitting the reaction from the other body) must intersect the axis of the hinge, for otherwise, rotation would ensue. If the forces all lie in the plane of the paper, and the axis of the hinge is at right angles thereto through a point a, this resultant must pass through a. If there is friction on the hinge, the resultant must be tangent to what is termed the *friction circle*, which has a radius $= r \sin \varphi$, instead of passing

through the center of the hinge. For structural purposes it is generally close enough to assume frictionless hinges.

If one body is connected to another body by a ball bearing, *i.e.*, a sphere about which either body is free to turn in any direction without restraint, then the resultant force acting upon either body (omitting the reaction from the other body) must act through the center of the sphere.

The force through a hinge or through a ball bearing may be either a tensile force tending to pull the two bodies apart, or a compressive force tending to push them together, if the connection is suitably designed. The effect of these connections is simply to fix the condition that the force must pass through a given line or a given point in order to secure equilibrium.

In order to secure the above results, the hinge or the ball bearing must be frictionless. Such a connection is, of course, impossible, in strictness, as some friction will exist, even in a lubricated bearing. The friction, however, is generally neglected.

If a flat bar is connected at its end by a pin at right angles to its length, the pin forms a hinge, and the resultant force on the bar must act through the pin, and is balanced by the reaction exerted by the pin upon the bar.

18. Kinetics: Momentum, Energy, Work.—With this branch of the subject of mechanics, the structural engineer has comparatively little to do, since his problems deal mainly with bodies at rest. Nevertheless, he should thoroughly understand the fundamentals. He should clearly understand the effect of force on a particle of matter or upon a finite body; and a few principles may be enumerated here.

An unbalanced force acting upon a particle always produces motion in the direction of the force. *The force multiplied by the time during which it acts equals the change of momentum*, which is the product of the mass and the velocity. The mass is the quantity of matter in the particle—not its weight, which is the force with which the earth attracts it, and which varies from point to point of the earth's surface—but the quantity of matter, which is invariable. If, then, a force F acts for a time t upon a body of mass m, and if the velocities in the direction of the force at the beginning and end of the time are respectively V_1 and V_2, then $Ft = m(V_2 - V_1)$.

The force multiplied by the distance through which it moves in the direction of its action equals is the work done by the force, and equals the change in energy of the particle, which is equal to $\frac{mV^2}{2}$. Hence if d is this distance

$$Fd = \frac{m}{2}(V_2^2 - V_1^2).$$

If a force is balanced by an equal and opposite force in the same line, the body will not move, but the force will still have effects. It will produce internal stresses between the particles of the body, and these will produce corresponding deformations, though the body as a whole will not move, as when a rubber ball is compressed between the fingers. In this case the outer forces do work, because their points of application move; but this work is exactly equal to the internal work done by the inner forces in altering the distances between the particles. If the body is elastic, the particles are ready to assume their original positions as soon as the applied forces are relaxed, and in doing so they will do exactly the same work that was originally done upon them. The principle of the *conservation of*

energy is that no energy is lost, but only transformed; either into some other form, such as heat or electricity, or into latent or *potential energy*, as of a compressed spring which in resuming its original length can do the same amount of work done in compressing it (less what may be lost by heat), or a weight lifted from the earth which in its raised position can do in falling the same amount of work spent in lifting it (potential energy of position), or into *kinetic energy* or a change of $\frac{mV^2}{2}$, which shows itself in an increase of velocity.

The principles of work find frequent application in structural engineering, as will later be shown.

References

There are many excellent works on mechanics, and one of these should be in the library, and more important, in the mind, of every structural engineer. Some of those most available may be mentioned:

Church, Irving P.: "Mechanics of Engineering," John Wiley and Sons, N. Y., is an elaborate work, treating not only of Mechanics, but of the Strength of Materials, and of Hydraulics and Pneumatics as well.

DuBois, A. Jay: "The Mechanics of Engineering," vol. 1, John Wiley and Sons, 1902, is a more elaborate work, covering in admirable style, if somewhat abstrusely, the principles of Mechanics and of the Strength of Materials.

Fuller and Johnson's "Applied Mechanics," John Wiley and Sons, N. Y., 1915. Volume 1 of this excellent work treats of Statics and Kinetics, and gives all that the student ordinarily needs of these subjects.

Johnson, L. J.: "Statics," John Wiley and Sons. Excellent on this branch.

Maurer, E. R.: "Technical Mechanics," third edition, John Wiley and Sons, 1909, is well adapted for the structural engineer, and is in every way admirable.

Slocum, S. E.: "The Theory and Practice of Mechanics," Henry Holt and Co., 1913, is an excellent work, very clear and concise, with many engineering applications, avoiding the abstract and abstruse character of some more pretentious works.

There are many other good books in English, French, and German, but one of the above will be sufficient for the reader of this book.

Problems

1. A force which is completely given is to be resolved into two components: In each of the following cases answer the question, or state whether a solution is possible, and, if so, whether more than one, and how many; and indicate the method of solution:
   - (*a*) What conditions must the components fulfil?
   - (*b*) Both lines of action are given.
   - (*c*) The line of action of one component is given, and a line parallel to the line of action of the other.
   - (*d*) In (*c*) there is given, in addition, the *sense* of one component.
   - (*e*) The line of action of one component and a point on the line of action of the other are given.
   - (*f*) The line of action of one component and the magnitude of the other are given.
   - (*g*) The magnitudes of both components are given.

2. A given force is to be resolved into three components:
   - (*a*) Must all three lie in one plane?

   Suppose they all lie in one plane:

(*b*) The lines of action of all three are given.

(*c*) The lines of action of two and a point in the line of action of the other are given.

(*d*) The lines of action of two and a line parallel to the line of action of the other are given.

(*e*) The lines of action of two and the magnitude of the third are given.

(*f*) The line of action of one and the magnitude of the other two are given.

(*g*) The line of action of one, a point in the line of action of the other, and the magnitude of the third are given.

(*h*) The magnitudes of all three components are given.

3. A given force is to be resolved into four components in a plane.

(*a*) How many of the components must act in a plane with the resultant?

(*b*) How many must intersect the line of action of the resultant?

(*c*) The lines of action of all four components are given.

(*d*) The lines of action of three and the magnitude of one are given.

(*e*) The lines of action of three and a point on the line of action of the fourth are given.

(*f*) The lines of action of two and the magnitudes of two are given.

4. A body rests upon an inclined plane. Find, in terms of the necessary quantities, the expression for the horizontal force just sufficient to make it slide up-hill.

The teacher can indefinitely extend examples of this kind if necessary, and should assign concrete cases of given forces to be combined, and of a given force to be resolved.

CHAPTER III

GEOMETRICAL RELATIONS

GEOMETRICAL PROPERTIES OF SURFACES AND LINES

1. In the study of structures, a knowledge of certain geometrical properties of lines and surfaces is essential. For instance, in the study of stress, in which it is necessary to consider the body as cut in two parts by a plane, and to study the relations between the external forces acting upon one of the parts and the internal forces or stresses acting upon the cut surface, termed the *section*, the geometrical properties of this plane area must be understood.

The properties to be considered are: (1) the length of a curved line; (2) the *area* of a surface; (3) the *center of gravity* of a plane or surface, and sometimes of a volume; (4) the *moment of inertia* of an area; (5) the *radius of gyration* of an area; (6) the *product of inertia* of an area. These will be considered in this chapter.

(1) LENGTH OF CURVED LINES

2. Length of a Circular Arc.—If s is the length of the arc ACB (Fig. 14), $a = AB$, and $b = AC$, the value of s is given very closely by the formula

$$s = \frac{8b - a}{3}. \tag{1}$$

The error by this formula is less than one-tenth of 1 per cent, even if s represents an arc of 60°.

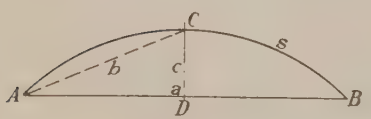


Fig. 14.

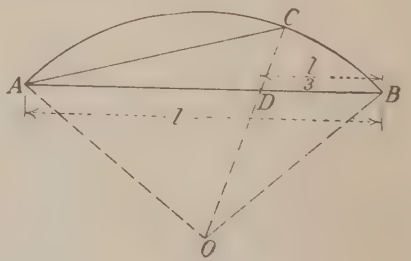


Fig. 15.

A simple graphical method of finding the length of a circular arc is shown in Fig. 15. Let AB be the arc whose length is desired, and O the center. Let $DB = \frac{1}{3}AB$ and draw the radius ODC. Then length of arc $ACB = \frac{3}{2}$ · length of chord AC. This construction is said to involve an error of one-half of 1 per cent, when the angle $AOB = 90°$, and of only one-hundredth of 1 per cent when AOB is 40°.[1]

[1] Taylor and Thompson, "Concrete: Plain and Reinforced," third ed., p. 728.

The length of a circular arc ($2r\alpha$ in radians) can, of course, be easily found if r and α are known; but frequently, in engineering problems, the given quantities are the chord a and the rise c, from which b can be found, and then s by the above formula.

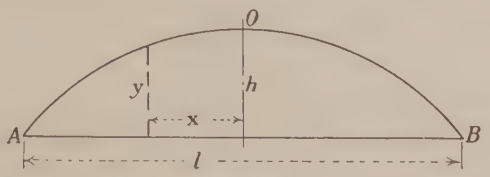


FIG. 16.

3. Length of a Parabola (Fig. 16).—The length of an element of the curve ds is

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The value of y is $h - \frac{4hx^2}{l^2}$, hence $\frac{dy}{dx} = -\frac{8hx}{l^2}$;

hence the length of the curve AOB is

$$\begin{aligned} L &= 2\int_o^{\frac{l}{2}}\left[1 + \frac{64h^2x^2}{l^4}\right]^{\frac{1}{2}}dx \\ &= \frac{l}{2}\left(1 + \frac{16h^2}{l^2}\right)^{\frac{1}{2}} + \frac{l^2}{8h}\log_e\left[\frac{4h}{l} + \left(1 + \frac{16h^2}{l^2}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{2}$$

This gives the exact length of OB, but is somewhat complicated. The approximate length may be obtained by expanding the value of L under the integral, and integrating each term; thus

$$\begin{aligned} L &= 2\int_o^{\frac{l}{2}}\left(1 + \frac{32h^2x^2}{l^4} - \frac{512h^4x^4}{l^8} + \ldots\right)dx \\ &= l\left(1 + \frac{8}{3}\cdot\frac{h^2}{l^2} - \frac{32}{5}\cdot\frac{h^4}{l^4} + \ldots\right) = l\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \ldots\right) \end{aligned} \tag{3}$$

if $n = \frac{h}{l}$.

In most engineering problems n is small, and it is quite accurate enough to write

$$L = l\left(1 + \frac{8}{3}n^2\right). \tag{4}$$

The error in using this formula, as compared with the result by including the next term is, if $n = 0.1$, only $\frac{3}{100}$ of 1 per cent; and if n is even as large as $\frac{1}{5}$, it is only 1 per cent, which is much less than the usual error in engineering computations.

(2) AREAS

4. The reader is assumed capable of finding with sufficient accuracy the area of any plane figure, whether bounded entirely by straight lines, or in part by curved lines.

A figure bounded by straight lines may always be divided into parallelograms and triangles, or entirely into triangles. A figure bounded by curved lines may

be divided approximately in the same way, by considering the curve to be a succession of straight lines; and while in many cases such a division will be sufficiently accurate, greater accuracy may be attained by taking into account the segment between the chord and the arc; so that for our purposes all areas may be divided into parallelograms, triangles, and segments of curves, or into triangles and segments. A segment may be considered a circular or a parabolic segment.

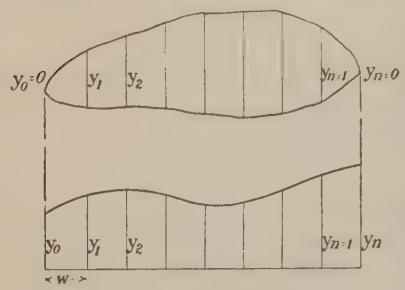


FIG. 17.

5. Frequently, however, the above procedure is inconvenient, as, for instance, where the curved boundary is very irregular (Fig. 17). In such a case, the area is ascertained by dividing the surface into strips of equal width by parallel lines (drawn perpendicular to some properly assumed axis), and using one of the following rules, which are themselves deduced from geometrical considerations:

Trapezoidal rule:

$$A = w(\tfrac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \tfrac{1}{2}y_n). \tag{5}$$

Simpson's rule:

$$A = \frac{w}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n) \tag{6}$$

(this requires an *even* number of strips of equal width).

Durand's rule:

$$A = w(0.4y_0 + 1.1y_1 + y_2 + y_3 + \ldots + y_{n-2} + 1.1y_{n-1} + 0.4y_n). \tag{7}$$

6. Area of Circular Segment (Fig. 18).—Length of curve ACB (in radians) $= 2r\alpha$

Length of chord $AB = 2r \sin \alpha$
$OD = r \cos \alpha$
$AF = r \tan \alpha$

$$\text{Area } OADB = \frac{r^2 \sin 2\alpha}{2} \tag{8}$$

$$\text{Area } OACB = r^2\alpha \tag{9}$$

$$\text{Area } ACBD = r^2(\alpha - \sin\alpha\cos\alpha) \tag{10}$$

$$\text{Area } ACBF = r^2(\tan\alpha - \alpha). \tag{11}$$

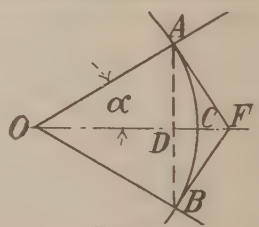


FIG. 18.

7. Area of a Parabolic Segment (Fig. 19).—If AB is the axis of a parabola, and A its vertex, the area of any symmetrical segment like $ADBG$ equals *two-thirds of the rectangle* $EDGF$.

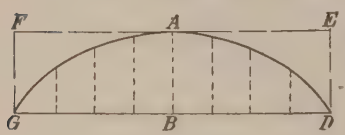


FIG. 19.

At any point, A', of a parabola (Fig. 20) let a tangent $F'A'E'$ be drawn, a chord $D'G'$ parallel thereto, and a line $A'B'$ parallel to the axis AB. $A'B'$ will bisect $G'D'$, and the area of the segment $A'D'C'G'$ equals two-thirds of the area of the parallelogram $E'D'G'F'$, in which $E'D'$ and $F'G'$ are parallel to AB.

In almost all practical cases, a flat arc of a curve (ACB in Fig. 14) may be considered to be a parabola, and the area of a segment $ACBD$ between the

curve and a chord AB may be taken as two-thirds of the product of AB and the ordinate CD perpendicular to AB at its center. Even for a circular arc this will be very close for central angles (2α) as great as 60°, in which case the ratio of the segment ACB to the enclosing rectangle will actually be 0.676 instead of 0.667. For $2\alpha = 90°$, the ratio will still be less than 0.69, and for a semicircle ($\alpha = 90°$) it will be 0.7854. The ratio for flat arcs will clearly be very close to two-thirds.

Tables of the exact areas of circular segments are given in engineering handbooks, but in structural work it is rarely necessary to use them.

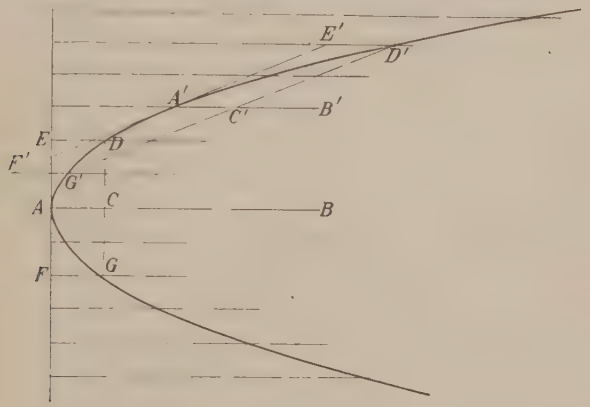


FIG. 20.

(3) CENTER OF GRAVITY

8. The center of gravity of a body is the point in the body through which the resultant weight of the body *i.e.*, the resultant of the weights of all its particles passes, no matter how the body is turned in space; *i.e.*, it is the point through which the resultant of a set of parallel forces acts, no matter what their direction is, provided they always are parallel, all have the same sense, and with the points of application unchanged. The reader may easily prove by the principles of statics that there is such a point.

The center of gravity of a plane area is the center of gravity of a slab, of infinitesimal thickness, having the given surface. The center of gravity of a line is the center of gravity of a rod of infinitesimal cross-section having the line for an axis. The centers of gravity of areas and lines may obviously be determined as though they had no thickness, using the areas and lengths respectively as if they were weights. The center of gravity of a line, surface, or volume, considered independent of weight, is called the *centroid*.

To determine the centroid, therefore, it is only necessary to suppose all the given forces to act in any assumed direction, and to find the line of action of their resultant by taking moments about two rectangular axes in a plane perpendicular to the direction of the forces. The moment of the resultant about each of these axes must equal the algebraic sum of the moments of the separate forces; hence the coordinates of a point in the resultant may be found. Then the given

forces are supposed to act in some other assumed direction, and the line of action of the resultant again found. The centroid will be at the point of intersection of the two resultants, and even if the forces are in space (not in a plane), the lines of action of all such resultants will always actually intersect in the same point of the body.

If the forces are all in one plane, which is the plane of the paper, two rectangular axes may be taken in that plane, the forces first supposed to act parallel to the axis OY, and the distance x of the resultant from the axis is found by taking moments about O; then the forces may be supposed to act parallel to OX and the distance y of the resultant from that axis found by taking moments about O, and the center of gravity of the forces will have the coordinates x and y.

If the forces are the weights of the particles of a body, the centroid becomes the center of gravity, so that the center of gravity is the centroid of the weights of all the particles; or the center of gravity of an area or a line is the centroid of all the elementary areas or lengths of which it is composed.

To find the distance x_o of the center of gravity of any given area A at right angles to any straight line in its plane, let x be the perpendicular distance from that line of any infinitesimal part of the area, da. Then

$$x_o = \frac{\int x da}{\int da} = \frac{\int\int x dy dx}{\int\int dy dx} = \frac{M}{A} \tag{12}$$

M being the moment of the area about the line. From this it follows that if $M = 0$, $x_0 = 0$, or the *moment of any plane area about any straight line passing through its center of gravity is zero.*

The following principles result:

(1) If a line, surface, or solid, has an axis or plane of symmetry, the center of gravity lies in that axis or plane; for an axis or plane of symmetry is a line or plane such that for every small area or volume in the figure, on one side of it, there is another similar area or volume at the same distance on the other side of it, the line joining the two points being at right angles to the axis or plane.

(2) If a line or surface has two axes of symmetry, the center of gravity is at their intersection.

(3) If a solid has two planes of symmetry, its center of gravity lies in the intersection of those planes; if it has three planes of symmetry, its center of gravity is at the intersection of those planes.

(4) If a figure (line, surface, or solid) be divided into any two parts, the center of gravity of the whole is on the line joining the centers of gravity of the parts.

(5) An area (such as a triangle) may have a line (such as the line from any vertex to the center of the opposite side) such that for each and every small area on one side of that line there is an equal area on the other side, at the same perpendicular distance from that line (as in a strip parallel to the base). In such case, the center of gravity lies upon the line that possesses this property, for this line passes through all the centers of gravity of the strips into which the figure may be divided.

Principles (1) and (5) may be more generally expressed thus: If for each and every infinitesimal portion of a line, surface, or solid, there is an equal portion (*i.e.*, equal in length, area, or volume) so situated that the line joining the centers of gravity of these two portions is in every case bisected by a fixed line or plane,

then the center of gravity of the entire figure (line, surface, or solid) lies upon this line or in this plane.

In finding centers of gravity, the above fundamental principles, if kept in mind, will enable the student to work out any problem that may be presented, even though he may have forgotten the short methods of solution which may be worked out by a more elaborate study. In this way, by thoroughly mastering fundamental principles, the student should seek to relieve his memory. Memory of processes should always, if possible, be subordinated to mastery of principles. For instance, from the fundamental principles the student should easily be able to find the center of gravity of a trapezoid, perceiving that it must lie on the line joining the centers of the two parallel sides, and then that it must be on the line joining the centers of gravity of the two triangles into which the trapezoid may be divided; hence it must lie at the intersection of these two lines.

9. Examples of Center of Gravity (*abbreviated c.g., lettered C*) (*a*) *Lines.*—The c.g. of a straight line is at the center of its length.

The c.g. of two straight lines is on the line joining their centers, and divides this line in a proportion inverse to the ratio of the lengths of the lines (Fig. 21). Thus if C_1 and C_2 are the centers of AB and BD, the c.g. is at C, such that

$$C_1C : CC_2 :: BD : AB; \text{ or } C_1C : C_1C_2 :: BD : BD + AB.$$

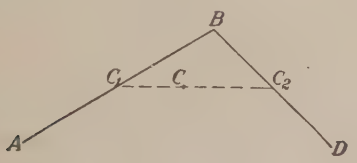


Fig. 21.

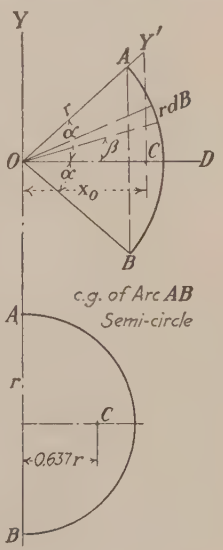


Fig. 22.

Arc of a Circle AB (Fig. 22).—The c.g. is on the axis of symmetry OD bisecting the angle AOB.

Any little element of the arc, $rd\beta$, is distant from OY a distance $r \cos \beta$. Taking moments about OY

$$OC = x_o = \frac{\int_{-\alpha}^{\alpha} r^2 \cos \beta d\beta}{2r\alpha} = \frac{r \sin \alpha}{\alpha} \tag{13}$$

$$\text{If } 2\alpha = 90^\circ : x_o = \frac{2r\sqrt{2}}{\pi} = 0.9r.$$

$$\text{If } 2\alpha = 180^\circ : x_o = \frac{2r}{\pi} = 0.637r.$$

To find the c.g. of other curved lines, like the parabola, is complicated, and will not be undertaken. Generally in practice it is sufficiently close to consider a flat curve as a circular arc.

10. (*b*) *Areas.*

1. *Rectangle.*—The c.g. lies at its center, *i.e.*, at the intersection of the diagonals, or of the lines joining the centers of parallel sides. The same is true of a *parallelogram.*

2. *Triangle.*—The c.g. lies on the median line, or the line joining any angle with the center of the opposite side (Fig. 23), because this line passes through the c.g. of every strip *ab* parallel to *BD*; hence the c.g. of the triangle is at the intersection of the three median lines, and is one-third the distance from any side to the opposite angle. Since any figure bounded by straight lines can be divided into

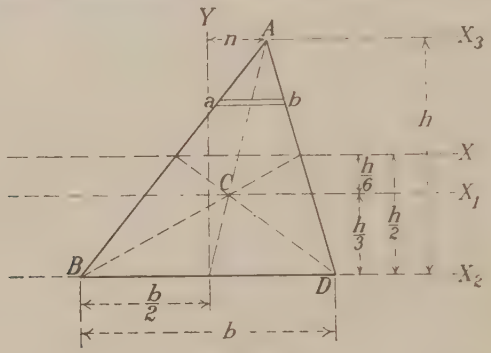


FIG. 23.

triangles, the c.g. of any such figure can be found. Special constructions apply, however, to certain figures.

3. *Trapezoid.*—The c.g. may be found by various constructions:

First construction (Fig. 24):

Lay off on either side of b the distance b_1 (b and b_1 being the parallel sides) and on the other side of b_1 lay off b. Join the points so located by a straight line, and where it meets the line joining the centers of b and b_1 will be the c.g. If the

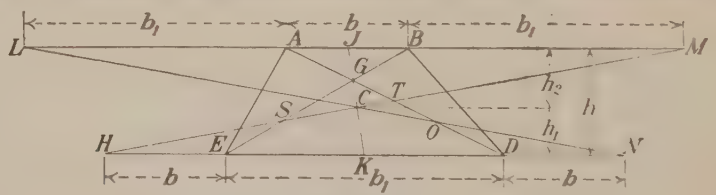


FIG. 24.

figure is divided into two triangles by the line EB, as shown, then assuming C to be the c.g. and taking moments about the base b_1

$$\frac{b + b_1}{2} \cdot h \cdot h_1 = \frac{bh}{2} \cdot \frac{2}{3} h + \frac{b_1 h}{2} \cdot \frac{h}{3},$$

from which

$$\frac{h_1}{h} = \frac{2b + b_1}{3(b + b_1)}. \tag{14}$$

Similarly

$$\frac{h_2}{h} = \frac{2b_1 + b}{3(b + b_1)} \tag{15}$$

therefore

$$\frac{h_1}{h_2} = \frac{2b + b_1}{2b_1 + b} = \frac{HN}{LM} \tag{16}$$

which conforms to the above construction.

The line HM cuts EB in S, such that from the triangles HES and SBM

$$ES : SB :: b : b_1.$$

But from triangles ABG and EDG

$$GB : GE :: b : b_1$$
$$\therefore ES = BG.$$

Similarly $DO = AG$ if O is where LN cuts AD.
Also, HM bisects AD since $AM = HD$; hence HM bisects GO.
Hence follows the
Second construction (Fig. 24):

Draw the diagonals AD and EB, and the median line JK. The c.g. must lie on JK. Lay off $DO = AG$, and $ES = BG$. Bisect GO and GS, and join the points so found with S and O. These lines will both meet JK at C.

The triangles AES and AGB have equal areas, and the c.g. of each, and of the triangle ASG, is on a line parallel to EB; and the c.g. of AES and AGB are equidistant from the c.g. of ASG; hence the c.g. of ASG coincides with the c.g. of AEB. Similarly, the c.g. of DSG coincides with the c.g. of DEB. Hence the c.g. of ASD coincides with the c.g. of the trapezoid, and is at the intersection of ST and JK.

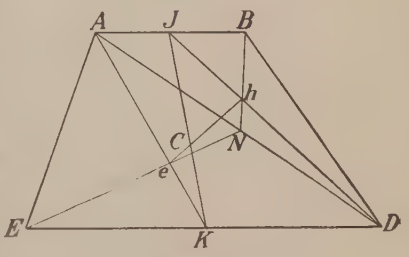


FIG. 25.

This is a special case of the quadrilateral explained below.
Third construction (Fig. 25):

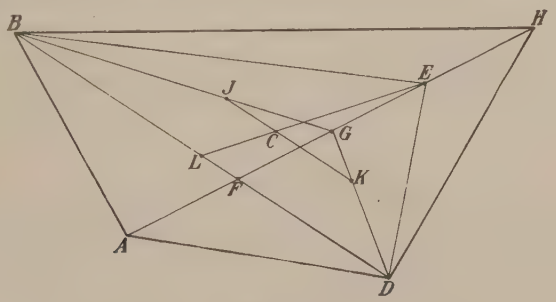


FIG. 26.

Find the c.g. (h) of the triangle ABD by drawing DJ and BN. Find the c.g. (e) of the triangle AED by drawing AK and EN. Where he cuts JK is C, the c.g. of $ABDE$. In dividing into triangles, care must be taken to secure good intersections, the lines being as nearly at right angles as possible.

4. *Quadrilateral* (Fig. 26).— To find the c.g. of any quadrilateral, proceed as follows:

Select either diagonal, say AH, lay off $HE = AF$, draw EB and ED. The c.g. of the triangle EBD will also be the c.g. of the quadrilateral. For let G be the center point of FE; it will also be the center of AH. The c.g. of the triangle BFE will be on BG, at J, such that $BJ = \frac{2}{3}BG$, and this will also be the c.g. of the triangle BAH. Similarly, the c.g. of the triangles DFE and DAH coincide at K such that $DK = \frac{2}{3}DG$. Hence the c.g. of the triangle EBD lies on JK, at C, such that $JC : CK :: \Delta BFE : \Delta FED$; and since $\Delta BFE : \Delta FED :: \Delta BAH : \Delta DAH$, the c.g. of the quadrilateral lies at the same point. Find C by drawing EL, L being the center point of BD.

11. Figures Bounded Partly by Curved Lines.

5. *Circular Sector OAFB* (Fig. 27).—This sector is made up of an infinite number of triangles, the c.g. of each being $\frac{2}{3}r$ distant from O. Hence the c.g.

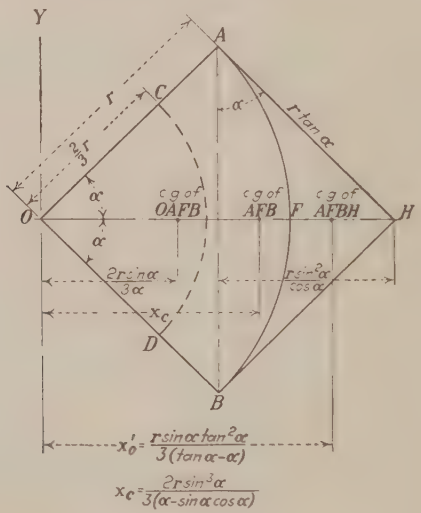


Fig. 27.

of the sector lies at the c.g. of the circular arc CD, drawn with a radius $r' = \frac{2}{3}r$. This c.g. will lie on the line OF, bisecting the arc AB, and at a distance from O equal to

$$x_o = \frac{r' \sin \alpha}{\alpha} = \frac{2r}{3\alpha} \sin \alpha. \tag{17}$$

The area A of the sector is αr^2 (18)

(α is of course expressed in radians).

$$\text{If } 2\alpha = 90°;\ x_o = \frac{4r\sqrt{2}}{3\pi} = 0.6r;\ A = \frac{\pi r^2}{4}$$

$$\text{If } 2\alpha = 180°;\ x_o = \frac{4r}{3\pi} = 0.425r;\ A = \frac{\pi r^2}{2}.$$

6. *Circular Segment AFB* (Fig. 27).—This is the difference of the sector $OAFB$ and the triangle OAB. Hence

$$\text{Area} = \alpha r^2 - r^2 \sin\alpha \cos\alpha = r^2(\alpha - \sin\alpha\cos\alpha). \tag{19}$$

$$x_c = \frac{\alpha r^2 \cdot \frac{2r\sin\alpha}{3\alpha} - r^2 \sin\alpha\cos\alpha \cdot \frac{2}{3} r\cos\alpha}{r^2(\alpha - \sin\alpha\cos\alpha)}$$

$$= \frac{2r\sin^3\alpha}{3(\alpha - \sin\alpha\cos\alpha)}. \tag{20}$$

7. *Outside Circular Segment AFBH* (Fig. 27).—The area $OAHB = r^2 \tan\alpha$. Therefore, the area

$$AFBH = r^2 \tan\alpha - \alpha r^2 = r^2(\tan\alpha - \alpha). \tag{21}$$

To take moments of the area $OAHB$ about OY, divide it into two triangles, OAB and ABH. Then the moment of $OAHB$ about OY

$$= r^2 \sin\alpha\cos\alpha \cdot \tfrac{2}{3} r\cos\alpha + r\sin\alpha\left(\frac{r\sin^2\alpha}{\cos\alpha}\right)\left(r\cos\alpha + \frac{r\sin^2\alpha}{3\cos\alpha}\right)$$

$$= \frac{r^3\sin\alpha}{3}(1 + \sec^2\alpha).$$

The moment of the sector $OAFB$ about $OY = \alpha r^2 \cdot \frac{2r\sin\alpha}{3\alpha} = \frac{2}{3} r^3 \sin\alpha$.

Hence the distance of the c.g. of $AFBH$ from OY is

$$x_o' = \frac{\frac{r^3\sin\alpha}{3}(1 + \sec^2\alpha) - \frac{2r^3\sin\alpha}{3}}{r^2(\tan\alpha - \alpha)}$$

$$= \frac{r\sin\alpha}{3} \frac{\sec^2\alpha - 1}{\tan\alpha - \alpha} = \frac{r\sin\alpha\tan^2\alpha}{3(\tan\alpha - \alpha)}. \tag{22}$$

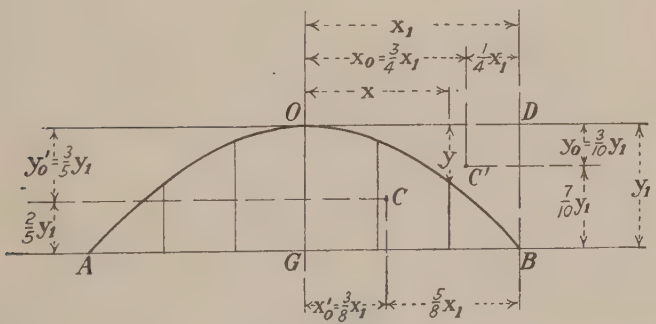


FIG. 28.

8. *Parabolic Segment* (Fig. 28).—If O is the origin of coordinates and the vertex of the parabola, and OG the axis, the curve is such that deflections from OD vary as the squares of the distances from O, or $y = ax^2$.

Hence the area of the external segment ODG is

$$\int_o^{x_1} y\,dx = a\int_o^{x_1} x^2\,dx = \frac{ax_1^3}{3} = \frac{x_1y_1}{3} = \frac{1}{3} \cdot OG \cdot OD.$$

Hence the area of OGB is $\frac{2}{3} OG \cdot GB$, and the area of AOB is $\frac{2}{3} \cdot AB \cdot OG$.

The distance of the c.g. of ODB (C') from OG is

$$x_o = \frac{3\int_o^{x_1} xy\,dx}{x_1y_1} = \frac{3a\int_o^{x_1} x^3\,dx}{x_1y_1} = \frac{3}{4}\frac{ax_1^3}{y_1} = \frac{3}{4}x_1. \tag{23}$$

The distance of the c.g. of OGB (C) from OG is found by taking moments of the rectangle and external segment, or

$$x_1y_1 \cdot \frac{x_1}{2} - \frac{x_1y_1}{3} \cdot \frac{3}{4}x_1 = \frac{2x_1y_1}{3} \cdot x_o'$$

$$x_o' = \frac{3}{8}x_1. \tag{24}$$

The distance of the c.g. of ODB from OD is

$$y_o = \frac{3\int_o^{x_1} ydx \cdot \frac{y}{2}}{x_1y_1} = \frac{3\int_o^{x_1} y^2dx}{2x_1y_1} = \frac{3a^2\int_o^{x_1} x^4dx}{2x_1y_1} = \frac{3a^2x_1^4}{10y_1} = \frac{3}{10}y_1. \tag{25}$$

The distance of the c.g. of OGB from OD is found from

$$x_1y_1 \cdot \frac{y_1}{2} - \frac{x_1y_1}{3} \cdot \frac{3}{10}y_1 = \frac{2}{3}x_1y_1 \cdot y_o'$$

$$y_o = \frac{3}{5}y_1. \tag{26}$$

The c.g. of AOB is, of course, in OG, and at a distance $\frac{3}{5}y_1$ from O.

9. *The Curve of Sines* (Fig. 29).—The equation of this curves is $y = \sin x$; x being expressed, of course, in radians, *i.e.*, 360° = 2π = 6.28318, or 1° = 0.0174533, or 57°17′45″ = 1 radian. Hence the curve is composed of a series of equal waves, each 2π in length, of which one-half of one is shown in Fig. 30. Only one-half of one wave is used in this book, and this sometimes finds application. In this case if the length is l (Fig. 30) the equation, to be a curve of sines, must be

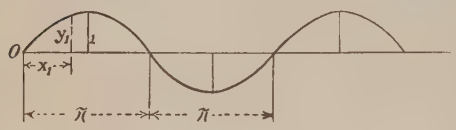


Fig. 29.

$$y = d \sin \frac{x}{l}\pi. \tag{27}$$

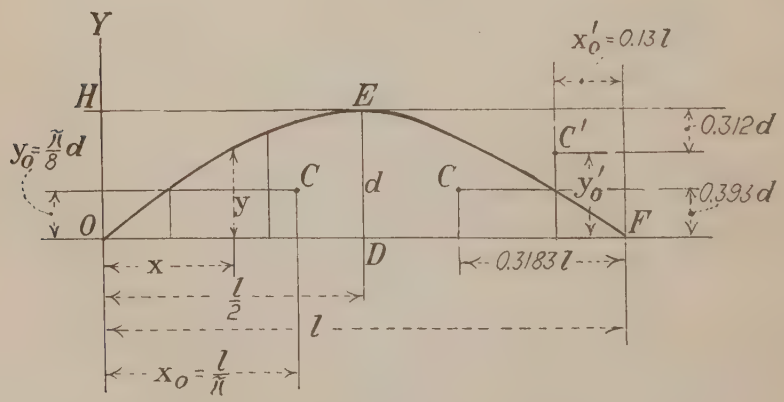


Fig. 30.

For if in Fig. 29 coordinates be called x_1 and y_1, and in Fig. 30 x and y,

$$y_1 = \sin x_1 \text{ (Fig. 29)}$$
$$x : l :: x_1 : \pi$$
$$y : d :: y_1 : 1$$
$$\therefore x_1 = \frac{x}{l}\pi$$
$$y_1 = \frac{y}{d} = \sin\left(\frac{x}{l}\pi\right)$$
$$\therefore y = d \sin\left(\frac{x}{l}\pi\right).$$

Comparing this curve with a parabola, for the same l and d

$x =$	$\frac{l}{8}$	$\frac{l}{4}$	$\frac{3l}{8}$	$\frac{l}{2}$
Parabola: $y =$	$0.4375d$	$0.75d$	$0.94d$	d
Sine curve: $y =$	$0.38d$	$0.707d$	$0.92d$	d

The sine curve therefore lies below the parabola.

The area OEF is

$$A = \int_o^l y dx = d\int_o^l \sin\left(\frac{\pi x}{l}\right)dx = \frac{dl}{\pi}\int_o^l \sin\left(\frac{\pi x}{l}\right)\frac{\pi}{l}\cdot dx$$
$$= -\frac{dl}{\pi}\cos\pi + \frac{dl}{\pi}\cos 0 = \frac{2dl}{\pi}. \qquad (28)$$

The area of the circumscribing rectangle is

$$A_1 = dl$$
$$\therefore A = A_1 \cdot \frac{2}{\pi} = 0.6366A_1 \qquad (29)$$

or a little less than if the curve were a parabola.

The area of the external portion of the rectangle is $0.3634A_1$.

The c.g. of the semi-area ODE is found from

$$x_o \cdot \frac{dl}{\pi} = \int_o^{\frac{l}{2}} y\, dx \cdot x = d\int_o^{\frac{l}{2}} x \sin\left(\frac{x\pi}{l}\right)dx$$
$$= \frac{dl^2}{\pi^2}\int_o^{\frac{l}{2}} \frac{\pi}{l}\cdot x \sin\left(\frac{x\pi}{l}\right)\frac{\pi}{l}dx.$$

Calling $\frac{\pi x}{l} = u$, the integral has the form

$$\int u \sin(u)\, du = -u\cos u + \sin u$$

or, putting in the limits

$$x_o \cdot \frac{dl}{\pi} = \frac{dl^2}{\pi^2}\left[-\frac{\pi x}{l}\cos\left(\frac{\pi x}{l}\right) + \sin\left(\frac{\pi x}{l}\right)\right]_{x=0}^{x=\frac{l}{2}} = \frac{dl^2}{\pi^2}$$
$$\therefore x_o = \frac{l}{\pi} = 0.3183l. \qquad (30)$$

To find the distance of this c.g. from OE,

$$y_o \cdot \frac{dl}{\pi} = \int_o^{\frac{l}{2}} y dx \cdot \frac{y}{2} = \int_o^{\frac{l}{2}} \frac{d^2 \sin^2\left(\frac{\pi x}{l}\right)}{2} dx$$

$$= \frac{d^2 l}{2\pi} \int_o^{\frac{l}{2}} \sin^2\left(\frac{\pi x}{l}\right) \frac{\pi dx}{l}$$

$$= \frac{d^2 l}{2\pi}\left[\frac{\pi x}{2l} - \frac{1}{2}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)\right]_{x=0}^{x=\frac{l}{2}} = \frac{d^2 l}{8}$$

$$\therefore y_o = \frac{d\pi}{8} = 0.3927d. \tag{31}$$

This is also the distance from OX of the c.g. of the entire area OEF, which of course lies on the axis of symmetry ED.

For the external segment OHE, to find the c.g. C'.

$$x_o' \cdot dl\left(\frac{1}{2} - \frac{1}{\pi}\right) = \frac{dl}{2} \cdot \frac{l}{4} - \frac{dl}{\pi} \cdot \frac{l}{\pi}$$

$$x_o' = l\frac{\pi^2 - 8}{4\pi^2 - 8\pi} = 0.13l. \tag{32}$$

$$y_o' \cdot dl\left(\frac{1}{2} - \frac{1}{\pi}\right) = \frac{dl}{2} \cdot \frac{d}{2} - \frac{dl}{\pi} \cdot \frac{d\pi}{8}$$

$$y_o' = \frac{d\pi}{4\pi - 8} = 0.688d. \tag{33}$$

These results are shown on Fig. 30, and may be compared with those for the parabolic curve.

The centers of gravity of curved surfaces and of solids are not often required in structural work. If necessary in any part of the book, they will be discussed when occasion arises.

12. Moment of Inertia, Radius of Gyration and Product of Inertia.—The *moment of inertia of a plane area about a straight line lying in its plane, and called the axis of inertia, is defined as the sum of the products of each infinitesimal area multiplied by the square of its perpendicular distance from the axis.*

Algebraically, if I is the moment of inertia, dA an elementary area, and y its distance from the axis of inertia

$$I = \int y^2 dA = \int\int y^2 dx dy.$$

Any line may be chosen as the axis, and the value of I will vary accordingly. The moment of inertia of a plane area is only found for an axis lying in its plane.

Similarly, *the moment of inertia of a line about an axis of inertia lying in its plane is the sum of the products of each differential length multiplied by the square of its perpendicular distance from the axis.*

Again, *the moment of inertia of a volume (or of a solid body) about any plane is the sum of the products of each differential volume (or weight) multiplied by the square of its perpendicular distance from that plane.*

In structural engineering we have to do almost exclusively with the moment of inertia of plane areas. It is necessary to investigate, however, the relations between moments of inertia about different axes. The moment of inertia will always be called I, with subscripts or superscripts.

13. Relation between Moments of Inertia about Parallel Axes.—Let the moment of inertia, I, about an axis BB (Fig. 31) be known, and let it be required to ascertain the moment of inertia, I_1, about a parallel axis B_1B_1, distant a from BB. Let the area of the figure be A.

Then
$$I = \int y^2 dA$$
$$I_1 = \int (y \pm a)^2 dA$$
$$= I + a^2A \pm 2a\int ydA.$$

If BB passes through the center of gravity of the area, $\int ydA = 0$, hence in this case, I_o being the moment of inertia about a parallel axis through the center of gravity.

$$I_1 = I_o + Aa^2. \tag{34}$$

Expressed in words, *the moment of inertia of any plane area about any axis in its plane equals its moment of inertia about a parallel axis through its center of gravity plus the area times the square of the distance between the two axes.*

From this it follows that *the moment of inertia about an axis through the center of gravity is less than that about any other parallel axis.*

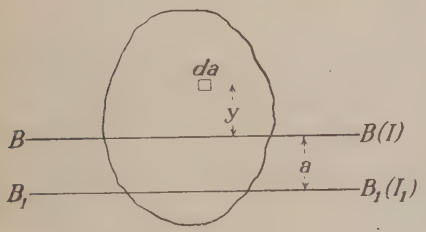


FIG. 31.

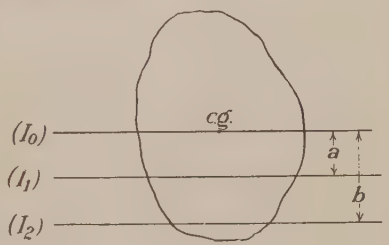


FIG. 32.

Suppose the moment of inertia, I_1, is given about some axis distant a from the center of gravity (Fig. 32), and it is desired to find the moment of inertia I_2 about a parallel axis distant b from the center of gravity. Clearly

$$I_2 = I_1 - Aa^2 + Ab^2. \tag{35}$$

14. The radius of gyration, r, of an area about a given axis, about which its moment of inertia is I, is defined by the equation

$$r^2 = \frac{I}{A}. \tag{36}$$

Hence, if r is the radius of gyration about an axis passing through the center of gravity, the radius of gyration, r_1, about a parallel axis at a distance a will be

$$r_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{I + Aa^2}{A}} = \sqrt{r^2 + a^2}. \tag{37}$$

Similarly, about a second parallel axis, distant b from the center of gravity

$$r_2 = \sqrt{r_1^2 - a^2 + b^2}.$$

These values may be constructed, if desired, by drawing a right triangle.

It will be observed that the moment of inertia of a plane area is a quantity representing a linear dimension to the fourth power, or it is an area times the

square of a distance; while the radius of gyration, as its name implies, is a linear distance.

15. In this book we shall have occasion to use only moments of inertia of plane areas. In many cases one of these values is deduced from another by means of the principles just explained. Thus, the moment of inertia of a rectangle (Fig. 33) about an axis through its center of gravity parallel to the side b is

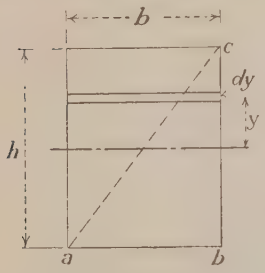


Fig. 33.

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} by^2 dy = b\left(\frac{h^3}{24} + \frac{h^3}{24}\right) = \frac{bh^3}{12}.$$

From this it follows that the moment of inertia of the triangle abc about the same axis is $\frac{bh^3}{24}$, and about its center of gravity it is $\frac{bh^3}{24} - \frac{bh}{2} \cdot \frac{h^2}{36} = \frac{bh^3}{36}$, and from this it further follows that the moment of inertia of the triangle about its base ab is $\frac{bh^3}{3} + \frac{bh}{2} \cdot \frac{h^2}{9} = \frac{bh^3}{12}$ or the same as that of the rectangle about the axis through its center of gravity.

16. The product of inertia of a plane area referred to any two rectangular axes in its plane, which we designate as K, is defined as *the sum of the products of each infinitesimal area multiplied by both its coordinates;* or algebraically

$$K = \int xydA = \int\int xydxdy.$$

Similar definitions may be given for the product of inertia of a line, and as in the case of moment of inertia, we may examine the relation between the values of K for different axes.

17. Relations between Values of K for Parallel Axes.—Let K be the product of inertia referred to any rectangular axes OX and OY (Fig. 34). Then for parallel axes O_1X_1 and O_1Y_1 as shown, the coordinates of O_1 being x_o and y_o, we shall have for the product of inertia about the axes OX_1 and OY_1 (K_1)

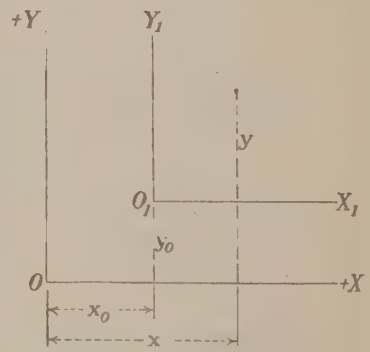


Fig. 34.

$$K_1 = \int (y - y_o)(x - x_o)\, dA$$
$$= K + x_o y_o A - y_o \int xda - x_o \int yda.$$

If O is the center of gravity of the area and K for this case is called K_o,

$$K_1 = K_o + x_o y_o A. \qquad (38)$$

Either x_o or y_o, or both, may be positive or negative. Hence *the value of the product of inertia for axes through the center of gravity* (K_o) *is numerically less than for parallel axes meeting at any point in the first or third quadrants* (x_o and y_o both + or both −); *and numerically greater than for parallel axes meeting at any point in the second or fourth quadrants* (either x_o or y_o negative, but not both).

As in the case of moment of inertia, the value of K_o being given, K_1 may easily be found for any set of parallel axes.

It is also clear that *for any pair of rectangular axes, either one of which is an axis of symmetry,* $K = 0$ (since for every particle on one side of the axis of symmetry there is an equal particle on the other side).

Further, *for any pair of rectangular axes, either one of which goes through the c.g., the value of K is the same as for parallel axes both going through the c.g.* (x_o or y_o is zero). *In other words, if one of the axes goes through the c.g. the value of K is the same no matter where the other axis is.*

18. Polar Moment of Inertia.—The polar moment of inertia, referred to any rectangular axes, is the sum of the products of each elementary area multiplied by its distance from O, and will be called I_p. Thus

$$I_p = \int (x^2 + y^2)\, da = I_x + I_y. \tag{39}$$

The polar moment of inertia of an area *about any point* in its plane, therefore, equals the sum of the moments of inertia about any two rectangular axes in its plane passing through the point, this sum being the same for all pairs of rectangular axes, through the same point, or

$$I_x + I_y = I_x' + I_y'. \tag{40}$$

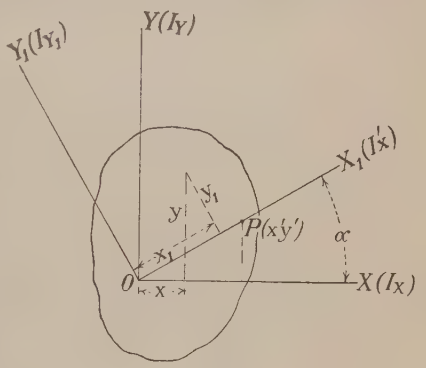


Fig. 35.

19. Relations between Moments of Inertia and Products of Inertia of an Area about Rectangular Axes in Different Directions in Its Plane, through the Same Point.—Let O (Fig. 35) be *any point* in the plane of a given area, and let I_x and I_y be the moments of inertia about the rectangular axes OX and OY respectively. Let it be desired to find the relation between these moments of inertia and those about another set of rectangular axes through O, OX_1 and OY_1, which are designated by I_x' and I_y', the angle between OX and OX_1 being α.

The relations between the coordinates are, by projecting x and y on OX_1 and OY_1,

$$\begin{aligned} x_1 &= y \sin\alpha + x\cos\alpha \\ y_1 &= y\cos\alpha - x\sin\alpha \\ \therefore I_x' &= \int (y\cos\alpha - x\sin\alpha)^2 da \\ &= I_x\cos^2\alpha + I_y\sin^2\alpha - K\sin 2\alpha \end{aligned} \tag{41}$$

in which K is the product of inertia $\int xyda$.

Thus, if I_x, I_y and K, for any pair of axes, are given, the moment of inertia for any other axis through the same origin may be found.

Similarly, the product of inertia is given by the equation

$$K' = \int x_1 y_1 da = \frac{(I_x - I_y)\sin 2\alpha}{2} + K\cos 2\alpha. \tag{42}$$

20. Ellipse of Inertia.—To study the variation of I_x with α, lay off on each axis (Fig. 35) a distance inversely proportional to $\sqrt{I}$, or $OP = \frac{c}{\sqrt{I_x'}} = s$,

c being any constant, and let x' and y' be the coordinates of P referred to OX and OY.

Then $\sin \alpha = \frac{y'}{s}$; $\cos \alpha = \frac{x'}{s}$; and substituting these values in Eq. (41)

$$I_{x'} = \frac{I_x x'^2}{s^2} + \frac{I_y y'^2}{s^2} - \frac{2K x' y'}{s^2} \tag{43}$$

This is the equation of the locus of P as α is varied, and being of the second degree is clearly a conic section. Since no value of $I_{x'}$ can be zero, and therefore no value of OP infinite, the curve must be an ellipse.

This ellipse is called the *ellipse of inertia* for the point O. There is an ellipse of inertia for every point in the plane of the given area.

To find the axes of this ellipse make $\frac{dI_{x'}}{d\alpha} = 0$, or

$$-2I_x \cos \alpha \sin \alpha + 2I_y \sin \alpha \cos \alpha - 2K \cos 2\alpha = 0$$

$$\tan 2\alpha = \frac{2K}{I_y - I_x}. \tag{44}$$

Comparing this equation with Eq. (42), it is obvious that when Eq. (44) is fulfilled (*i.e.* for the axes of the ellipse of inertia at any point) $K' = 0$. The following principle results:

For every point in the plane of a given area there is an ellipse of inertia of that area, and the product of inertia is zero for the axes of any such ellipse; that is to say, for the axes of greatest and least moment of inertia.

Since $K = 0$ for any axis of symmetry, it follows that *if an axis through any point O is an axis of symmetry of the figure, the product of inertia for that axis and one perpendicular to it anywhere is zero, and those axes are the axes of the ellipse of inertia for the point of intersection of the axes.*

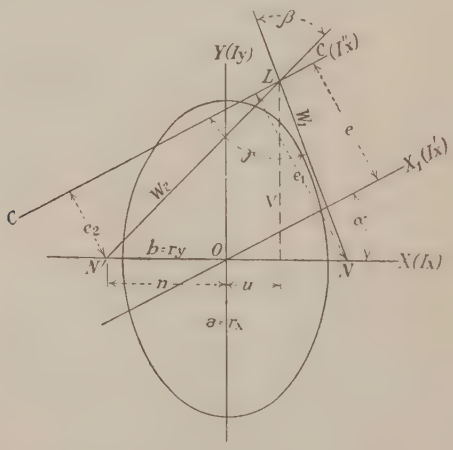


Fig. 36.

21. At any given point O take as original axes OX and OY the axes of the ellipse of inertia for that point, for which $K = 0$. Then the moment of inertia for a new axis OX_1 (Fig. 36) will be by Eq. (41),

$$I_{x'} = I_x \cos^2 \alpha + I_y \sin^2 \alpha \tag{45}$$

and the product of inertia for axes OX_1 and OY_1 will be, by Eq. (42)

$$K' = \frac{(I_x - I_y)\sin 2\alpha}{2}. \tag{46}$$

In Art. 17 it was shown that if K_o is the product of inertia for any pair of rectangular axes *through the c.g.*, the value of $K(K_1)$ for any axes parallel to these is $K_1 = K_o + x_o y_o A$, in which x_o and y_o are the coordinates of the new axes referred to the original one. If the original axes are the axes of the ellipse for the c.g., $K_o = 0$.

$$\therefore K_1 = x_o y_o A \tag{47}$$

or, in words,

For any pair of rectangular axes parallel to the axes of the ellipse fyr the c.g., the product of inertia equals the area multiplied by the product of the coordinates of the origin referred to said axes.

Suppose, *for any point*, the ellipse of inertia drawn (Fig. 37), and at the points where it intersects the axis OX_1 draw tangents to the ellipse. Draw also the

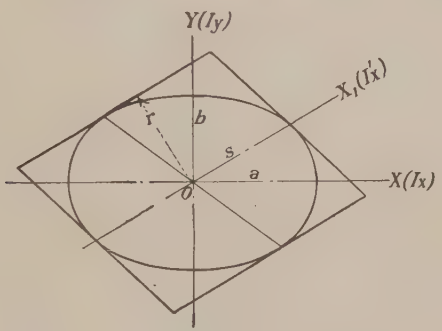


FIG. 37.

conjugate axis, and tangents at its extremities, forming a parallelogram, as is known from the properties of the ellipse. The area of this parallelogram is $4sr$, if r is the perpendicular from O upon one of the tangents at the end of the conjugate axis. But we know that all parallelograms formed by tangents at the ends of conjugate axes of an ellipse have the same area, so that if a and b are the semi-axes of the ellipse $sr = ab$.

$$r = \frac{ab}{s} = ab \cdot \frac{\sqrt{I_x'}}{c} = \frac{c}{\sqrt{I_x}} \cdot \frac{c}{\sqrt{I_y}} \cdot \frac{\sqrt{I_x'}}{c} = \frac{c\sqrt{I_x'}}{\sqrt{I_x I_y}}. \tag{48}$$

Since I_x and I_y are constant quantities for any given ellipse, while I_x' varies with α, this equation shows that r is proportional to $\sqrt{I_x'}$, that is to say, r is proportional to the radius of gyration of the area about OX_1. The constant c may be taken at will, and if it is taken as $\sqrt{\frac{I_x I_y}{A}}$ (A being the area of the given figure), we find that

$$r = \sqrt{\frac{I_x'}{A}} = \text{the radius of gyration about } OX_1. \tag{49}$$

In this case the semi-axis a represents the radius of gyration about OY, and the semi-axis b represents the radius of gyration about OX. If drawn in this manner, the ellipse of inertia will have the property that *the perpendicular distance from any diameter OX_1 to the tangent parallel to that diameter equals the radius of gyration of the area about OX_1. Each semi-axis represents the radius of gyration about the other.*

22. Central Ellipse—Fixed Points.—When the point O is the c.g. of the given area, and the semi-axes are taken as the radii of gyration, the ellipse of inertia for that point is called the *central ellipse*, and the axes of this ellipse are called the *principal axes* of the area. *The major principal axis, which is the minor axis of the ellipse*, is the one for which the value of I is greater than for any other axis through the c.g. (though of course not greater than for some axes not through the c.g.). *The minor principal axis, which is the major axis of the ellipse*, is the one for which the value of I is less than for any other axis, whether passing through the c.g. or not. Any axis of symmetry must pass through the c.g. and must be an axis of the central ellipse.

In Fig. 36, the central ellipse is shown, OX being the major and OY the minor principal axes of the section. The major axis of the ellipse is $a = r_x =$ radius of gyration about OX, and the minor axis is $b = r_y =$ radius of gyration about OY. Equations (45) and (46) give the values of I_x' and K' for any axis OX_1 making an angle α with OX. For any axis parallel to OY and distant d from it, the moment of inertia is $I_y + Ad^2$. Hence there are two points N and N' on OX_1 equidistant from O, for which the moment of inertia about an axis parallel to OY equals I_x, and the value of d for this point is given by the equation

$$I_x = I_y + Ad^2$$
$$\therefore ON = ON' = d = \sqrt{r_x^2 - r_y^2}. \qquad (50)$$

For rectangular axes through these points, one of which is OX, the moments of inertia are equal, hence *the moments of inertia about all axes through these points are the same.*

The product of inertia for the axes OX, OY, is zero. Hence from Eq. (47) it is zero for parallel axes through N or N' (since $y_o = 0$). Hence from Eq. (42) it is zero for any pair of rectangular axes through N or N'.

These interesting and important results may be stated as follows:

In every plane area there are two points lying on the major principal axis, equidistant from the c.g., and at a distance $\sqrt{r_x^2 - r_y^2}$ from it (or the distance from the c.g. to the foci of the central ellipse), which have the singular property that about any axis through either of them the moment of inertia is the same, and that for any pair of rectangular axes through either of them the product of inertia is zero. These points are called the fixed points of the area. The ellipse of stresses for either fixed point is a circle.

23. Relation of the Fixed Points to the Ellipse of Inertia for Other Points.—Let OX_1 (Fig. 36) be any axis through the c.g. making an angle α with OX, the major principal axis. Let L be any point in the plane of the area, with coordinates u and v. Draw a line CC through L parallel to OX_1 and distant e from it, and let I_x'' be the moment of inertia about CC. Then from Eqs. (34) and (41), since $K = 0$ for the principal axes,

$$I_x'' = A(r_x^2 \cos^2 \alpha + r_y^2 \sin^2 \alpha + e^2) = A[r_x^2 + e^2 - (r_x^2 - r_y^2) \sin^2 \alpha]. \qquad (51)$$

But the abscissa n of the fixed point $N = \sqrt{r_x^2 - r_y^2}$, hence

$$I_z'' = A(r_x^2 + e^2 - n^2 \sin^2 \alpha) \tag{52}$$

which is perhaps the simplest expression for the moment of inertia about any axis.

From N and N' draw perpendiculars of length e_1 and e_2 to CC.

Then

$$e_1 = e + n \sin \alpha$$
$$e_2 = e - n \sin \alpha$$
$$e_1 e_2 = e^2 - n^2 \sin^2 \alpha.$$

Hence

$$I_z'' = A(r_x^2 + e_1 e_2) \tag{53}$$

which is still more simple if e_1 and e_2 are known.

Join L with N and N'; and let $LN = w_1$, $LN' = w_2$, and the angle $NLN' = \beta$. Let the variable angle between CC and LN be γ.

Then

$$e_1 = w_1 \sin \gamma$$
$$e_2 = w_2 \sin (\gamma - \beta).$$

Hence

$$I_z'' = A[r_x^2 + w_1 w_2 \sin \gamma \,.\, \sin (\gamma - \beta)]. \tag{54}$$

With γ as the variable (that is, as the axis CC rotates around L), this expression will have maximum and minimum values for

$$\gamma = \tfrac{1}{2}\beta \text{ or } \tfrac{1}{2}\beta + 90°.$$

In other words, given O and the fixed points N and N', then *the axes of the ellipse of inertia for any other point L will bisect the angles between the lines NL and N'L.*

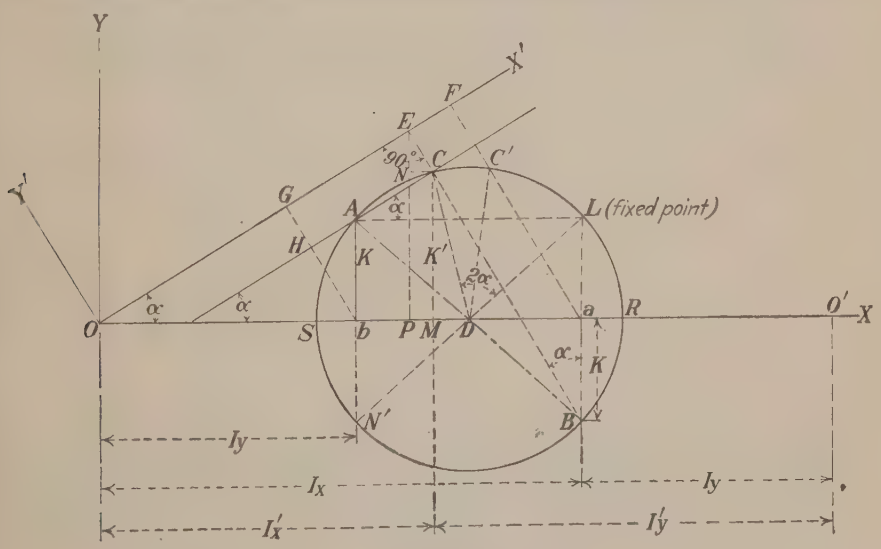


FIG. 38.

24. Circle of Inertia.—If I_x, I_y, and K are known for any pair of rectangular axes, Eqs. (41) and (42) give the values for another pair making an angle α with the first. These equations lead to an interesting geometrical construction.

Let I_x be the greater of the two moments of inertia about the given axes. In Fig. 38 lay off, on OX, $Oa = I_x$, $Ob = I_y$, $aO' = I_y$; and, at right angles to OX,

$bA = K$, laid off above or below OX according as it is positive or negative. Lay off also aB below OX, equal numerically to K. With AB as a diameter, draw a circle, whose center will be at D, on OX. Then $\frac{I_x + I_y}{2} = OD$; $\frac{I_x - I_y}{2} = bD = aD$. Draw through A the line AC parallel to the axis OX', making the angle α with OX. Draw CB, which will be at right angles to AC, since AB is a diameter. Draw AL parallel to OX and produce Ab to N'; then numerically $bN' = aL = K$, and the angle $CDL = 2\alpha$. Produce BC to E, and draw bG and aF perpendicular to OX'. Draw CM and EP perpendicular to OX. Then

$$OM = OP + PM = OE \cos \alpha + CE \sin \alpha;$$
$$MC = EP - EN = OE \sin \alpha - CE \cos \alpha;$$

but
$$OE = OF - FE = I_x \cos \alpha - K \sin \alpha;$$
$$CE = Gb - Hb = I_y \sin \alpha - K \cos \alpha.$$

Hence
$$OM = I_x \cos^2 \alpha + I_y \sin^2 \alpha - K \sin 2\alpha = I_x' \text{ (by Eq. (41))}$$
$$CM = \frac{I_x - I_y}{2} \sin 2\alpha + K \cos 2\alpha = K' \text{ (by Eq. (42))}.$$

Hence by drawing AC the values I_x' and K' are found at once. Also, since $I_x + I_y = I_x' + I_y'$,

$$I_y' = MO'.$$

The maximum moment of inertia is OR, on a plane parallel to AR;

The minimum moment of inertia is OS, on a plane parallel to AS.

These are the principal axes; they are at right angles; and for them $K = 0$.

25. Values for Structural Shapes.—In the shapes of steel sections used for structures, given in the Carnegie Pocket Companion, the areas, the position of the c.g. and the values of the moment of inertia for several axes, are indicated. Thus

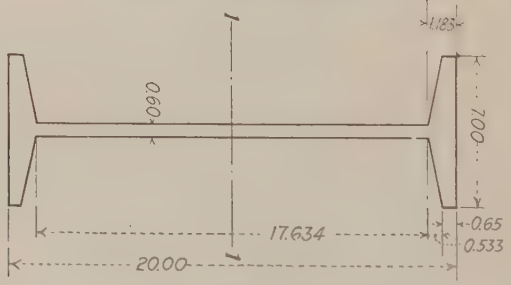


Fig. 39.

for I-beams, one principal axis is at right angles to the web at mid-height, and the other at right angles to the first through the center of the web, both being axes of symmetry. For channels, the principal axes are indicated. For bulb angles, the axes for which the moment of inertia are given are not principal axes. For angles and Z-bars, values are given for axes parallel to the sides, and for the principal axis about which I is a minimum, but the position of this axis is sometimes not indicated. For two angles taken together, the values are given for the principal axes, for various distances between the two angles.

Observe that it is only necessary to know the center of gravity and the values of I_x, I_y, and K for any two rectangular axes through the c.g. in order to be able to

find quickly the position of the principal axes and the principal moments of inertia by the formulae above given.

(1) Let us check the value of I for one of these shapes, as an example. Take the 20-inch I-beam weighing 81.4 pounds per foot, for which the value of I about axis 1-1 is given as 1,466.3. The shape is indicated in Fig. 39. To find I, take the I of the rectangle 7 by 20 inches, and subtract that for two rectangles and four triangles.

Thus

$\frac{1}{12} \times 7 \times \overline{20}^3 =$		4,666.67
$\frac{1}{12} \times 6.4 \times \overline{17.634}^3 =$	2,924.50	
$2 \times \frac{1}{36} \times 6.4 \times \overline{0.533}^2 =$	0.05	
$2 \times 6.4 \times \frac{1}{2} \times 0.533 \times \overline{8.995}^2 =$	275.98	3,200.53
I of section =		1,466.14

This takes no account of the rounding of the corners, and agrees closely with the value given. Whether the latter has been corrected for these curves is uncertain, and the difference is so small as to be immaterial anyway.

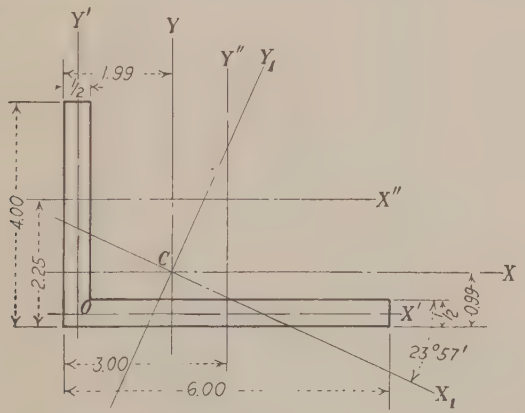


FIG. 40.

(2) As a second example, take the angle shown in Fig. 40, $6 \times 4 \times \frac{1}{2}$ inches. The area is $(6 + 3.5)\frac{1}{2} = 4.75$, as given in the table, evidently not allowing for rounding the corners. To find the c.g.,

$$\frac{3 \times 0.25 + 1.75 \times 2.25}{4.75} = 0.99$$

$$\frac{3 \times 3 + 1.75 \times 0.25}{4.75} = 1.99.$$

To find values of I about axis CX (2–2 in the table), take I for each rectangle about its own c.g. and transfer to new axes; thus,

$$\tfrac{1}{12} \times 6 \times 0.5^3 + 3 \times \overline{0.74}^2 = 1.70+$$

$$\tfrac{1}{12} \times 0.5 \times 3.5^3 + 1.75 \times \overline{1.26}^2 = 4.56+$$

I_x about CX = 6.27 (given as 6.3)

For I about CY;

$$\tfrac{1}{12} \times 0.5 \times 6^3 + 3 \times \overline{1.01}^2 = 12.06$$

$$\tfrac{1}{12} \times 3.5 \times \overline{0.5}^3 + 1.75 \times \overline{1.74}^2 = 5.33+$$

I_y about CY = 17.39+ (given as 17.4).

For K about XY axes, take axes first through centers of the two rectangles ($X'Y'$). For these axes $K = 0$, since Y' is an axis of symmetry of one rectangle, and X' an axis of symmetry of the other. Then in Eq. (38) the coordinates of O with reference to C are

$$x_o = -1.74;\ y_o = -0.74;\ \text{hence}$$
$$O = K + 4.75 \times 1.74 \times 0.74; \quad K = -6.12.$$

The result would be the same if we had begun with the axes $X''Y''$, which are also each axes of symmetry for the separate rectangles. Here x_o would be $+1.01$, and y_o would be $+1.26$, and $1.01 \times 1.26 = 1.74 \times 0.74$ (barring slight errors due to inaccuracy in the values 1.99 and 0.99).

To find the principal axes

$$\tan 2\alpha = \frac{2K}{I_y - I_x} = -\frac{12.24}{17.39 - 6.27} = -1.1007$$
$$2\alpha = -47^\circ\ 54' \text{ or } 132^\circ\ 6'$$
$$\alpha = -23^\circ\ 57' \text{ or } 66^\circ\ 3'$$
$$\sin \alpha = -0.4059;\ \cos \alpha = 0.9139;\ \sin 2\alpha = -0.742.$$

The moment of inertia for one principal axis (CX_1) will be

$$I_x' = 6.27 \times \overline{0.9139}^2 + 17.39 \times \overline{0.4059}^2 - 6.12 \times 0.742$$
$$= 3.57 = \text{minimum } I.$$

Since
$$I_x + I_y = 23.66 = I_x' + I_y'$$
$$I_y' = 23.66 - 3.57 = 20.09 = \text{maximum } I.$$

The minimum radius of gyration, about CX_1, will be

$$\min r = \sqrt{\frac{3.57}{4.75}} = 0.87 \text{ (as given in table).}$$

26. Moments of inertia and centers of gravity of areas may be found by purely graphical methods, as will be explained in discussing Graphical Methods in a later volume.

27. To Draw a Line through a Given Point and through the Intersection of Two Lines Which Do Not Meet on the Paper.—This problem, which frequently arises, may be easily solved by the following construction (Fig. 41): Let a and b be the given lines and C the given point. Draw through C any two lines AB and $A'B'$; draw $A'B$ and AB', meeting at D; draw through D any line $DB''A''$; draw AB'' and $A''B'$ (or $A'B''$ and $A''B$) meeting in C'' (or C'''); then the line $CC''C'$ will pass through the intersection of a and b. The construction applies equally well if C is not between a and b (as in Fig. 41), but outside (as in Fig. 42).

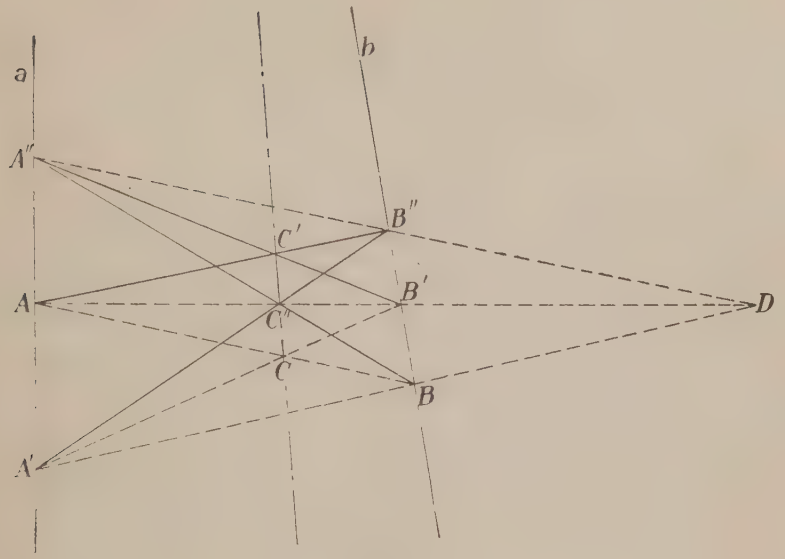


FIG. 41.

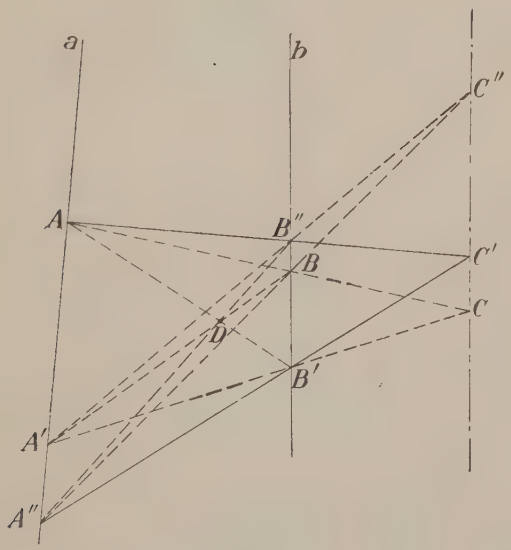


FIG. 42.

CHAPTER IV

BEHAVIOUR

STRESS AND STRAIN. BEHAVIOUR OF MATERIALS UNDER LOADS

1. Stress.—If a solid body, in equilibrium, is acted upon by external forces applied to its surface as well as by the force of gravity applied to its particles, and if it be imagined cut in two by a section, the part of the body on one side of the section will exert, in general, a force upon the part on the other side of the section. Stress is an *internal force* exerted by the molecules or particles upon each other, and is produced by the outer forces, though, as we shall see, there may be stresses without external or outer forces.

Suppose the body shown in Fig. 43 to be acted upon by forces as indicated, and let it be imagined cut in two parts A and B by the imaginary plane CD.

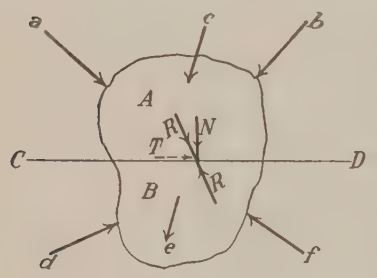


FIG. 43.

If the body is in equilibrium, all the forces acting upon it—a, b, c, d, e, f—balance, and the resultant of a, b, and c is equal and opposite to the resultant R of d, e, and f. The resultant of a, b, and c is the force which the part A exerts *on* the part B, and the equal and opposite resultant of d, e, and f is the force which the part B exerts *on* the part A. Either of these is the total stress to the cut section CD, and is distributed over the section in some manner that will be studied later. The stress on a section, therefore, acts in one direction or in the opposite direction, according to the part of the body, A or B, which we are considering acted upon by the stress.

The direction of the total stress R may be normal to the section or inclined to it. If inclined, it may be resolved into two components, N and T, one normal to the section and the other along the section. There can be only these two kinds of stress, *normal* and *tangential*.

The normal stress may either act away from the section, tending to separate A and B, or to pull them apart, in which case it is called *tension;* or it may act toward the section, tending to press A and B together, in which case it is called *compression*.

The tangential force tends to slide the part A along the part B in some direction, and the part B along the part A in the opposite direction, and is called a *shear*, or *shearing stress*.

There are, therefore, only three kinds of stress, *tension*, *compression*, and *shear*. There can be no others.

A straight rod or rope with a pull at each end (Fig. 44*a*) is the simplest example of tension, and a post or column carrying a load is the simplest example of com-

pression. A pair of scissors cutting a body is the simplest example of shear; or three bars acting as in Fig. 44*c*.

If the force R be divided by the area of the cut section CD, the quotient is the average stress per unit of area, or the average *unit stress*. This, however, will not be the same or every unit of area except under certain conditions; the unit stress will vary from point to point of CD. The real unit stress or *intensity of*

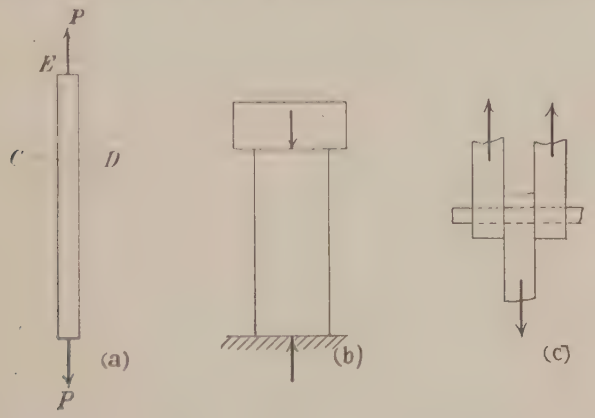


Fig. 44.

stress (or *stress intensity*) at any point would be found by taking an infinitesimal area at that point and dividing the total (infinitesimal) stress on that area by the area itself. At each point of a plane section there is a certain intensity of normal stress, a certain intensity of shearing stress, and a certain intensity of resultant stress.

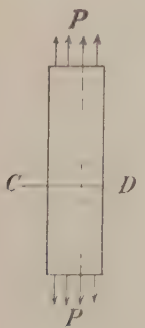


Fig. 45.

Fig. 45*a*.

Fig. 46.

If the resultant stress on a section is normal, and acts at the center of gravity of the section, as, for instance, if the rod in Fig. 45 has tensile forces at the ends uniformly distributed over the ends, the stress on CD may be assumed to be uniformly distributed, so that the intensity at all points is the same. Even if the forces at the ends take hold of the rod at the center of the end areas (where there may be supposed to be eyes, to which hooks may be attached), it is generally

assumed that the stress on CD is uniformly distributed, since the resultant force on CD goes through the center of gravity of the area. It is easy to see, however, that it is not necessarily true that if the resultant force on an area goes through its center of gravity, the stress is uniformly distributed; for in Fig. 45*a* this is not true at the end E, and there will be some distance below E in which the stress on a horizontal section will not be uniformly distributed. It will require a certain distance before the force P, applied at the center of the end, will be uniformly distributed over the section.

The distribution of a resultant stress over a section will require careful study hereafter. For the present we may confine ourselves to straight pieces acted upon by uniformly distributed forces at the end (Fig. 45), so that the stress on a plane section is uniformly distributed over it.

The section considered need not be plane, but may be curved. It is often necessary to study the stress or load on a curved section or surface, as for instance, where a body rests upon a pin, as in Fig. 46.

When a piece is bent (exposed to *flexure*) (Fig. 47) *bending stresses* are said to be produced. This does not mean that there are any other than the three kinds of stress that have been named; bending stresses are merely those that occur in flexure, and they are tension, compression, and, almost always, shear. Similarly, when a piece is twisted (exposed to torsion), torsional stresses are produced, but these are merely tension, compression and shear, which are produced by twisting.

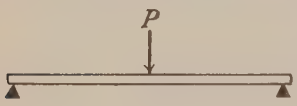


FIG. 47.

Intensity of stress is expressed as force per unit of area, and as pounds per square inch is often written lbs./in.^{2}

A stress always acts upon some area or surface. *Stress should never be thought of without perceiving the area or surface on which it acts.* If the reader will adhere to this rule, the subject will be easier for him. The surface may be plane, curved, or broken, but a section taken for the study of stress is generally taken as plane.

The stress on a plane section taken—or imagined taken—through a body will generally vary from point to point. The intensity at any point may be represented by a line in the direction of the stress, drawn from the point and having a length proportional to the intensity. If we are studying normal stress, the line representing stress intensity is normal to the surface. If we are studying shear, the line cannot be drawn parallel to the force, and the distribution of shear is therefore also represented by drawing at each point a line *normal to the surface* representing the shearing intensity at that point. Hence stress distribution is in both cases shown by normal lines. Thus, in Fig. 48, if ab represents the surface considered, and if cd represents the stress distribution, the intensity at any point of ab is the normal from ab to the line cd. Generally speaking, when a line is erected at each point of a surface A, normal to that surface, and proportional to the stress intensity at the point in question, the other ends of those lines will form a surface B which represents the stress distribution. The total stress on A will be represented by the volume between A and B, and the resultant stress will act through the point of A where A is cut by the normal through the center of gravity of the volume. For suppose the stress normal, and let p be its intensity on a little area dA; then the total stress is $\int p dA$, which is the volume between A and B;

and the distance of the resultant from any line in A, if x is the distance from the line of the point where p acts, is

$$\frac{\int pxdA}{\int pdA}$$

which is the distance from the line to the point where A is cut by a normal through the center of gravity of the volume. In Fig. 48, cd may be considered to show the distribution on ab if the stress intensity is the same at all points along any line of that surface perpendicular to the paper. The shape of cd will show the distribution of stress, the area $abdc$ will represent the total stress if the surface ab is a

Fig. 48.

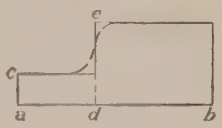


Fig. 49.

rectangle with a dimension unity perpendicular to the paper, and the resultant stress (if normal) will go through the center of gravity of $abdc$ one-half a unit from the paper. If cd is parallel to ab, the stress is uniformly distributed, or has the same intensity at every point. If cd is a straight line not parallel to ab, the stress is uniformly varying.

While the stress intensity on a surface may vary from point to point, it will be shown that it cannot change abruptly; thus on ab (Fig. 49) it may be ac at a, but cannot change suddenly at d to de; the change must be gradual, by some line like the dotted line; depending upon the yielding of the material, though the change may be rapid. An external load or pressure, however, may begin, or stop, or change suddenly at a point or line. The principles governing the distribution of stress over a plane surface are fully discussed in Chap. IX.

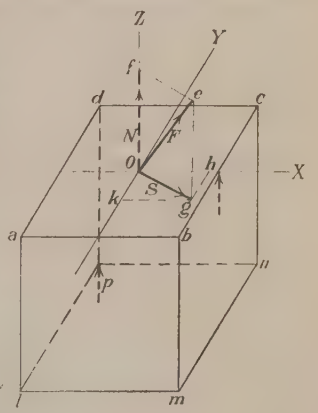


Fig. 50.

In Fig. 50, let the surface $abcd$ represent an infinitesimal area on which a stress $Oe = F$ acts, and let F be resolved into *normal* and *tangential* components N and S. Clearly there can be only these two kinds of stresses. Normal stresses are distinguished into tension and compression because the former tends to tear the body apart, while the latter tends to compress it. Shearing forces need not be distinguished if the body is homogeneous, for the action is the same no matter what the direction of S. If the body is not homogeneous, like wood, S may be classified, as shearing along the grain and shearing across the grain, and the strength may be very different in these two directions. S may be resolved into components acting along any two lines at right angles in $abcd$, and it may act in either sense along either axis. The stress F may thus always be resolved into three components along rectangular axes OX, OY, OZ; one of these components will be a normal stress, the other two shearing stresses.

Stresses may therefore be classified as follows:

$$\text{Stresses} \begin{cases} \text{Normal} \begin{cases} \text{Tension} \\ \text{Compression} \end{cases} \\ \text{Tangential (shearing).} \end{cases}$$

2. Strain or Deformation.—Every stress produces a corresponding *deformation*, or *strain*, or change of shape. No body is absolutely rigid, but any stress, however small, produces some change of shape. The deformation will be greater the greater the stress, but not necessarily in proportion. A tension produces elongation or stretch; a compression produces contraction; a shear produces lateral distortion.

A material is called *elastic* if, when the stress is removed, the deformation disappears entirely, and the original shape, in an unstressed state, is regained. If the stress is great enough, the deformation may only partly disappear when the stress is removed; a part of the deformation remaining as a permanent distortion or *permanent set*. Sometimes this permanent set may partly, or wholly, disappear after a sufficient time. Such materials are called *partially elastic*, or elastic up to a certain limit of stress. In some materials, even a small stress may produce a deformation that will not entirely disappear when the stress is removed. A material is called *plastic* when the entire deformation produced by a stress remains as a permanent set when the stress is removed. No structural materials are plastic.

The laws of stress and strain, and the relations between them, must be thoroughly understood by the engineer. This is the field of the subject of strength of materials. The purpose of this subject is to show how to determine the stress and the strain at any point of a body due to the action of given outer forces or loads; it also deals with certain physical properties of solid bodies, and the relation of these properties to stress and strain; it studies the stress or strain necessary to produce fracture or permanent injury, the manner in which failure occurs, and the phenomena which accompany stresses of different kinds under different conditions.

3. Properties of Solid Bodies.—The properties of solid bodies that are important for the engineer are the following:

Elasticity is the property by which a body, if deformed by the action of a force, will resume its original shape when the force is removed. Most engineering materials are elastic to some degree; that is to say, if the force is not too large the deformation will disappear when the force is removed. If a small force produces a relatively large deformation, which disappears when the force is removed, the body is very elastic. Elasticity is measured by the ratio of the deformation which disappears to the force producing it, all expressed in proper units.

Plasticity is the property by which a body is deformed by the action of a force, without recovering its original shape when the force is removed, but remaining entirely in the deformed shape. Some bodies, when deformed by a force, recover a part of the deformation, but not all, when the force is removed; they are partly elastic and partly plastic. Plasticity may be measured by the ratio of the permanent deformation to the applied force, all expressed in proper units. If a body recovers none of the change of shape produced by a force, it is perfectly plastic. Some materials, such as metals, which are far from plastic under usual conditions, become more or less plastic at certain temperatures, or when the

applied force becomes large enough. Tresca found that lead and similar plastic metals would flow under certain conditions, and that such plastic flow under the action of a single force, such as tension or compression, involves no change of density; that is to say, the volume of a particle remains unchanged. Under the action of such a force as a hydrostatic pressure, however, the volume may be changed, the particles of the body being forced closer together or farther apart.

Ductility is the property by which a body may be deformed by stretching, without recovery when the stretching force is removed. A bar of ductile material may be drawn out into a fine wire without fracture.

The difference between ductility and plasticity is that ductility refers to stretching, or the action of a tension, as when copper is drawn into wire; while plasticity has reference to compression, as when clay is moulded. A plastic material may have little or no ductility, or it may have a great deal. Ductility may be measured by the ratio of permanent stretch to the applied force. It is, however, generally given a different and more definite and restricted meaning, namely, the ratio of the total stretch of a given length, before fracture, to the original length. Thus, if a bar of copper, 1 foot long, could be stretched so as to be 1.5 feet long before fracture, but no longer, the ductility would be 50 per cent. These two meanings must be carefully distinguished. The latter should be termed "ultimate elongation" or elongation before fracture, expressed as a ratio to the original length.

Malleability is the property by which a body may be hammered out into a thinner sheet without fracture. Thus, gold may be hammered out into a very thin sheet; it is very malleable. Following the second definition of ductility, malleability could be measured or defined as the ratio of the original thickness of a sheet to the thinnest sheet into which it could be hammered without fracture. Malleability clearly depends on ductility and softness.

Stiffness (or rigidity) is the property by which a body deforms little under a load. A body is stiff if a large load produces a small deformation, independent of whether the deformation is elastic or permanent. It is measured by the ratio of load to deformation, all expressed in proper units. The larger E is (see Art. 4) the greater the stiffness. Sometimes it means a small deformation before fracture. The reverse of stiffness is *flexibility*.

Hardness is an important property, but a vague one, and difficult to define. Sometimes it means resistance to indentation by a ball or cone of a standard substance; sometimes it means resistance to cutting or scratching. If one substance will scratch another but is not scratched by it, the former is the harder, and substances may be arranged in a scale of hardness on this principle, every substance scratching those above it on the list but not those below, the lowest on the list being the hardest. Resistance to indentation and to scratching, however, are not always proportional. Another meaning of hardness is resistance to abrasion, and this may be measured by the amount of material rubbed off or abraded by some standard treatment. A fourth meaning often given to hardness is a high elastic limit, that is, a high value of the stress below which the stress is proportional to the deformation, and no permanent set occurs; or, a high resistance to permanent or plastic deformation. This meaning has no direct relation to the others.

Hardness does not mean brittleness. A hard steel may be tough and ductile.

Resilience is the amount of work or energy that can be stored up *in an elastic body* by loads applied to it, and which will be given up by the body in recovering its original shape when the loads are removed. Sometimes it means the total work that must be done to produce fracture, but this is an improper use of the term. If used, the two should be distinguished as "elastic resilience" and "total resilience" (see Art 27).

When a force produces distortion it does work. Thus, if a tensile force P, stretches a bar a distance Δ, and if the force P is gradually applied, beginning with zero, and increasing proportionally to the stretch so that the average force is $\frac{1}{2}P$, the work done will be $\frac{1}{2}P\Delta$. If the entire load P were applied suddenly at the beginning, the piece would be thrown into vibration as a helical spring would be by a sudden load.

Brittleness implies a small deformation before fracture, small resilience and plasticity, and little resistance to a sudden blow. It is not the same as hardness. It is opposed to ductility and toughness.

Toughness means that a large amount of work must be done upon the body to fracture it. It implies large strength, ductility, and total resilience. A tough body has great resistance to blows.

Strength is the greatest unit stress of a specified kind that the body will sustain without fracture if gradually applied, so as to avoid blows and vibration. It should be measured in pounds per square inch on the section that breaks, strictly speaking, but ordinarily it is measured in pounds per square inch on the original area of the section that breaks (see Art. 18).

In studying the behaviour of a body under stress, a distinction must be made between materials that are ductile and those that are brittle, as will be pointed out.

4. Modulus of Elasticity.—When a straight bar of uniform section is exposed to tension (Fig. 45) it is stretched, and, if it is homogeneous all parts of it are stretched equally up to a certain limit. The total stretch produced by a given load P will be proportional to the length; and, if P is varied, the total stretch will be proportional to P up to a certain limit called the *proportional limit.* If the cross-section varies, the stretch per unit length at any point will be proportional to the stress intensity $\frac{P}{A}$ up to the same limit. If E is a constant for the material in question,

Intensity of stress $= E \times$ strain per unit length

$$\text{or } E = \frac{\text{stress per unit area}}{\text{strain per unit length}}.$$

The strain per unit length, or the total strain in a given length divided by the length, may be called the *unit strain;* hence

$$E = \frac{\text{unit stress}}{\text{unit strain}}. \tag{1}$$

The quantity E is called the *modulus of elasticity*, or Young's modulus. Since E is a force per unit area divided by the ratio of a distance (total elongation or shortening) to another distance (original length), E is a force divided by an area, and is expressed in pounds per square inch (or, in the metric system, kilograms per square centimeter).

If the unit strain were unity, that is, if a unit length were stretched by another unit length, so that its original length were doubled, E would equal the unit stress necessary to produce this result. In other words, E is the stress intensity necessary to double the length of a piece *if this could be done without exceeding the limit of elasticity*.[1] In practice, for engineering materials, the limit of elasticity is reached when the unit strain is very small, so that this conception of E is purely imaginary, though it is sometimes convenient.

A compressive stress shortens the length of a piece, and the action is similar to that for tension if no bending occurs. The modulus of elasticity for compression may or may not be the same as for tension. For steel it is practically the same, and for most engineering materials it is assumed to be the same.

The law that the *unit stress varies as the unit strain* is called *Hooke's law*, and was originally stated in 1678 by Hooke as "ut tensio sic vis." It is nearly correct for some materials, such as mild steel, up to the so-called limit of elasticity, but is not correct for other materials such as cast iron or wood, and is perhaps not *exactly* correct for any material.

For steel, since the unit stress varies as the unit strain up to the elastic limit, E is constant up to that limit. Above that limit, or for materials that do not conform to Hooke's law, E varies with the stress. (When the word stress is used without qualification it should be understood to mean unit stress.)

If a load P be applied to a piece whose length is l and whose area of section is A (the section at right angles to the force), the total elongation Δ may be found from the formula if E is known. It is better, however, not to use the formula, but to build up the result from a knowledge of the principles, thus: P divided by A will be the unit stress, this divided by E will be the unit strain, and this multiplied by l will be the total strain, or, if $e = \dfrac{P}{AE}$ = strain per unit length,

$$\Delta = \frac{Pl}{AE} = el. \tag{2}$$

The same result is obtained from Eq. (1).

5. Values of E.—The values of E, for iron and steel up to the elastic limit, and for materials that have no elastic limit the value when the stress is small, is given in the following table, approximately:

VALUES OF E

Material	Value of E in pounds per square inch, for tension or compression
Wrought iron	27,000,000
Steel	29,000,000 to 30,000,000; varying slightly with the grade.
Cast iron	About 15,000,000; with considerable variations.
Timber, along the grain.	1,000,000 to 2,000,000; with great variations.
Concrete	2,000,000 to 4,000,000; with great variations.

[1] No distinction is commonly made between proportional and elastic limit, though strictly speaking the proportional limit is that up to which stress and strain are proportional, while elastic limit is that up to which there is no permanent set (see Art. 17).

The value of E may vary considerably, even for material of the same kind. For wrought iron it may generally be assumed in engineering calculations as 27,000,000, and for steel as 30,000,000. For cast iron, it is more variable. For timber, it is very variable, with moisture, quality, defects, etc. For concrete, it is also very variable, with age, proportions of the ingredients, character of mixing, etc., but is quite generally assumed as 2,000,000, or $\frac{1}{15}$ that of steel. Further figures will be found in Vol. II. The above are sufficient for the present.

A steel bar with a unit tensile stress of 10,000 pounds per square inch would thus be elongated $\frac{1}{3,000}$ of its length, and if the elastic limit were 30,000 pounds per square inch, it would at that stress be elongated $\frac{1}{1,000}$ of its length.

6. Lateral Contraction Accompanying Normal Stress; Poisson's Ratio.—When a cylindrical piece (Fig. 51) is stretched by a longitudinal force P, it contracts laterally in all directions. The lateral contraction per unit of lateral dimension will be a fraction $\frac{1}{m}$ of the longitudinal extension per unit of longitudinal dimension. Thus, if the cross-section is a rectangle with a width t in the plane of the paper, and t' at right angles to the paper, the lateral contraction in the direction of t will be $\frac{Pt}{mAE}$, and in the direction of t' it will be $\frac{Pt'}{mAE}$. The unit contraction is equal in all lateral directions if the material is homogeneous in all directions.

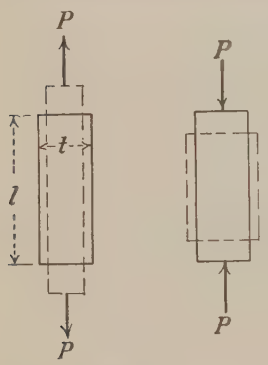


FIG. 51.

If the force P is a compression, the lateral dimensions are increased according to the same principles, with practically the same value of m.

The quantity $\frac{1}{m}$ is called *Poisson's ratio*. It is a fraction less than unity, and for iron and steel varies from 0.25 to 0.33, or m is between 3 and 4. A fair value of $\frac{1}{tm}$ is 0.3, but it is frequently taken as 0.25.

A steel rod stressed to 30,000 pounds per square inch would thus have the lateral dimensions reduced by about $\frac{1}{4,000}$ of their original values.

The determination of m obviously requires very delicate measurement, and the results are not uniform. The following table[1] gives average values (see Art. 15).

Material	Poisson's ratio	Material	Poisson's ratio
Glass	0.2451	Delta-metal	0.3399
Steel	0.2686	Lead	0.4282
Copper	0.3270	Stone	about 0.25
Brass	0.3275	Concrete	about 0.08 to 0.18

[1] JOHNSON'S "Materials of Construction," pp. 4, 255, 480.

Since a longitudinal pull produces lateral contraction, it is often thought by students that there must be a lateral force exerted on a longitudinal plane like the one represented by the dotted line in Fig. 45. For it may be argued that if the particles move inward laterally, some force must be present to produce this motion. Such a force could be a compression pushing the particles in, or a tension pulling them in. It can easily be seen that it could not be compression; for if a particle to the right of the center line were pushed in to the left by a particle to the right of it, that particle would be pushed by another to its right, and so on till the surface at D is reached, where there is no compressive force; consequently the molecular force cannot be compresssion. It is conceivable, however, that a particle to the right of the center line might be pulled in to the left by a particle to the left of it.

All this, however, is fallacious. For taking the entire section shown by the dotted line, the total stress on it is zero, because the resultant force to the right of this section is zero. It follows that if there is tension on any part of this section there must be compression on some other part; and this we cannot conceive. There is just as much reason for tension to exist on one part as on any other. Furthermore, what resistance is there to this inward lateral motion or contraction? None. The particles are drawn apart longitudinally, and there is room for them to draw in laterally. What lateral force would it take to move the particles inward against no resistance? Clearly an infinitesimal one, or really none at all.

Hence we see that statically we must assume or perceive that although there is a lateral contraction there is no lateral stress on a longitudinal plane. Similar reasoning holds regarding lateral expansion under a longitudinal compression.

Therefore, there may be a strain in a given direction with no stress in that direction (i.e., on a plane perpendicular to that direction). There may also be a stress in a given direction with no strain in that direction.

7. Biaxial Loads.—If stresses exist at a point in a body, along two or three rectangular axes X, Y, Z, each stress will cause change of length along all three axes, which can easily be calculated by the foregoing principles. Such a case is called one of *biaxial or triaxial loads*. Let there be unit stresses as follows:

tension f_x along X,
tension f_y along Y,
compression f_z along Z;

then along X there will be a *unit elongation* of

$$e_x = \frac{f_x}{E} + \frac{f_z}{mE} - \frac{f_y}{mE}, \tag{3}$$

and similarly for other cases.

8. If at a point in a body there are three planes at right angles to each other, on each of which there is only normal stress, the intensities being n_x, n_y, n_z along axes X, Y and Z, the resultant *unit strains*, e_x, e_y, e_z, will be (tension being +)

$$\left.\begin{aligned} e_x &= \frac{n_x}{E} - \frac{n_y}{mE} - \frac{n_z}{mE} \\ e_y &= -\frac{n_x}{mE} + \frac{n_y}{E} - \frac{n_z}{mE} \\ e_z &= -\frac{n_x}{mE} - \frac{n_y}{mE} + \frac{n_z}{E} \end{aligned}\right\} \tag{4}$$

From these equations the values of n_x, n_y, n_z are

$$\left.\begin{aligned} n_x &= \frac{mE[e_x + e_z + (m-1)e_x]}{m^2 - m - 2} \\ n_y &= \frac{mE[e_x + e_z + (m-1)e_y]}{m^2 - m - 2} \\ n_z &= \frac{mE[e_x + e_y + (m-1)e_z]}{m^2 - m - 2} \end{aligned}\right\} \tag{5}$$

For $m = 4$,

$$\left.\begin{aligned} n_x &= \tfrac{2}{5}E(3e_x + e_y + e_z) \\ n_y &= \tfrac{2}{5}E(3e_y + e_x + e_z) \\ n_z &= \tfrac{2}{5}E(3e_z + e_x + e_y) \end{aligned}\right\} \tag{6}$$

The single stress intensity n_x', along X, which would produce the unit strain e_x (unaccompanied by stresses along Y or Z) would be

$$n_x' = n_x - \frac{n_y}{m} - \frac{n_z}{m}. \tag{7}$$

This is the *imaginary simple stress* along X which, acting alone, would produce the same *strain* that is due to the actual stresses. It is sometimes called the *ideal stress intensity*.

If the case is one of biaxial stress, parallel to a plane XY, $n_z = 0$, and

$$\left.\begin{aligned} e_x &= \frac{n_x}{E} - \frac{n_y}{mE} \\ e_y &= \frac{n_y}{E} - \frac{n_x}{mE} \\ e_z &= -\frac{n_x}{mE} - \frac{n_y}{mE} \end{aligned}\right\} \tag{4a}$$

$$\left.\begin{aligned} n_x &= \frac{mE}{m^2 - 1}(e_y + me_x) \\ n_y &= \frac{mE}{m^2 - 1}(e_x + me_y) \end{aligned}\right\} \tag{5a}$$

For $m = 4$:

$$\left.\begin{aligned} n_x &= \tfrac{4}{15}E(e_y + 4e_x) \\ n_y &= \tfrac{4}{15}E(e_x + 4e_y) \end{aligned}\right\} \tag{6a}$$

9. Shear on Planes at Right Angles at a Point.—Let $Oabc$ (Fig. 52) be a rectangular prism with dimensions $Oa = dx$, $Ob = dy$, and $Oc = dz$, along rectangular axes OX, OY, OZ. Let the stresses on the six sides be resolved into normal and tangential components parallel to the three axes. Thus, on the bottom, or plane of XY, the normal stress intensity is n_z, the subscript denoting that it is on the Z plane, that is, on the plane perpendicular to the Z axis; and the shearing intensities are S_{zx} and S_{zy}, meaning that they act on the Z plane, one in the direc-

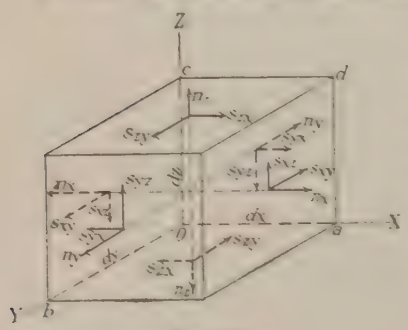


Fig. 52.

tion of OX and the other in the direction of OY. The stresses on planes parallel to XY may vary in intensity as we go along the Z axis, and if we are studying this variation, the normal intensity on the bottom plane being n_z, that on the top plane, dz distant from the bottom, would be

$$n_z + \frac{\partial n_z}{\partial z} dz$$

where $\frac{\partial n_z}{\partial z}$ is the rate of change of n_z with z, that is to say, the change in a unit distance if the change were uniform and at the same rate as at O. In mathematical language, it is the partial differential coefficient or derivative of n_z with respect to z. If, however, as in the present case, we are studying the relations between stresses at O, on different planes, the intensities on the top and bottom are to be taken as the same, as they differ by an infinitesimal quantity, since the prism has infinitesimal dimensions.

The prism is therefore acted upon by nine different pairs of equal, parallel, and opposite forces, namely:

on each of the Z planes,
- a normal stress, $n_z dx \cdot dy$;
- a shearing stress parallel to OX, $s_{zx} dx \cdot dy$;
- a shearing stress parallel to OY, $s_{zy} dx \cdot dy$;

on each of the X planes,
- a normal stress, $n_x dy \cdot dz$;
- a shearing stress, parallel to OZ, $s_{xz} dy \cdot dz$;
- a shearing stress parallel to OY, $s_{xy} dy \cdot dz$;

on each of the Y planes,
- a normal stress, $n_y\, dx \cdot dz$;
- a shearing stress parallel to OZ, $s_{yz} dx \cdot dz$;
- a shearing stress parallel to OX, $s_{yx} dx \cdot dz$.

The weight of the particle, being a finite weight per cubic unit multiplied by three infinitesimal dimensions, while the other forces are each a finite stress intensity multiplied by two infinitesimal dimensions, disappears from consideration.

The normal stresses, being in equal and opposite pairs all acting through the center of the prism, balance among themselves. The shearing stresses, being in equal and opposite pairs, balance along the three axes, but as they do not act in the same lines, they have moments about the three axes. There are thus only three of the six equations of equilibrium which remain to be considered, namely, that the sums of the moments about the three axes are zero.

The only couples having moments about the X axis are those formed by the forces whose intensities are s_{zy} and s_{yz}, and the condition that the sum of the moments about the X axis is zero is (with arrows directed as shown)

$$-s_{zy} dx \cdot dy \cdot dz + s_{yz} dx \cdot dz \cdot dy = 0$$

or

$$s_{zy} = s_{yz}.$$

Obviously, for equilibrium these stresses must be directed either both towards or both away from the edge of the prism which they both intersect.

Similarly, taking moments about OY,

$$s_{xz} = s_{zx}$$

and they are both directed as shown or both in a reverse direction. Also, about OZ,

$$s_{xy} = s_{yx}.$$

This result is of exceeding importance. It may be stated thus:

(1) *On any two planes at right angles to each other, at any given point, the shearing stresses, in directions perpendicular to the line of intersection of the planes, are of equal intensity; and they act either both toward the line of intersection or both away from it.*

Or the first part of this may be expressed as follows: *At any point in a body, if three planes are taken at right angles to each other, the tangential components of the stresses on any two of these planes, in directions parallel to the third, must be of equal intensity.*

In other words, there cannot at a point be a shearing force on one single plane passing through the point; there must be a shearing force also on at least one other plane, and it will later be seen (Chap. V) that there are shearing forces on still other planes. This does not mean, however, that, at a given point, on any two planes at right angles to each other the total shears are of equal intensity; it is only the component shears in certain directions that are equal in intensity. It does not mean that if there is a shear on a given plane there is a shear of equal intensity, or any shear at all, on every plane at right angles to the first. There may be shear on a plane and no shear at all on another plane at right angles to it, if that other plane is parallel to the shear on the first plane.

10. Special Case: Stress Parallel to a Plane.—A particular case, very common in practice, arises when all the stresses act parallel to a plane, as, for instance, the plane XZ. Suppose that on the upper and lower surfaces of the prism, in Fig. 52, the shear $s_{xy} = 0$, then $s_{yz} = 0$ also, and suppose that n_y, s_{xy}, and s_{yx} are also zero, then all stresses act parallel to the plane XZ and there is no stress on Y planes. In this case we need only consider the projection of the prism on the XZ plane, which is taken as the plane of the paper, in Fig. 53. In this figure the axes are called X and Y instead of X and Z. Here it is easy to see that the shearing forces must form two couples which balance, so that $s\,dx \cdot dy = s_1\,dx \cdot dy$; or $s = s_1$; and they must act as shown or in opposite directions.

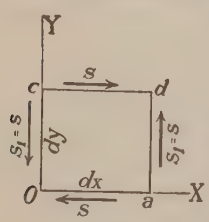


FIG. 53.

In the case of stresses parallel to the plane of the paper there is no stress on the plane of the paper or planes parallel to it. The dimension of the particle perpendicular to the paper is unity or any other value. The top and the bottom of the particle are rectangles. The thickness of the particle, perpendicular to the paper, is immaterial.

11. Shear Alone on Two Planes at Right Angles for Stress Parallel to a Plane, Involves Tension and Compression on Planes at 45° with These.—Let there be no normal stress on the surfaces of the particle in Fig. 53, but only shears, and let the particle be a cube with sides equal to unity. Then $s_1 = s$, as shown above. The stress on the plane Od, as in Fig. 54, will be a normal stress $s\sqrt{2}$, and the area Od will be $\sqrt{2}$, so that the intensity will be s, and the stress will clearly be *compression*. On ca, at right angles to Od, the stress will be a *tension*, also of intensity s. If

the shears act in directions opposite to those indicated, the stress on *Od* will be tension and on *ca* compression.

Hence it follows that: *in the case of stress parallel to a plane, if there is only shear on any plane, and hence a shear of equal intensity on another plane at right angles to the first, there must be pure tension and pure compression of the same intensity as the shear acting on planes inclined at 45° to the planes of shear.*

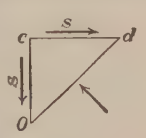


Fig. 54.

The case would be different if on *Oa* and *Oc* in Fig. 53 there were normal stresses, either with or without shear. This general case is treated in Chap. V.

This principle is of great importance, and should be fully grasped by the reader.

12. Deformation Produced by Shear.—If shear only exists on the sides of a prism or cube, with stress parallel to the plane of the paper (Fig. 55), the face *abcd* will be distorted into the parallelogram *a'b'c'd*, or, if the side *dc* be supposed to be kept unchanged, *abcd* will become *a'b'cd* in Fig. 56. If *a* moves to *a'*, the length *da'* is greater than the length *da*, which would mean tension on the plane *cd*, which is contrary to the supposition made that there is only shear on *cd*. If *a* moves in an arc of a circle with *d* as a center, *a'b'* is slightly below *ab* (Fig. 57).

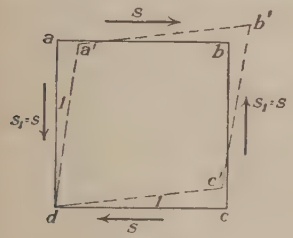


Fig. 55.

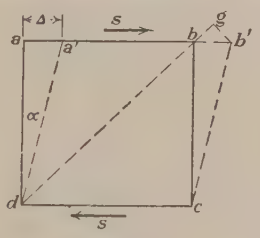


Fig. 56.

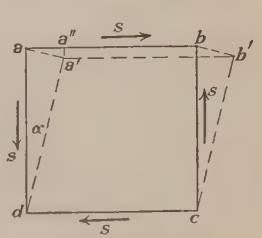


Fig. 57.

The arc $aa' = ad \cdot \alpha$, if α is the angle of distortion ada'. The arc is almost exactly equal to the chord aa', since the angle α is extremely small. The angle $a''aa' = \frac{\alpha}{2}$ and $a'a'' = aa' \cdot \frac{\alpha}{2} = ad\frac{\alpha^2}{2}$, very closely. Since α is very small, α^2 will be negligible; hence it is always assumed that *a* moves to *a'* in Fig. 56. The shearing deformation is $\Delta = aa' = \alpha \cdot ad$, and is thus proportional to *ad*, and also, of course, to the intensity of shear. The unit shearing strain or deformation is Δ divided by *ad*, or the angle α, or its tangent. (It cannot be dependent on *ab*.) The *modulus of elasticity for shear*, often called the *modulus of rigidity*, being the unit shearing stress divided by the unit shearing strain, is $\frac{s}{\alpha}$, or

$$\text{modulus of elasticity for shear} = E_s \text{ (or } G) = \frac{s}{\alpha} = \frac{s \cdot ad}{aa'}. \qquad (8)$$

Since a pure shear is the same as a tension and a compression of the same intensity of the shear, on planes at right angles to each other and at 45° to the planes of shear, the diagonal *db* is elongated to *db'*, and *ac* is shortened to *a'c*; and there is obviously a discoverable relation between *G* and *E*. The elongation of *db* per unit of length will be

$$\frac{s}{E} + \frac{s}{mE} = \frac{s}{E} \cdot \frac{m+1}{m}$$

and the total elongation of db will therefore be

$$bg = \frac{s}{E} \cdot \frac{m+1}{m} \cdot ad\sqrt{2}.$$

Since this elongation is very small, by drawing a line from b' perpendicular to ab' to g, it is obvious that this elongation equals bg, and $bg = \frac{bb'}{\sqrt{2}} = \frac{aa'}{\sqrt{2}}$, since the figure represents a cube. Hence

$$E_s = G = \frac{s \cdot ad}{aa'} = \frac{s \cdot ad}{bg\sqrt{2}} = E \cdot \frac{m}{2(m+1)}. \tag{9}$$

For steel, with $m = 4$;

$$G = \tfrac{2}{5}E. \tag{10}$$

The angle of distortion produced by a shear is

$$\alpha = \frac{s}{G}. \tag{11}$$

If, as would really be the case, the particle $abcd$ in Fig. 55 is distorted into $a'b'c'd$, α is the difference, in radians, between the angle adc and the angle $a'dc'$, or, since the deformation is very small, it is $cdc' + ada'$.

Since pure shear merely changes the surface $abcd$ to $a'b'cd$, both having the same area (neglecting α^2 as explained), it is obvious that under this assumption pure shearing does not alter the volume of the particle. Hence, *a tensile stress on a given plane together with an equal compressive stress on a plane at right angles* (which are the same as pure shearing on two other planes) *does not change the volume.*

13. Change of Volume Produced by Stress.—Since shearing stress parallel to the X plane produces no change of volume, the same is true of shear parallel to the Y or Z planes; hence any shearing stresses produce no change of volume. Only normal stresses do this; and it remains to find what change of volume is produced by normal stresses.

A rectangular prism of length l and lateral dimensions a and b, subjected to a tensile force P along l, would have the original dimensions changed as follows, calling $\frac{P}{E} = k$:

$$l \text{ would become } l + \frac{Pl}{AE} = l + \frac{kl}{ab}$$

$$a \text{ would become } a - \frac{Pa}{mAE} = a - \frac{k}{mb}$$

$$b \text{ would become } b - \frac{Pb}{mAE} = b - \frac{k}{ma}.$$

Hence the new cross-sectional area will be

$$\left(a - \frac{k}{mb}\right)\left(b - \frac{k}{ma}\right) = ab - \frac{2k}{m} + \frac{k^2}{m^2ab}.$$

If a and b are respectively dy and dx, so that we are finding the change of volume of a small particle, and if $\frac{P}{A} = f$, the new area is

$$dydx\left(1 - \frac{2f}{mE} + \frac{f^2}{m^2E^2}\right).$$

Since $\frac{f}{E}$ is very small, the last term may be neglected, and the new area is almost exactly $ab - \frac{2k}{m}$. The increase of volume will be

$$\left(ab - \frac{2k}{m}\right)\left(l + \frac{kl}{ab}\right) - abl = kl\left(1 - \frac{2}{m}\right) - \frac{2k^2l}{mab}.$$

For the same reason as before, the last term may be neglected, and the increase of volume may be taken as $kl\left(1 - \frac{2}{m}\right)$. The increase of unit volume will be this divided by lab, or

$$\text{Increase of unit volume} = \frac{k}{ab}\left(1 - \frac{2}{m}\right) = \frac{P}{AE}\left(1 - \frac{2}{m}\right). \tag{12}$$

If $m = 4$; increase of volume $= \frac{f}{2E}$.

The volume would be unchanged if $m = 2$.

14. Volume Modulus.—Besides the direct modulus of elasticity E, and the shearing modulus E_s, there is a volume modulus, E_v, which is defined as a unit *hydrostatic pressure* (or tension) on a particle, such as a cube, divided by the change of volume produced by such pressure. Let it be required to find the change of volume of a cube having all sides unity, exposed to a tensile stress intensity p on all six sides (Fig. 58). First, let us add two tensions p on the top and bottom, and two equal compressions on the sides, as shown by the dotted lines, since by the theorem demonstrated in the last article the volume is not changed by so doing. Then let us add two more tensions p on the top and bottom and two equal compressions on the front and back (not shown) since for the same reason the volume is not changed thereby. Then on every side except the top and bottom the resultant is zero, while on the top and bottom the unit stress is $3p$, which causes the same change of volume as the equal tensile stresses p on each of the six sides. According to Art. 13 the change of unit volume produced by this stress of intensity $3p$ is, since $\frac{p}{E} = e$, the unit strain,

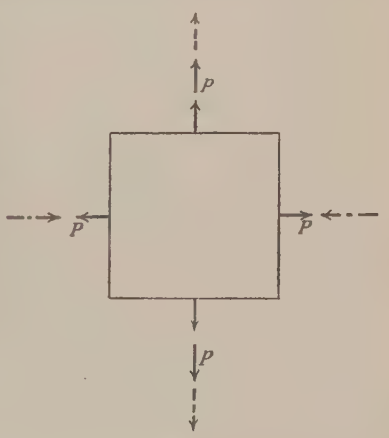


FIG. 58.

$$\frac{3p}{E}\left(1 - \frac{2}{m}\right) = 3e\left(1 - \frac{2}{m}\right)$$

and consequently the volume modulus is

$$E_v = \frac{p}{3e\left(1 - \frac{2}{m}\right)} = \frac{Em}{3m - 6}. \tag{13}$$

15. Relations between E, E$_s$, and E$_v$.—The relation between E and E_s has been found in Art. 12 to be

$$E_s = E\frac{m}{2(m+1)}. \tag{9}$$

Equation (13) gives the relation between E_v and E. Substituting in Eq. (13) the value of E from Eq. (9),

$$E_v = \frac{2E_s(m+1)}{3m-6}. \tag{14}$$

It will be found from these relations that

$$E_v = \frac{E\,E_s}{9E_s - 3E}. \tag{15}$$

$$E = \frac{9E_sE_v}{E_s + 3E_v}. \tag{16}$$

$$m = \frac{2E_s}{E - 2E_s}. \tag{17}$$

These relations make it possible to find m without direct measurement of the change of transverse dimensions under a longitudinal load, which is difficult and uncertain because the changes are so minute. But it is comparatively easy to measure E and E_s, and from these m may be found by Eq. (17).

16. Allowable Stresses.—The reason for finding the maximum stresses in a structure is because the actual stress must not be allowed to exceed a certain value, called the *working* or allowable stress, otherwise the requisite safety will not be present. The unit stress which would break the piece is called the *ultimate stress* or strength. The allowable stress must not be anywhere near the ultimate, but is taken as the ultimate divided by a number called the *factor of safety*. The ultimate itself will vary with circumstances, such, for instance, as whether the load is quiescent or *dead*, or moving; whether the load is repeated many times, etc. The factor of safety, or ratio of ultimate to allowable, is necessary to provide for the following uncertainties:

(1) Inaccuracy of calculation of stress.
(2) Possible increase of load in the future.
(3) The effect of shocks, vibration, etc., which cannot be calculated.
(4) Variations in quality of material.
(5) Defects in material.
(6) Deterioration of material, as by rust or decay.
(7) The possible serious effects of a failure of the material.

These will be discussed in detail in a later chapter. They are more or less arbitrary. The determination of the factor of safety and the allowable stress is not a simple matter. The latter is standardized in specifications representing the judgment of those preparing them, but they are subject to change from time to time, as we gain in experience and in knowledge of materials, and processes of design and of calculation are perfected. Examples are given in the chapters dealing with the various materials.

17. Stress-strain Diagram.—The behaviour of a material under stress is best studied by testing a piece with an original length L and cross-section A, gradually increasing the load, measuring the deformation or change in the length

L at intervals, and plotting a "stress-strain diagram" in which the stress intensity is represented by vertical distances and the strain per unit length (change in L divided by L) by horizontal distances.[1] The shape of such a line will vary greatly according to the kind of material, the heat treatment or mechanical treatment to which it has been subjected, and the kind of stress. Such a diagram for mild steel in tension is shown in Fig. 59. As we have seen, under a tensile stress the cross-section diminishes as the stress is increased, and under compression it increases. Stress-strain diagrams as almost universally plotted show the stresses *per unit of the original (not the actual) area.* They therefore do not show the real unit stresses, which are at every point greater than those shown, if the stress is tension, since the actual area is less than the original area; and the reverse in compression. For scientific study, the actual stresses should be used, but in

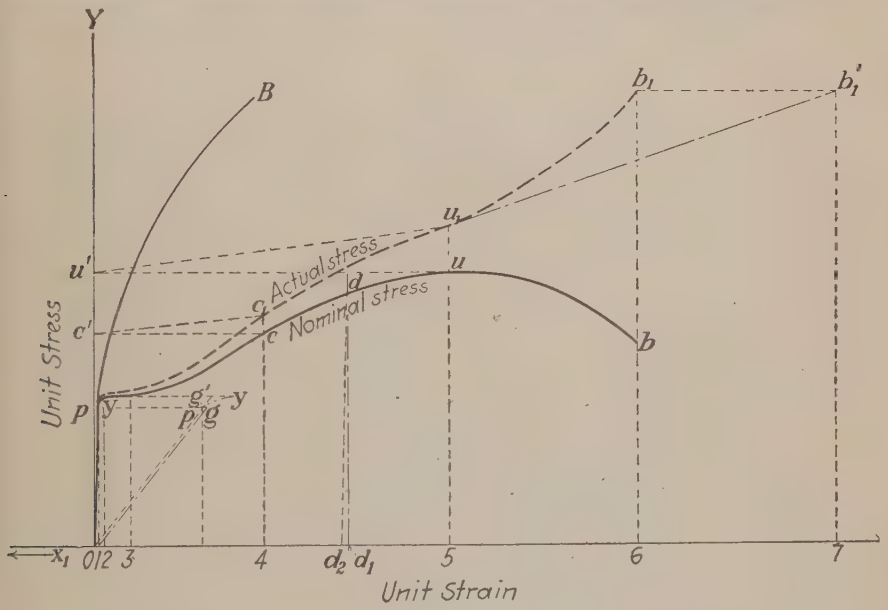


FIG. 59.

practice the engineer is only concerned with the original areas which he should provide in order to resist the stresses, and the changes of cross-section are too small to be considered, within the limits of allowed stress.

As the stress is increased, the piece stretches nearly uniformly over its whole length, if homogeneous, and the stretch is proportional to the stress up to the point p, called the *proportional limit.* (The deformations are very small at first, and are shown also on an enlarged scale in the figure.) Up to some other point there will be no permanent set when the load is removed; this is the *elastic limit,* and for a ductile material like mild steel is practically the same as the proportional limit. Even in ferrous metals, the exact set limit is found sometimes above the proportional limit and sometimes below. It is sometimes stated that, if the measurements could be made with sufficient accuracy, all materials would show

[1] In England it is quite common to plot stress horizontally and strain vertically.

some slight permanent set under any load. This, however, if true, is probably not due to any real permanent set of the piece as a whole, but to the straightening of slight original bends in the piece (which gives a local permanent set at the bend); or to slight yielding or adjustment in the wedges or other means by which the piece is held in the machine; or to small defects or lack of homogeneity of the material; or to uneven distribution of the load; or to the presence of initial stresses. These initial stresses, as will be shown in Chap. XIX, may themselves be as high as the elastic limit, so that even small applied loads may raise the stress to above the elastic limit. There is every reason to believe that a ductile material, like soft steel, if without initial stress, and if straight and homogeneous, and if accurately tested, would be perfectly elastic up to the usual elastic limit.

At some point y, above p, there will be (in the material considered) a quite sudden and large increase of stretch with a very slight increase of load, or with no increase at all, or even with a small decrease of load; this is called the *yield point*, and in structural steel is commonly taken as the elastic limit in commercial testing. The yield point is the point at which the material becomes partially plastic and begins to "flow" with little or no increase of stress; indeed, when once the flow is started, it may even continue with a slightly decreased load, and the stress-strain diagram is often shown (and if autographically drawn records it) with a drop at the yield point (for a ductile material) as in Fig. 60, though very probably if the load were increased with extreme slowness there would be no such drop. The yielding at the yield point is still uniformly distributed over the length of the piece. This yielding is large in proportion to the total stretch up to just before the yield point is reached. In structural steel with an elastic limit of about 30,000 pounds for square inch, the stretch at this load (with $E =$ 30,000,000) is $\frac{1}{1{,}000}$ of the length, while the yielding at the yield point may be many times this amount, depending upon the material, as much as 10 or even 20 times for soft steel. The harder the steel, the higher the yield point and the less the yielding at that point.

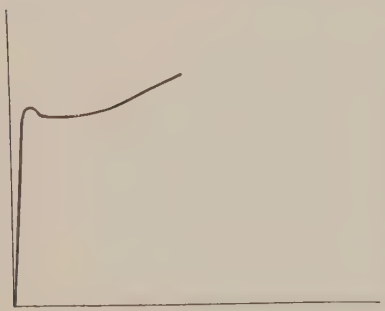


Fig. 60.

If a stress is applied up to a point g, which is above the elastic limit but below the yield point, when the load is removed the line will follow back parallel to Op, and there will remain a small so-called "permanent set" $g'g$, which, however, may wholly or partially disappear in time, if the piece is allowed to rest. If the same load is immediately reapplied, the elastic limit will be found to have been raised to the value of the load.

Beyond the yield point, after the yielding has occurred, the stress-strain curve resumes its upward course, but the strain increases faster than below the elastic limit, until the maximum or "ultimate" load is reached at u. Here the stretch begins to be no longer uniformly distributed over the length of the piece, which begins to "neck down" and draw out at some point, with large reduction of

area at that point, as in Fig. 61. Sometimes there is a tendency to neck down at more than one point before the piece finally selects the weakest point and breaks there. The breaking load occurs at b, and is smaller than the ultimate, but is never measured because it cannot be.

Brittle materials like cast iron, brick, stone, concrete, and also wood and hard steel, do not neck down, and they show no definite yield point.

Morley states that "in the case of rolled aluminum slowly and continuously loaded, at very low stresses the strains increase faster than the stresses, and yet practically all the strain disappears after the removal of the stress; hence the elastic limit cannot be found from an inspection of the stress-strain diagram."

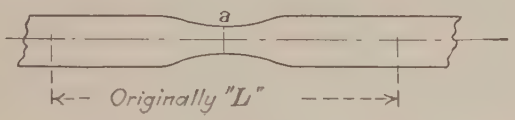


FIG. 61.

18. Real Stress Intensity in Stress-strain Diagram.—The diagram gives the stress per unit of *original* area. The actual area might be measured at various points of the curve, but this is never done in commercial testing. After fracture (in tension) the parts are put together, and the area of the fractured section measured in order to obtain the "reduction of area." The actual area and the real stress intensity, however, may be determined if it is assumed that *in plastic deformation* the volume is unchanged. We have seen in Art. 13 that below the elastic limit there would be no change of volume if $m = 2$; but for steel and most construction materials m is between 3 and 4, so that the volume is slightly increased by a tensile stress and decreased by a compressive stress, though the change below the elastic limit is too small to be considered. When the plastic or partially plastic stage is reached, however, at the yield point, and particularly when necking begins, the plastic or permanent deformation is much greater than the elastic deformation, though there is still elastic deformation up to the ultimate. If a load up to d (Fig. 59) is removed, for instance, the stress-strain unloading curve will follow down dd_2, closely parallel to Op, and the distance Od_2 will be permanent or plastic deformation, while d_2d_1 will be the elastic deformation. If dd_1 is 50,000 pounds per square inch d_2d_1 will be $\frac{1}{600}$ of the length of the piece; while Od_1 may for such material be 0.15 or more. In plastic deformation it is nearly true that the density is unchanged, that is, that a permanent deformation produces no change of volume (see Art. 3). This being assumed, if the piece of original length L and area A has its length increased to $L + \lambda$ and the area reduced to A', then *as long as the piece remains cylindrical, that is, up to the point at which necking begins,*

$$\frac{A'}{A} = \frac{L}{L + \lambda}.$$

whence,

$$\frac{\text{contraction of area}}{A} = \frac{A - A'}{A} = \frac{\lambda}{L + \lambda} \tag{18}$$

and,

$$\text{real unit stress} = \frac{P}{A'} = \frac{P}{A}\left(1 + \frac{\lambda}{L}\right). \tag{19}$$

Since the value of L is unity in the stress-strain diagram as defined, the real stress intensity at any point, c, of the diagram may be computed by Eq. (19), or plotted by laying off to the left of O the distance $OX_1 = 1$, to the same scale as the unit elongations (or about 1,000 times $O1$) projecting c on OY at c', and drawing $X_1c'c_1$; then $4c_1$ is the real unit stress. This holds only up to u, where the real unit stress is $5u_1$ as above shown. The curve of actual stress lies entirely above the usual stress-strain curve.

Beyond u, where necking begins, the real stress could only be found by dividing the load by the actual area; at fracture it is $\frac{P}{A'} = 6b_1$; it can only be found for any other point between u and b by measuring the most contracted area under a given load. But the area cannot be measured, for the piece is in process of breaking.

19. Relative Elongation and Contraction of Area.—Results of tests state the relative elongation of the gauged length expressed as a percentage of the original length or

$$\text{percentage elongation} = 100\,\frac{\lambda}{L}.$$

The elongation at fracture, λ, is made up of two parts, one a general elongation, approximately uniform over the length L, and therefore proportional to L and independent of the sectional area, and the other a local elongation due to the necking, and independent of L but increasing with the sectional area. Hence approximately, if k is a constant, and c a quantity which is not constant, but increases with A,

$$\lambda = c + kL,$$

or,

$$\text{percentage elongation} = 100\left(\frac{c}{L} + k\right). \tag{20}$$

The percentage elongation is therefore greater the greater the sectional area and the smaller L, and in order that tests may be comparable, test pieces must have standard lengths and cross-sections.

Obviously, if two test pieces of the same material are geometrically similar, they will have the same relative elongations. This is *Barba's "Law of Similarity."*[1] It means that L should vary with the square root of the area, or, for cylindrical bars, with the diameter. This condition, however, is not usually strictly observed. If it is not, on any given length L, the percentage of elongation is greater the greater the section, because the local contraction will be greater. Barba found, for steel bars, that when $\frac{\text{length}}{\text{diameter}}$ was 20, the per cent elongation on a given length was about two-thirds what it was when the ratio was 5, and that it was very nearly constant when the ratio was constant.

Unwin found for mild steel plates the values of the constants in Eq. (20) to be, approximately, $c = 66$, $k = 17$; varying with the thickness and the direction of the test piece with relation to the direction of rolling. They will of course vary greatly with the percentage of carbon.

When necking occurs, the relative elongation is different at different points along the piece (Fig. 61), being greatest at the smallest section. Its value here

[1] "Mémoires de la Société des Ingénieurs Civils," 1880, pt. 1, p. 682.

may be found if the contracted area is known, assuming that the volume does not change. Let L_1 be the length that the original length L *would have assumed* if it had remained cylindrical and with the contracted area A_1; then if the volume is unchanged,

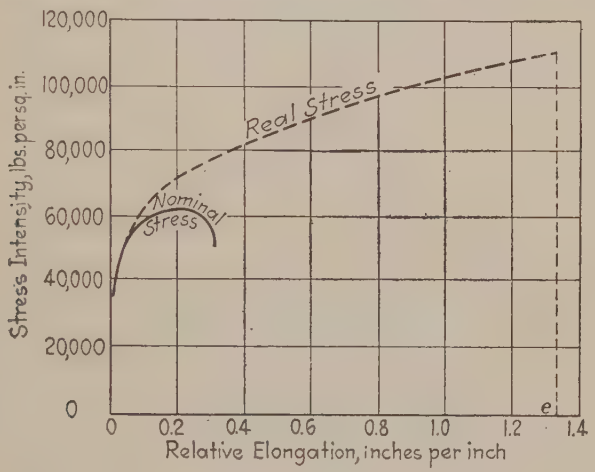
$$AL = A_1 L_1; L_1 = \frac{AL}{A_1}.$$

$$\text{Increase of length} = L_1 - L = L\left(\frac{A}{A_1} - 1\right).$$

$$\text{Relative increase of length} = \frac{A}{A_1} - 1. \qquad (21)$$

This is the physical or real relative elongation at the contracted section, and is much greater than the relative elongation as usually stated.

The portion $u_1 b_1$ of the curve of actual stress in Fig. 59 may now be drawn so that it will not be misleading. To the left of u, any ordinate $4c_1$ is the real unit stress corresponding to the real relative elongation $O4$, because the elongation is



(*Upton: The Structure and Properties of the More Common Materials of Construction.*)

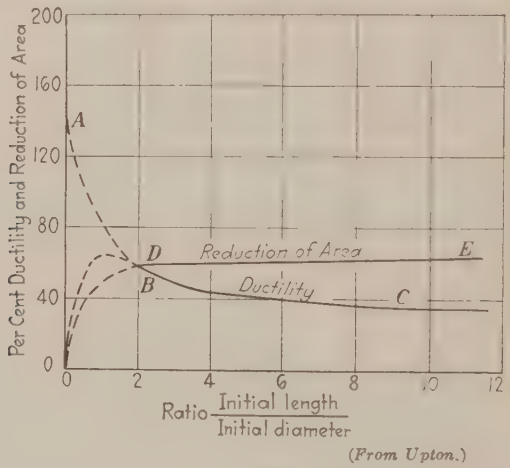
FIG. 62.

the same at all points in the length L. The real stress $6b_1$ at fracture does not correspond to a real relative elongation $O6$ (since the latter is not the elongation at the contracted section), but corresponds to a real unit elongation given by Eq. (21); hence we should lay off $O7 = \frac{A}{A_1} - 1$, and $7b_1' = 6b_1$ to find the true point b_1'. The line $u_1 b_1'$ is the correct line; intermediate points may be found only by measuring the actual area, which is difficult if not impossible.[1] The unit elongation of structural steel at fracture, referred to the original length L, may be 25 per cent and the reduction of area 50 per cent, and, for these values, $O6 = 0.25$, $O7 = 1$, or 4 times $O6$. The real shape of the line of actual stress should therefore be something like that in Fig. 62.

[1] In Eq. (19), if for $\frac{\lambda}{L}$ is substituted its value when necking occurs, from Eq. (21), the value of the real stress intensity becomes $\frac{P}{A_1}$, as it should.

The relative contraction of area *at fracture* being almost independent of L and of the sectional area, gives a better indication of the ductility of the material than the relative elongation at fracture. There is no necessary relation between elongation and contraction of area.

The real ductility and reduction of area, for a ductile material like mild steel, will obviously be as shown in Fig. 63. The value OA is that given by Eq. (21),



(*From Upton.*)

FIG. 63.

representing the real unit elongation at the contracted section. The ductility curve becomes asymptotic to a horizontal line whose ordinate is the unit elongation in the part of the piece which remains cylindrical, namely the value of $100k$ in Eq. (20) (making $L = \infty$).

The length of the test piece being greater than that of the portion which draws down, it is clear that the right end of the diagram, (Fig. 59) from u to b, will depend

FIG. 64.

on the length of the specimen tested. The greater the length of the specimen, the shorter the distance 56. As the specimen is progressively shortened, L being gradually reduced, it ultimately becomes a grooved section, C, or a notched section, D (Fig. 64C, D) for each of which L is really zero; that is, they are not comparable with ordinary test pieces. If pieces like C or D are tested, while they break at the grooved or notched section, the necking is wholly or partially prevented, and

the ultimate unit stress on the notched section will be greater than in a straight cylindrical piece. The ductility curve of Fig. 63, therefore, means little when the length is less than twice the diameter, and is drawn dotted. The dotted part of curve *CBO* may be considered to represent the relative elongation of a short length. The notched piece, *D* , for which $L = 0$, would break at the notch with practically no elongation.

A ductility percentage obviously means nothing unless the length and diameter are stated, or made standard; and the form of the stress-strain curve will vary greatly with the dimensions adopted.

The stress-strain diagram for the stress at a sharp notch cannot be measured, for the length of the piece having the minimum section is zero. If it could be drawn for a piece of infinitesimal length, it would not be at all like the line *opyub* of Fig. 59, but would no doubt be more like the line marked *B*. It might even be practically a straight prolongation of *op* to the breaking point, with little curvature. If lateral contraction is hindered, the usual curve is profoundly modified; if contraction is prevented, the curve may be almost a straight line. From such a line, the change would be gradual to the line *opyub* as the length of the groove is increased, becoming *opyub* when the straight part is long enough to contain all the necking.

Materials differ greatly in the necking which occurs when tested in tension. Wrought iron necks less and stretches more uniformly than ordinary or high-carbon steel; and manganese steel stretches more uniformly than wrought iron.

It should also be noted that $\frac{P}{A_1}$ will not be the actual *maximum* stress intensity at the most contracted section, but is the *average* stress there. On account of the curve of the necking, the load P will not be quite uniformly distributed over the area A_1, but will be greatest at the outside.

The fracture of a ductile material in tension at the necked section is often cup-shaped, sloping at the edges at an angle of about 45° or more (perhaps 55°) with the direction of pull, and then breaking nearly square across at the center. The fractured ends can be put together and the contracted area measured, though not as accurately as the extension can be measured.

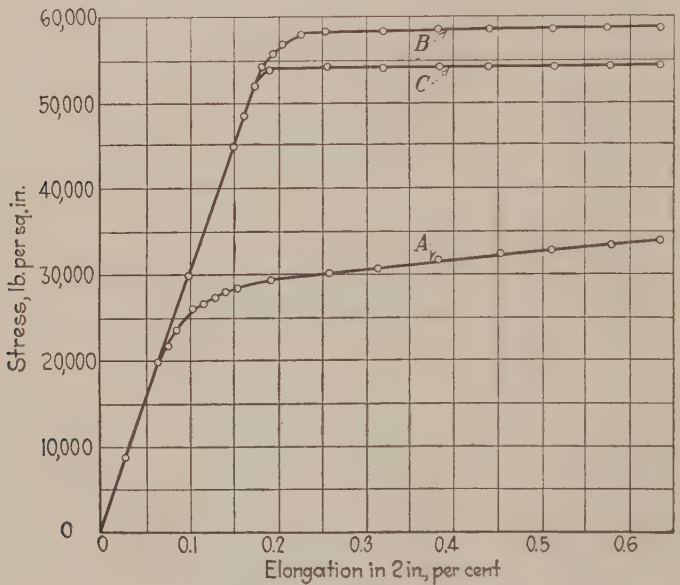
20. A ductile material, however, does not always show a sudden yielding at the yield point. There may be a gradual yielding, as in *A*, Fig. 65. The shape and dimensions of the curve may be infinitely varied, even in the same material, depending on heat treatment and mechanical treatment, as in Fig. 65.[1]

Brittle materials like cast iron generally give a diagram which is curved from the start, with no necking and no yield point, with a small set from the beginning, and with E continuously varying; though generally the curvature is small for some distance from the origin, and E may be properly assumed nearly constant up to a considerable unit stress.

The diagram for compression is in general similar to that for tension, but the phenomena are different. The cross-section increases in compression, and there is not the sudden yielding at the yield point, unless the piece bends. It is difficult to keep long pieces straight during compression, and hence it is necessary to test only short cylinders or prisms in studying pure compression. A ductile material will bulge and assume a barrel-shaped form, and the friction of the ends on the

[1] See *Proc. A.S.T.M.*, 1916, pp. 403–454.

heads of the testing machine will resist the tendency to bulge there, and so will increase the apparent strength. Brittle materials tested in compression will fail by shearing on a plane or planes making angles less than 45° with the direction of the compression; hence such materials should not be tested in cubes, but the length should be two or more times the lateral dimension. Compression tests, however, do not show phenomena like the local yielding and necking down that occurs in tension, and the reduction of stress at the end of the diagram which occurs in tension does not occur in compression. Indeed, it is often impossible to fix the ultimate stress in compression, especially if the length of the specimen is less than the diameter, and the material is ductile; continued deformation,



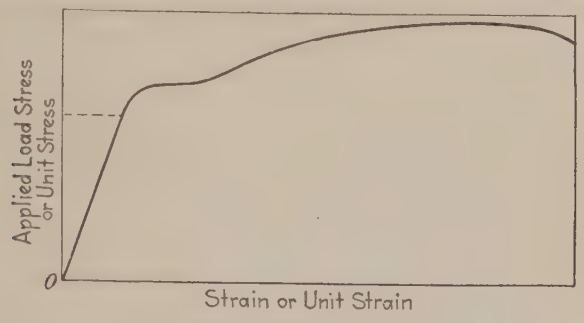
(*American Society for Testing Materials, 1916, p. 419.*)

FIG. 65.—Stress-strain diagrams for three specimens of cast steel from the same coupon:
A—As cast
B—After quenching in oil
C—After quenching in oil and drawing at 500° C.

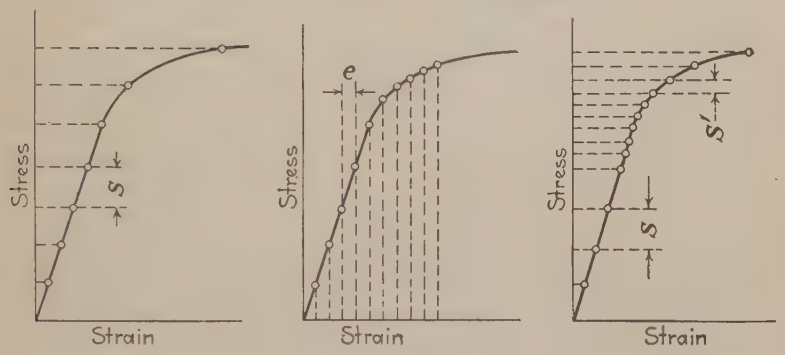
accompanied by bulging or by cracking, will occur, and the ultimate must sometimes be merely an arbitrary limit of deformation beyond which the usefulness of the material is gone. A ductile material will not break in two or more parts, as it does in tension; but a brittle material will shear into two parts or into conical and wedge-shaped parts.

21. Measurement of Elastic Limit and Yield Point.—The precise determination of the elastic limit, up to which strain is proportional to stress and there is no permanent set, depends upon the accuracy of the measuring instruments. The deformations are very minute and the opportunities for errors large (see Chap. XXI). If the strains are measured with equal increments of stress, the exact point where the line begins to depart from a straight line is difficult to fix (Fig. 66). If there are equal increments of strain, the point is more definitely

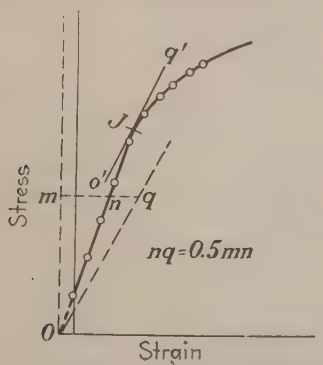
fixed, but still not accurately. Probably the best way is to use two increments of stress, the first up to near the point where the elastic limit is expected to occur, if this can be predicted, and a smaller increment until the yield point is passed.



A TYPICAL STRESS-STRAIN DIAGRAM



ILLUSTRATING THREE METHODS OF PLOTTING STRESS-STRAIN CURVES



ILLUSTRATING METHOD II FOR DETERMINING ELASTIC LIMIT

(*Am. Soc. Testing Mats., Vol. XXII, Pt. I, p. 518.*)

FIG. 66.

In any case there will be errors of observation, and the points up to the elastic limit will not lie exactly on a straight line, which increases the difficulty. This

has led to the suggestion by J. B. Johnson to define the elastic limit arbitrarily as the point where the slope of the curve with the axis of stress is 1.5 times (or some other ratio) the slope at the origin. Drawing the line as nearly as practicable as a smooth curve, straight for a distance from the origin till the curvature is evident, if mq is made 1.5 mn, and $o'q'$ drawn parallel to oq and tangent to the curve, the point of tangency, j, is taken as the elastic limit, and is called the "apparent elastic limit." This, of course, is purely empirical and the point j is not the true proportional limit, but the method is widely used. Sometimes, also, the elastic limit is arbitrarily found as the point where the set has a small fixed value: by laying off this value to the right of O and from this point drawing a straight inclined line parallel to On, it is thought that the intersection with the curve representing the measurements can be found more accurately than to try to find the exact point where the curvature begins. This may be true, but these methods clearly make the elastic limit *not* the proportional limit or the limit below which there is no set; and regarding them Professor Unwin remarks that "such a point has no significance."[1]

To define and fix the elastic limit as the point below which there is no permanent set requires repeatedly unloading the test piece in order to find whether there has been a permanent set, and this is time-consuming, objectionable, and impracticable in commercial testing.

If the stress-strain diagram shows no sudden or nearly sudden yielding at the yield point, but a gradually rounded curve, then there is, strictly speaking, no yield point, and if it is desired to provide one it must be arbitrarily defined; as, for instance, the point where the elongation is 0.5 per cent; but such a point has no significance, as Professor Unwin says, and the important point is the elastic limit. Most steels, however, show a decided yield point, and this is usually determined by the "drop of the beam." The usual testing machine is a *straining machine*, that is, the machinery in it strains or deforms the test piece at a tolerably uniform rate, while the weighing machinery enables the operator to increase the load to keep pace with the increased deformation or stretch. This he does by moving the balancing weight out on the arm of the lever so as to keep the lever balanced or "floating." As the strain increases he moves this weight out at a nearly uniform rate as long as the stress and strain are proportional; but when the point is reached at which the strain increases greatly and suddenly with little increase of stress, he will be moving the weight out as before and will overrun the load necessary to balance the strain, and the beam will drop. This is called the "drop of the beam," and is more noticeable the more rapid the speed of testing. This point is easily determined, and is usually taken in commercial testing as the elastic limit, though strictly it is the yield point, and the elastic limit is below it and not determined. Where there is a decided yield point, it, and not the elastic limit, is the most important point for the engineer to know.

The yield point is sometimes determined "by the dividers," by measuring the distance between the fixed gage points with a pair of dividers and noticing when there is a sudden or a stated increase in this distance. Capp's multiplying dividers may be used (see Chap. XXI).

[1] Professors Moore and Seely state that "the stress at which a very small but definite permanent set can be detected is probably the most nearly logical 'elastic limit' "; but to this the writer does not agree (see *Proc. A.S.T.M.*, 1916, pp. 426–433).

22. Effect of Time on the Stress-strain Curve.—Deformation, especially plastic deformation, is gradual, generally increasing with time, but at a diminishing rate. Hence if a load *Oc*, above the elastic limit (Fig. 67), is applied to a ductile material and allowed to remain, the elongation will gradually increase for a time. If the load is then increased, the stress-strain line will rise practically parallel to *Op* until it reaches the original line *Opb*. If, therefore, the load is applied, not gradually and continuously, but in a series of increments followed by intervals of rest, the curve above the elastic limit will consist of a series of notches, one side of each notch being horizontal and the other practically parallel to *Op*. If the periods of rest are short, the value of the ultimate will be affected little if at all; but, if the periods are long enough, it will be found that the ultimate is much reduced: in other words, a steady load considerably less than the usual ultimate will cause fracture of a ductile material if applied long enough. This only means that the length of the notch, given sufficient time, is great enough to cut off the top of the curve and pass over to the necking-down portion.

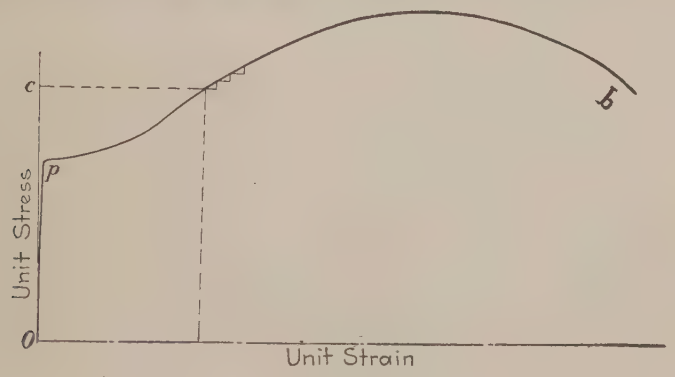


FIG. 67.

Ductile materials, therefore, which have a large plastic yield, generally or often have a larger ultimate strength for rapidly applied loads than for slowly applied or steady oads. This may be modified, however, by the raising of the elastic limit and of the ultimate by a stress above the yield point, discussed in Art. 24; and it may happen that a load above the yield point, if allowed to remain for a time, may raise the ultimate so that when the increase of load is resumed the ultimate may be above that found for a continuous and more rapid loading.[1] Generally, the stress-strain curve is flatter for slow application than for rapid application unless modified as first explained. It is generally considered that the ultimate elongation is decreased by rapid loading, which does not give time for the usual necking down; but this again may be reversed in some cases, which may be explained if it is assumed that, though the short local contraction is prevented by rapid application, there is a greater but more uniform extension over the whole length of the bar.[2] Unwin says that, "generally, the more slowly the load is applied, the shorter is the local contraction and the less the contraction of area."

[1] See UNWIN, "The Testing of Materials of Construction," Third edition, p. 113.

[2] See *Proc. Inst. C.E.*, vol. 89, p. 120; and Annales des Pouts et Chaussées, 1890, or UNWIN, pp. 112, 113.

Brittle and hard materials act differently. Cast iron is stated by Hodgkinson to be capable of carrying indefinitely a load closely approaching the ultimate for a single application. Timber will usually carry permanently only a much smaller load than that given as the ultimate.

Even below the elastic limit a very rapid (though not instantaneous) application of the load may produce the stress-strain line OA (Fig. 68), steeper than the usual line Op for slow loading, the point A soon moving to B on Op. Similarly, in unloading, the line may be BCO. A rapid application up to the point of fracture may raise and shorten the entire curve; and a very rapid application may produce a curve very different from $Opub$ of Fig. 59, and more nearly resembling AB, practically obliterating the yield point and raising it and the ultimate, because there would not be time for the development of the phenomena of a slow test as previously described.

The effect of a rapid application of the load in raising the ultimate is shown in brittle as well as in ductile materials, though it is probably more pronounced in the

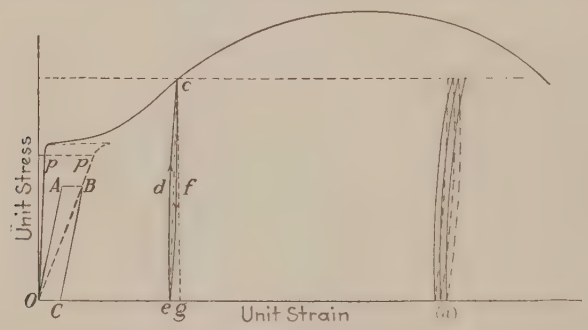


FIG. 68 and 68a.

latter. Professor D. A. Abrams has shown this effect for concrete,[1] having tested 270 concrete cylinders 6 inches in diameter and 12 inches long, at 28 days, with three different "mixes," namely, 1 volume of cement to 3, 5, or 9 volumes of mixed aggregate. "When shortened at the rate of 0.15 inches per minute, the cylinders showed a compressive strength 6, 14, and 20 per cent higher than when loaded at 0.006 inches per minute, for the 1:9, 1:5, and 1:3 mixes respectively." He also found that as much as 88 per cent of the ultimate could be applied rapidly and the remainder slowly, without affecting the ultimate. (This ratio would probably be quite different for a ductile material, for concrete has no definite yield point and little resilience.) He prefers a rate of loading of between 0.01 and 0.02 inches per minute, after applying from 50 to 75 per cent of the ultimate rapidly. The rate of application should mean the actual rate of deformation of the specimen, which is less than the speed of the machine head when moving idle and without load at the same speed of the machinery, due to the deformation of the levers, screws and other parts of the machine under increasing load.

The above facts make evident the reason why in standard testing the rate of application of the load should be specified.

[1] *Engineering and Cement World*, Jan. 15, 1918.

23. Hysteresis Loop. Above the elastic limit the loop *OABC* of Fig. 68 would be more pronounced with rapid application and unloading. The unloading curve would not follow back exactly parallel to *Op* in Fig. 59 but a little steeper and partly curved. This is called a hysteresis loop. In steel the loop is sometimes present, and sometimes not, depending upon the material and the rapidity of application of the load. For rapidly repeated stresses above the elastic limit, Bairstow's experiments indicated that the upper half *cf* of the descending line and the lower half *ed* of the ascending line were curved, while *fe* and *dc* were straight and parallel. If such a loop exists for repeated loadings and unloadings up to *C*, *the loop being closed*, it means that the repeated applications of the load produced no increase of the strain or of the permanent set; but any such loop in which the strain for a given load is slightly greater on the unloading curve than for the loading curve means that at each application of the load a certain amount of external work, represented by one-half the area of the loop, is done upon the piece, and disappears in heat or some other form of energy. For in loading along the curve *edc* by a gradually applied load the work done by that load is one-half the area *edcg*, while in unloading along the curve *cfe* the work restored by the elasticity of the material is one-half the area *efcg*; and the difference is consumed in some way at each application. If, however, the hysteresis loop were not closed, but if a new loop were formed at each application, with increased set (Fig. 68*a*), it is obvious that, if this continued, failure would ultimately occur (see Chap. XXII on Repeated Stress).

If a load *gc* (Fig. 68) above the yield point, be applied to a ductile material, removed, and *immediately* applied again, the limit of elasticity will be found to have been reduced very low, perhaps to zero, while the yield point has been raised to the previous load. This agrees with the fact that on repetition the line is the curved line *edc*, curved from the beginning. The material has by the first application been carried into the plastic stage, and it is now ready, if immediately tested again, to continue to deform at a faster rate than the stress increases. (This agrees with the amorphous cement theory of the constitution of steel; see Art. 32.) If, however, the piece is allowed to rest a short time, only a few minutes, or if it is gently heated by being immersed for a few minutes in boiling water, it will be completely elastic again and the stress-strain line will go straight from *e* to *c* or nearly so.

24. Raising of the Elastic Limit by Overstress.—As indicated at the close of the last article, if a ductile material like steel be stressed to *gc*, above the yield point, it will be found that the elastic limit, after a brief period of rest, or after heating, has been raised as high as the previous stress. Neglecting the hysteresis loop, therefore, if the load is removed at a point below the ultimate, the stress-strain line will travel down parallel to *Op* and back again, to pursue the curve to the ultimate almost as if there had been no interruption. But singularly enough, and especially if there is a longer period of rest after unloading, perhaps 24 hours, or over, a load *gc* will be found to have raised the original elastic limit and the yield point will be raised *even above c*, so that on a re-application of the load the line would be *cc'* (Fig. 69), and a new yield point would be found at *c'*, above *c*, at which the line would bend and return nearly to the original line. Sometimes, however, overstress not only raises the yield point higher than the

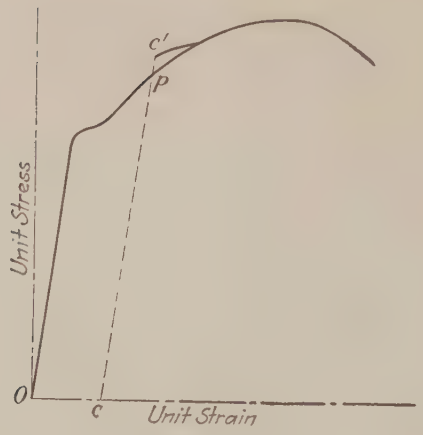
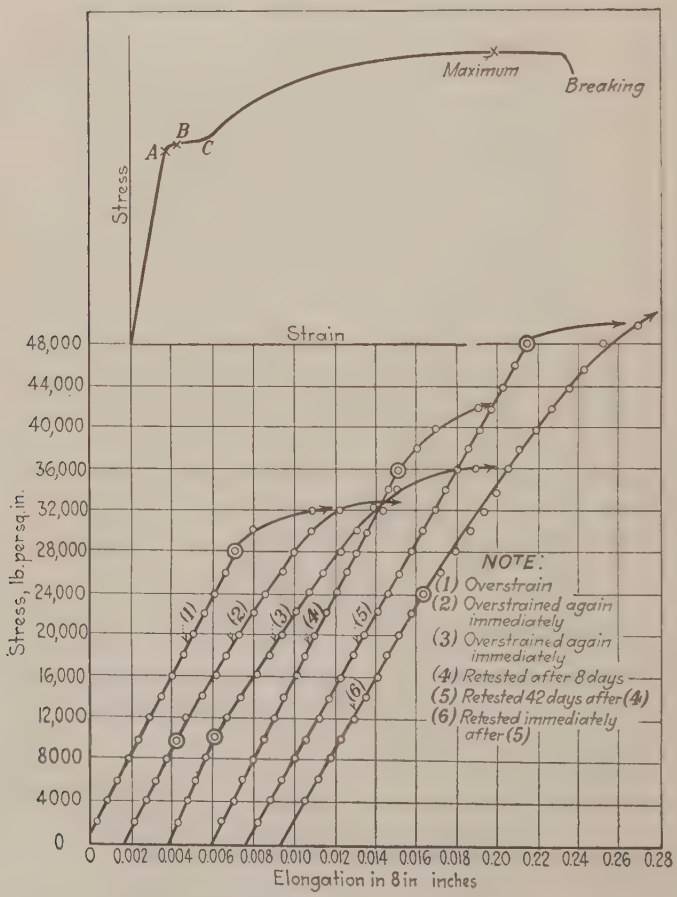


FIG. 69.



(From Hancock: "Strength of Steel from Structural Shapes," Am. Soc. Testing Materials, Vol. XI (1911), p. 484.)

FIG. 70.—Overstrain of steel from I-beam.

overstress, but also somewhat raises the ultimate also. Heating to 100° C. hastens the recovery. Figure 70 illustrates these facts.

25. "Hardening" of Steel by Overstress.—The fact that the yield point of steel may be raised, with time or heating, to a point equal to or higher than the applied stress, is often referred to as the hardening of steel by overstress. Clearly in this expression the term hardening is used in the fourth sense mentioned in Art. 3, as meaning an increased elastic limit (for this is raised also) and an accompanying decreased ductility. It does not necessarily mean greater resistance to indentation or to abrasion, though there may in some cases be a slight increase of these.

This hardening by overstress may be completely removed by annealing.

All commercial steel, except castings, has been subjected to mechanical work in rolling or forging which has involved stresses above the elastic limit of the material previous to those operations, and the elastic limit has thereby been artificially raised. It may be raised still higher, in some cases nearly to the ultimate, by further overstressing. *It may, however, be lowered by stressing in the opposite direction;* that is to say, the elastic limit in tension will be lowered by stressing in compression, and *vice versa.*

The material close to punched holes or sheared edges of steel is slightly hardened by the process, and rivet holes are frequently reamed, to remove this material.

Ewing's "Strength of Materials" contains the following statement.

"A remarkable experiment may be made by taking a bar of mild steel and stretching it in the first instance just up to the primitive yield point, then heating it for a few minutes up to 100° C. to produce elastic recovery, then stretching it again just up to its new yield point, then heating again to 100° C. and so on. Each step raises the elastic limit, and notwithstanding its naturally plastic quality the bar may in this way finally be caused to break with a fracture resembling that of hard steel, with comparatively little total extension or contraction of section at the fracture, and under a total load much greater than that which could be applied in an ordinary test."

26. "Original" and "Natural" Elastic Limits.—The elastic limit found for steel or wrought iron as received was termed by Bauschinger the "*original*" or *primitive* elastic limit. It is in a sense artificial or unstable, and by repeated alternations from tension to compression up to a limit even below the primitive elastic limit, this elastic limit may be lowered until equal limits in tension and compression are produced, these being called by Bauschinger the "*natural elastic limits.*" Such repetitions of stress from tension to an equal compression are called cyclical variations of stress. Cyclical variations, however, may be unequal in tension and compression, and by such variations both limits, in tension and in compression, may be raised or lowered.

27. Work and Resilience.—When a load is applied to a piece, and deforms it, work is done upon it. If the piece is perfectly elastic, and recovers its original form when the load is removed, it restores that work during the recovery. This power of restoration, or springiness, is called resilience, and is measured by the amount of work which the piece can so absorb and fully restore. If the load P, less than the elastic limit, is applied gradually, and causes an elongation e, the work done is the average load times the distance it moves through, or $\frac{Pe}{2}$, and since

the strain, e, is $\frac{Pl}{AE}$, and the volume of the piece is Al, the work per unit of volume is $\frac{P^2}{2A^2E} = \frac{p^2}{2E} = \frac{1}{2}px$, where p is the unit stress and x the corresponding elongation per unit length. As p and x are the quantities usually plotted in the stress-strain curve, the work done per unit volume up to the elastic limit is the area of the diagram up to the stress applied. The same is true up to any other point, for if the piece has a unit section and a unit length (unit volume), the work done up to any point will be $\int p \cdot dx$.

The resilience of a material is therefore properly the area of the stress-strain diagram up to the elastic limit, and as it refers to a unit volume it is called the *modulus of resilience*. Resilience really implies elasticity or power of restitution, but the term is sometimes used meaning the total work done up to fracture. The last should be termed (though illogically) "total resilience." To avoid ambiguity we may use the term *modulus of elastic resilience* and *modulus of total resilience*. The modulus of elastic resilience is thus the measure of the amount of work that a unit volume of the material can absorb, as by a shock or blow, up to the elastic limit; and the modulus of total resilience is a measure of the work that it can absorb up to fracture. It may thus be regarded as a measure of the shock-resisting power of the material.

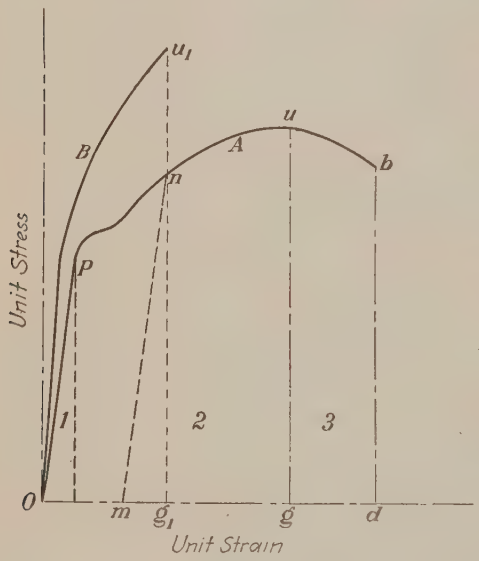


FIG. 71.

It is important to note, however, that the areas given by the usual static stress-strain diagram, may be very different from the resilience under a shock or blow, for the reason that the stress-strain diagram, if drawn for such a mode of application of the load, may be very different from the usual static diagram.

It has been shown in Art. 18 that the diagram may be very different for a rapidly applied load from what it is for a slowly applied load, as is usually the case in a static test. For a very rapidly applied or sudden load, the curve may be more like line B than like curve A (Fig. 71). This fact may serve to explain some of the discrepancies and discussions regarding impact testing (see Chap. XXI).

28. Significance and Importance of Ductility.—Structures and machines are not often exposed to purely quiescent, that is to say, static loads. The loads are moving, are accompanied by shocks and jars, and not infrequently fall upon the structure, in which case the kinetic energy must be resisted or absorbed. Resilience is, therefore, a very important quality of a material. It clearly depends upon both strength and ductility, not upon either alone. A material of high strength, if brittle and lacking in ductility, may be a very poor material to use.

Both qualities are important; their relative importance depending upon circumstances. The greater the strength and the elastic limit, the larger the unit stress that may be used, and hence the smaller the sections and the less the weight and cost of the structure or machine; but this may be an illusory gain if a sudden unexpected blow is likely to demolish it. For this reason engineers insist that the material used shall have a certain specified degree of ductility. But increased ductility generally goes with smaller strength and elastic limit, hence strength and ductility are contradictory to a certain degree.

In a short bridge, whose weight is small, the principal load is the moving load, and ductility is the more important quality. In a very long span, however, the moving load may be much less than the weight of the structure; here ductility is less important and a high ultimate and elastic limit more desirable. Hence for such structures the engineer is justified in using one of the alloy steels, or a higher carbon steel, even though some degree of ductility is sacrificed, for the dead weight may be greatly reduced by so doing.

Ductility is also of great value if the stress on a plane surface is not uniformly distributed over it, but varies from a maximum at one edge to a minimum at the center or at the other edge. If a test piece has the shape of Fig. 72, the stress at the smallest section, owing to the curve, will be greatest at the outside. If the stress at the outside should reach the yield point, and the material were ductile, yielding at the outside would occur, with no increase of stress, and the stress on the outside fibers would be distributed over fibers nearer the center. The stress at the outside, in such a material, could not go above the yield point without the yielding; and hence, as the load is increased, as soon as this stress reaches the yield point a redistribution of stress occurs which makes the load more uniformly distributed over the entire section; and, when fracture occurs, the ultimate will be much more nearly the average than with a non-ductile material.

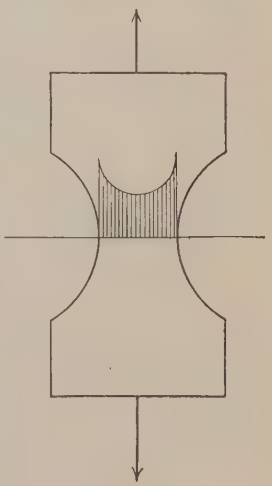


Fig. 72.

29. Modulus of Resilience.—This, as already shown, is the area of the stress-strain diagram (though not necessarily of the diagram as drawn for a static test). The total work to fracture, per unit volume, in Fig. 71, is the sum of the elastic resilience (area 1), the work done in a general plastic elongation of the bar (area 2), and the work done in necking down (area 3). Area 1 is the square of the elastic limit divided by $2E$. The total work or resilience is of importance, for it indicates the energy necessary for fracture. In an earthquake country the total resilience is most important; a structure built of a material in which it is high may survive where one in which it is low would be destroyed.

Assuming the static curve to apply, the modulus of total resilience is the area Obd; but often the area 3 is neglected, as it is uncertain. The stress-strain diagram is not often drawn, and the following rule is sometimes used, for the modulus of total resilience-R,

$$R = \tfrac{1}{2} \cdot ug \cdot Od.$$

This is not quite logical, but Og is seldom found in tests. Probably it is more nearly accurate to write

$$R = \tfrac{2}{3}ug \cdot Od.$$

The coefficient may be even greater than $\frac{2}{3}$ for a ductile material. For a curve like B, R may be quite closely $\frac{2}{3}u_1g_1 \cdot Og_1$.

Stressing above the yield point removes a part of the original toughness or resilience; thus in Fig. 71, a stress to the point n involves an elastic resilience mng_1, which remains, and a plastic resilience Onm, which is removed.

The preceding discussion will probably make clear the reason why so much attention has recently been devoted to the subject of impact testing, arising from the desire to ascertain the amount of work required to rupture a material by a sudden blow, and so to judge of its total resilience (see Chap. XXI).

30. Referring to the properties enumerated in Art. 3 as important for the engineer, the measure of these qualities may now be stated by reference to the stress-strain curve (Fig. 71).

Elasticity is greater the less the modulus of elasticity or the flatter the slope of the line Op with the horizontal; and the greater the elastic limit; in other words, it may be measured by the modulus of elastic resilience.

Plasticity increases with the area of the diagram beyond p, and particularly the area below ub.

Ductility increases with the distance Od; or perhaps more accurately with the distance Oe in Fig. 62.

Stiffness increases with E.

Hardness may mean a high elastic limit, but is a vague term (see Chap. XXI).

Resilience is measured by the area under Op, or under Ob, according as it is elastic or total (but see remarks in Art. 27).

Brittleness is greater the smaller Od.

Toughness means about the same as total resilience.

Strength, of any particular kind, means ug.

When these terms, or any terms, are used, their meanings should be clearly in mind. They are often used carelessly, and, therefore, considerable confusion has arisen in court trials on account of their being used with different meanings by different witnesses, or sometimes by the same witness. One may differ with the above definitions; the important thing is to know the sense in which one uses a term and to make this clear.

31. Sudden Application of Load.—If a load is suddenly applied to a piece, as to the tension bar shown in Fig. 73, it produces a greater strain and stress than if applied gradually. If applied gradually, starting with zero and increasing no faster than the bar can stretch in following the load, suppose that it produces the elongation $e = \dfrac{Pl}{EA}$. The work done will be $\frac{1}{2}P$ times this elongation, and will be represented by the area $OP'S$ in the stress-strain diagram (Fig. 74). The elastic work of stretching the bar equals the work done by the load. If, however, the load is applied suddenly, the entire load $P = OP$ at the beginning. When the elongation reaches e, the elastic work will be less than the work done by the load, the latter being Pe, or the area of the rectangle $OPP'S$. There is therefore an accumulated kinetic energy of the load, which must produce a further elongation to a point C, (Fig. 73) such that $AC = OS_1$. The load P will then have done the work

$OPP'''S_1$, and the elastic work of elongation will be $OP''S_1$, and these will be equal if the triangle $P'P''P'''$ equals the triangle OPP', or when OS_1 is twice $0S$ provided P'' is below the elastic limit. If OP is greater than half the elastic limit (Fig. 74a), the stretch will be to S_1, such that the area $P'P''P'''$ equals the triangle OPP' and if the elastic limit has been raised to S_1P'' the bar will ultimately come to rest with approximately the elongation OS_2. If the load has an accumulated energy represented by the area $OHKP$ when first applied, as it might have if it had fallen the distance OH, the area $P'P''P'''$ should equal the area $OHKP'$.

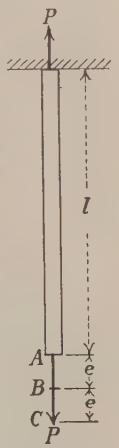


Fig. 73.

The subject may also be treated by using d'Alembert's principle, which is, that a set of particles in motion will satisfy the conditions of static equilibrium if, besides the given actual forces acting, there is imagined applied to each particle a force equal and opposite to its mass multiplied by its actual acceleration. When the load P is first applied to the bar in Fig. 73, there is no elastic resistance, since stress is proportional to strain, and there is no strain. Hence P moves downward with the acceleration of gravity, and it is necessary to imagine an *upward* force applied equal to $\frac{P}{g}\,g$, or P, and there is no stress in the bar. As stretch is produced, there is a corresponding stress, but it is less than P, and consequently P moves with an accelerated motion till the elongation is e. If at any point the acceleration downward is a, the stress in the bar is

$$P - \frac{P}{g} \cdot a.$$

The load P, therefore, is acted upon by its own weight, and by a force in the opposite direction less than its weight, which last force increases as a decreases. The acceleration a, of P, therefore, will not be g, since the resultant downward force on P is less than its weight. When the elongation is e, the stress in the bar

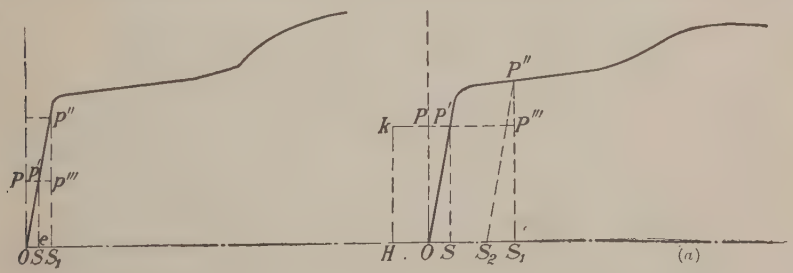


Fig. 74 and 74a.

equals P, and the acceleration is zero, and beyond e there is a retardation, or an upward acceleration, so that the stress in the bar is

$$S = P + \frac{P}{g}\,a.$$

The upward acceleration increases from B to C, and back to B, and is a maximum at C, where it is g, and $S = 2P$. Here the resultant force on P is an *upward* force equal to its weight. At B, when P is moving upward, $S = P$, the acceleration is

zero, and above B it is an acceleration downward again, up to A, or near A, and back to B.

The weight of the bar is neglected in the above.

The load vibrates above and below B, and finally comes to rest at that point, the energy of vibration being dissipated by molecular friction, producing heat.

The effect of suddenness of application, or of a load falling upon a bar, whether in tension or flexure, may be found from these principles, and formulæ obtained (see chapter on Impact). But the results are not accurate, because the stress-strain diagram for a static, gradually-applied, load, is not the same as for a sudden application.

It is often stated that when a weight falls upon a piece, the resulting stress $P''S_1$ (Fig. 74) will be such that the work of deformation up to that point (area of $OP''S_1$) will equal the kinetic energy of the weight at the moment of striking. This overlooks the fact that during deformation the weight does additional work $OPP'''S_1$, whatever that may be for the actual stress-strain diagram during impact.

If, when the load P is suddenly applied to the bar in Fig. 73, there is a quiescent dead load W already on it, that dead load will move with the weight P suddenly applied, and both will at each point have the same acceleration. The weight P must accelerate both its own mass and that of W, so that the acceleration will be less than if W were absent, but the acceleration times the accelerated mass will be the same.

32. Slip Bands. Lüder's Lines.—Bearing in mind that tension cannot occur without shear, or shear without tension (Art. 11), it is obvious that any permanent deformation must mean (or be accompanied by) a permanent shearing deformation. A piece exposed to tension tends to shear on planes making angles of 45° or more with the direction of the tension, and a piece in compression tends to shear on planes making angles of less than 45° with the direction of the compression. When the yield point in tension is reached, therefore, this permanent shearing deformation occurs, and the mill scale on steel is detached or disturbed along these lines, which are plainly visible on the surface of the piece. These are called *Lüders'* lines. They were also observed and described by Hartmann.[1] Even if the mill scale is removed, and the surface polished, these lines will be visible in a suitable light (see Fig. 75*a*).

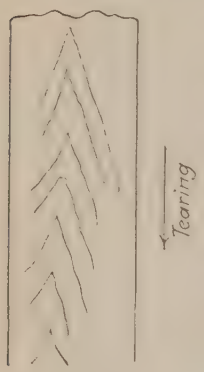


Fig. 75.

Slipping takes place in the individual crystals which make up steel, and forms what are observed as *slip bands* in the crystal surfaces when examined by the methods of metallography. These methods consist in cutting out a small portion of the metal, polishing one surface to a plane, etching it with a dilute acid to bring out the structure, and examining it through a microscope by reflected light. The crystals will be etched to somewhat different depths, depending upon their character, and the boundary lines may be distinguished.

When the crystals form, in cooling from the melted state, they are not all oriented alike, and do not fit together closely. According to the amorphous

[1] Hartmann, L., "Distribution des Deformations dans les Metaux soumis a des Efforts." Berger-Levrault, Paris, 1896.

cement theory of Beilby, which, while not yet generally accepted, seems to offer the best working hypothesis, the spaces between crystals, or practically surrounding them, are filled with an amorphous or non-crystalline cement which hardens quickly, and becomes *stronger than the crystals themselves*. This is shown by the fact that when a piece of steel breaks, the fracture goes *through* the crystals and not around them, unless at a high temperature, when the cement may become softened. The crystal boundaries are therefore strengthening surfaces.

When the metal is deformed, a slip takes place on different planes in the different crystals, depending upon their orientation. If the deformation is elastic, the molecules can adjust themselves when the stress is removed; but if the yield point is exceeded, the slip is permanent, and some of the molecules are ground to the condition of amorphous material, which sets and prevents further slipping on those planes. As the stress is increased further slip occurs on other planes, on which slipping is originally more difficult to produce. The amorphous cement theory thus offers a satisfactory explanation of the phenomena of strain, and of the so-called "hardening due to overstress."[1]

(By Permission of U. S. Bureau of Standards.)

FIG. 75*a*.—Method of R. S. Johnston for showing Lüder's lines on angle column section 10 × 3 feet of material for Delaware River Bridge.

33. Neumann Bands.—The engineer is liable to be called upon to investigate accidents and failures in structures or machines, and to determine the causes. It is often necessary to determine whether a piece failed suddenly, as from a blow or explosion, or gradually, as from a static load. For such problems an ultimate knowledge of the structure, properties and appearance of materials, and the effects produced by various kinds of forces, may be useful or necessary.

It is worth while to state here, therefore, that there are certain parallel bands, occurring in groups across the crystals of a metal like steel, and different from slip bands, which, in the opinion of many if not most metallurgists, are and can be produced only by a sudden blow or force. These are known as *Neumann Bands*. If these are present, it is considered good evidence that the fracture was not due to yielding from structural weakness, but from some other sudden cause.

Wherever it is important, therefore, to ascertain whether a failure arose from structural weakness, a competent metallographer should be employed to look for Neumann bands, and to consider all other pertinent facts. Here, as in all cases, however, judgment is necessary. A bridge may fail from structural weakness, and, in falling, some part may be struck and broken by a sudden blow and may thus show Neumann bands. But if it can be determined beyond

[1] In order fully to understand this subject, the student should read Rosenhain's "Introduction to Physical Metallurgy," especially Chap. XI on the Structure of Metals under Strain.

question where the initial fracture was, and if this fracture shows none, the case is different. In cases of engineering fractures it is often most difficult to distinguish cause from effect. A train may be wrecked and a broken axle found; it may be the cause or the effect of the wreck.

34. Herringbone Lines.—In examining a fracture of a steel plate, for instance, which has been torn across, it may sometimes be of importance to determine the direction in which it tore. For this purpose the appearance of the fractured surface may give some indications. If the material is ductile, and tore downward in Fig. 75, this surface will show markings or ridges, triangular in shape, pointing toward the origin of the tear, or opposite to the direction of tearing. These have been called herringbone lines, and are often distinguishable.

35. The following table shows usual requirements for steel for various purposes, as adopted by the American Society for Testing Materials.

SPECIFICATIONS FOR ROLLED STEEL ADOPTED BY AMERICAN SOCIETY FOR TESTING MATERIALS, STANDARDS 1921

Material		Tensile strength, pounds per square inch	Yield point, pounds per square inch	Elongation in 8 inches, per cent	Elongation in 2 inches, per cent	Reduction of area, per cent
Steel for bridges	Structural	55,000 to 65,000	0.5 tensile strength	$\frac{1,500,000}{\text{tensile strength}}$	22	
	Rivet	46,000 to 56,000	0.5 tensile strength	$\frac{1,500,000}{\text{tensile strength}}$		
Structural nickel steel	Rivet steel	70,000 to 80,000	45,000	$\frac{1,500,000}{\text{tensile strength}}$	..	40
	Plates, shapes, and bars	85,000 to 100,000	50,000	$\frac{1,500,000}{\text{tensile strength}}$	..	25
	Eye-bars, flats, and rollers, unannealed	95,000 to 110,000	55,000	$\frac{1,500,000}{\text{tensile strength}}$	16	25
	Eye-bars, flats, and pins, annealed	90,000 to 105,000	52,000	20	20	35
Billet steel for concrete reinforcement	Plain bars, structural grade	55,000 to 70,000	33,000	$\frac{1,400,000}{\text{tensile strength}}$		
	Plain bars, hard grade	80,000 minimum	50,000	$\frac{1,200,000}{\text{tensile strength}}$		
	Deformed bars, hard grade	80,000 minimum	50,000	$\frac{1,000,000}{\text{tensile strength}}$		

CHAPTER V

RELATIONS BETWEEN STRESSES AND BETWEEN STRAINS AT A POINT, ON DIFFERENT PLANES PASSING THROUGH THAT POINT

1. From the previous chapter it is clear that one kind of stress alone cannot, in general, exist throughout a body. At any given point there are generally stresses of different kinds and intensities and directions on different planes passing through that point. A study of the relations between these stresses and between the corresponding strains is of fundamental importance for the engineer, and forms the subject of this chapter.

2. It has been seen that there is *one universal method of studying stress*, namely, to imagine the body cut completely in two by some surface or combination of surfaces, then considering that the part of the body on one side (either side may be chosen) of this surface or section is removed, replacing on the cut section the stresses exerted by the part removed upon the part remaining, and studying the equilibrium of the part remaining, under the action of the known outer forces which act upon it and the unknown stresses on the cut section. If the section considered passes entirely through the body from one side to the other, so that, if the section were actually made, the two parts of the body could be separated, then the imposition of the conditions of equilibrium upon one of the parts will give *relations between outer and inner forces*, that is, between loads and stresses. This is the method, and the only method, of obtaining stresses from known loads.

If, however, the section considered cuts the body into two parts, one of which is entirely within the other, the conditions of equilibrium will give *relations between stresses* (not between loads and stresses), since these are the only forces, aside from its weight, acting on the part of the body entirely enclosed within the whole. If it is desired to find the relations between stresses at a point on different planes all passing through that same point, the part of the body conceived to be separated out must be an infinitesimal particle at the point in question. In this case the weight of the particle drops out of consideration, as shown in Art. 9 of the previous chapter; furthermore, since all dimensions of the particle are infinitesimal, the stress on any bounding plane surface will be uniformly distributed over that surface and its resultant will act at the center of gravity of the surface.

In Art. 9 of the previous chapter the relations between shears on two planes at right angles have been shown; in Art. 10 the usual case of stress parallel to a plane has been explained, and in Art. 11 it has been shown that for this case a shearing stress alone involves or causes tension and compression of equal intensity with the shear, on planes at 45° with the shear.

3. Of the nine intensities of stress acting upon the particle of Fig. 52, therefore, three are numerically equal to three others, hence there are only six different or independent stress intensities, namely,

$$n_x,\ n_y,\ n_z, s_{xy},\ s_{yz},\ s_{zx}.$$

If these six are given, the condition of stress at the given point O is completely determined. The total forces on any infinitesimal part of the rectangular planes can be found, and also the total stress, and its intensity and direction, on any other plane such as abc in Fig. 76, as will be clear from the following considerations:

Let $Oabc$ in Fig. 76 be an infinitesimal triangular pyramid, three of whose sides lie in the rectangular planes. The six independent intensities of stress being given, the resultant normal stress and the two components of shearing stress on the area Oac may be found, and they all act at e, the center of gravity of Oac. Similarly, the total stresses on the sides Obc and Oab may be found, acting at the centers of gravity of those areas. These nine forces, involving six independent known stress intensities, are in equilibrium with the resultant stress on abc, which acts at h, the center of gravity of abc, and which may be resolved into three components parallel to the axes X, Y, and Z. If we now imagine three rectangular axes meeting at h, parallel to OX, OY, and OZ, and refer the six equations of equilibrium to these axes, the three equations of moments about these axes merely give us the result obtained in Art. 9 of the previous chapter, namely, $s_{xy} = s_{yx}$, $s_{yz} = s_{zy}$, $s_{zx} = s_{xz}$ (numerically, with opposite tendencies to rotate the particle), so that the remaining three equations ($\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$) suffice to determine completely the three components of the stress on abc in terms of the six known stress intensities on the other sides of the pyramid.

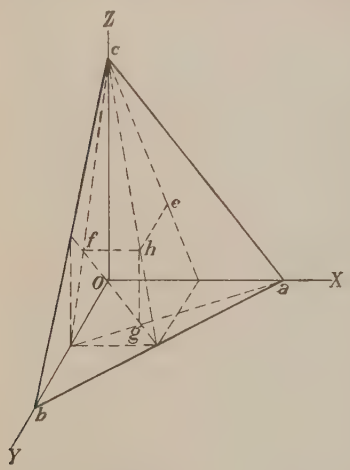


Fig. 76.

The state of stress at any given point is thus completely fixed if the six independent stress intensities on any three rectangular planes through that point are given.

For the special case of stress parallel to plane, it follows that if the normal and tangential stress intensities at a point, on any two planes at right angles through that point are given, the stress at that point on any other plane through that point may be found.

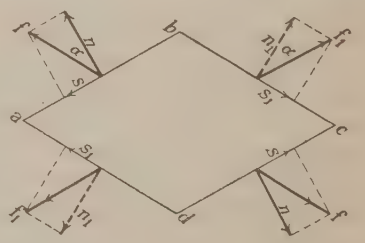


Fig. 77.

4. Conjugate Stresses (Fig. 77).—Still considering the case of stress acting parallel to the plane of the paper, let the parallelogram $abcd$ represent an infinitesimal particle with sides perpendicular to the paper, and suppose all sides to have a unit area. Let the stress f on the sides ab and cd act parallel to the sides bc and ad; then these stresses f balance each other, and the stresses f_1 on the sides bc and ad must also balance each other, that is, they must be equal in magnitude, in opposite directions, and act along the same line. Since each acts at the center of its area, these forces f_1 must therefore act parallel to ab and cd. The weight of the particle drops out, as explained.

The forces f and f_1 constitute what are called *conjugate stresses*. The facts may be stated thus:

At any given point in a body, if the stress on any plane A is parallel to a plane B, then the stress on the plane B is parallel to the plane A, and the stresses are conjugate. Two conjugate stresses may be both tension, both compression, or one tension and one compression.

The angle that the direction of the stress on a plane makes with the normal to that plane is called the *obliquity* of the stress. Clearly, each of two conjugate stresses has the same obliquity. Call the obliquity α (Fig. 77), assume the sides ab and bc unity, and resolve the forces f and f_1 into normal and tangential components, n, s, n_1, and s_1. Then since f and f_1 have the same obliquity,

$$\tan \alpha = \frac{s}{n} = \frac{s_1}{n_1}$$

$$\therefore \frac{n}{n_1} = \frac{s}{s_1}. \qquad (1)$$

Hence, *the intensities of the normal components of a pair of conjugate stresses have the same numerical ratio to each other as the intensities of the shearing forces.*

5. Stress on a Plane at the Free Surface of a Body.—Let *Oab* (Fig. 78) represent the surface of a body, upon which surface at the point *O* no outer force acts. This is termed a *free surface*. Let the planes *Oac* and *Obc* be both perpendicular to the surface and to each other, so that the lines *Oa*, *Ob*, and *Oc* are perpendicular to each other. Let *abc* be *any* other plane cutting the body.

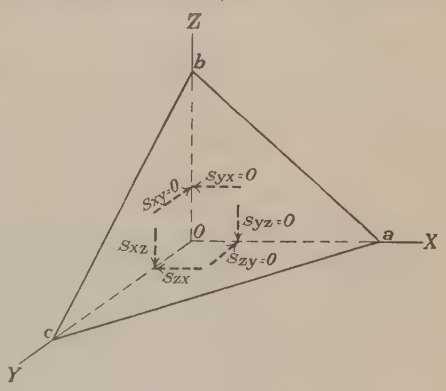


Fig. 78.

Consider the equilibrium of the infinitesimal particle *Oabc*. Since there is no force on *Oab*, the three forces on the three planes *Oac*, *Obc*, and *abc* must be in equilibrium, and hence must meet at a point. Since there is no shear on *Oab* in the direction of *Ob* there must be no shear on *Oac* in the direction of *Oc*; hence the stress on *Oac* must be in a plane parallel to the surface. Similarly the stress on *Ocb* must be in a plane parallel to the surface. These two planes must coincide, since the forces all meet at a point; and the stress on the plane *abc* must lie in the same plane and must be parallel to the surface. Hence,

If no external force acts upon the surface of a body at a given point, the stress at that point, on any plane passing through that point, is parallel to the surface. This is true whether the surface be plane or curved; if curved, the stress acts parallel to the tangent plane.

If the force acting on the surface at *O*, instead of being zero, is a *normal force*, then as in the previous paragraph the stresses on the planes *Oac* and *Obc* must be in a plane parallel to the surface. Since the particle is infinitesimal and the stress therefore uniformly distributed, this plane will pass through the c.g. of the triangles *Oac* and *Obc*. The stress on *abc* will in this case have a componnet

perpendicular to the surface which balances the normal force on the surface area *Oab*. Hence,

If at any point on the surface of a body there is a normal force acting on the surface, then on any plane normal to the surface the stress at that point is parallel to the surface, but on any plane not normal to the surface the stress is not parallel to the surface.

6. Application of the Previous Principles.—If Fig. 79 represents the vertical cross-section of a wall, and we imagine a horizontal plane *ab* cutting it, the weight *W* of the portion of the wall above *ab* is resisted by internal stress distributed over *ab* whose resultant is equal and opposite to *W*. As represented by the arrows, the direction of this stress varies, being parallel to the surface at *a* and *b*, and vertical at the center of *ab* if the section is symmetrical about a vertical line. The law of variation is unknown. The stress is frequently assumed to be vertical over the entire length *ab*, which is of course erroneous.

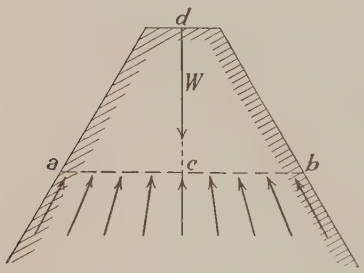


Fig. 79.

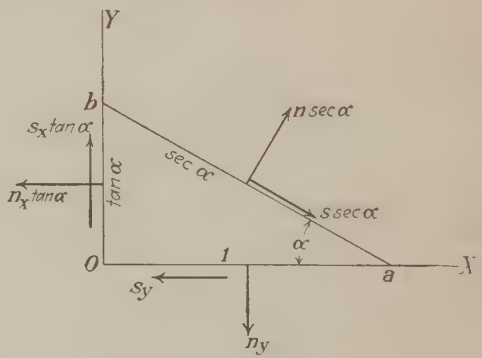


Fig. 80.

7. General Discussion of the Relations between Stresses Acting on Different Planes at a Given Point, for the Case of Stress Parallel to a Plane (Fig. 80).—Let *OX* and *OY* be two rectangular axes, and let the stress on all surfaces be parallel to the plane of the paper. Consider an infinitesimal particle of which *Oab* is a section, the ends being parallel to the plane of the paper, *ab* making an angle α with *OX* in the sense shown, *i.e.*, clockwise from the position *OX*. If a plane *XX* (Fig. 81), originally parallel to *OX*, be revolved about *O* clockwise until it reaches the position *XX* again, it will pass through all possible positions of a plane through *O* perpendicular to the paper, and α will vary from 0 to 180°. For any given position of this plane, *with* $\alpha < 90°$, the stresses at *O* may be studied by considering the infinitesimal triangular prism *Oab* in Fig. 80.

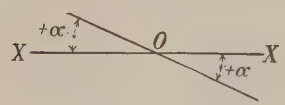


Fig. 81.

Let $Oa = 1$; then $Ob = \tan\alpha$, $ba = \sec\alpha$. Let n_y and s_y be the normal and tangential intensities of stress on the *Y* plane (*OX*), n_x and s_x those on the *X* plane (*OY*), and *n* and *s* those on the plane *ab*. Then the total forces (stresses) acting on the particle are:

On the *Y* plane, n_y and s_y,

On the *X* plane, $n_x \tan\alpha$ and $s_x \tan\alpha$,

On the *ab* plane, $n \sec\alpha$ and $s \sec\alpha$.

The weight of the particle is infinitesimal.

Let normal forces be called positive when they are tension, and shearing forces positive when they tend to cause clockwise rotation of the particle. Indicate all forces as if they were positive; then if they come out negative, they act in directions contrary to those assumed.

Since the particle is in equilibrium,

$$\Sigma X = 0; \Sigma Y = 0; \Sigma M = 0.$$

From the last condition, taking moments about the center of ab,

$$\frac{s_y \tan \alpha}{2} + \frac{s_x \tan \alpha}{2} = 0$$

$$\therefore s_y = - s_x \tag{2}$$

as already found, the signs showing that if s_x acts up (or away from O), s_y must act to the right (or also away from O).

The other conditions give us

$$n_x \tan \alpha + s_y - s \sec \alpha \cdot \cos \alpha - n \sec \alpha \cdot \sin \alpha = 0$$
$$n_y - s_x \tan \alpha + s \sec \alpha \cdot \sin \alpha - n \sec \alpha \cdot \cos \alpha = 0$$

from which, remembering that $s_x = -s_y$,

$$n_x \tan \alpha + s_y - s - n \tan \alpha = 0. \tag{3}$$
$$n_y + s_y \tan \alpha + s \tan \alpha - n = 0. \tag{4}$$

From these may be deduced

$$n = n_x \sin^2 \alpha + n_y \cos^2 \alpha + s_y \sin 2\alpha. \tag{5}$$

$$s = \frac{n_x - n_y}{2} \sin 2\alpha + s_y \cos 2\alpha. \tag{6}$$

Equation (5) may be written thus:

$$n = n_y - n_y \sin^2 \alpha + n_x - n_x \cos^2 \alpha + s_y \sin 2\alpha. \tag{5a}$$

Adding (5) and (5a)

$$n = \frac{n_x + n_y}{2} + \frac{n_y - n_x}{2} \cos 2\alpha + s_y \sin 2\alpha. \tag{7}$$

Equations (5) and (6), or (7) and (6) give values of n and s for any plane ab, in terms of α, n_x, n_y, and s_y.

Equations (7) and (6) show that

$$\frac{dn}{d\alpha} = 2s \tag{8}$$

which shows that as α increases, if s is positive, n increases.

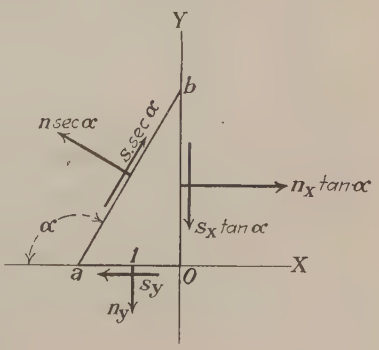


Fig. 82.

As soon as α exceeds 90°, the particle Oab will be on the other side of OY, as represented in Fig. 82, and with tension as positive and a positive shear as one which tends to rotate the particle in a clockwise direction, as before, positive intensities will act in directions shown; that is, for instance, s_y will act in the opposite direction from n_x, instead of in the same direction as in Fig. 80. But, since $\alpha > 90°$, $\cos \alpha$, $\tan \alpha$, and $\sec \alpha$ will be negative, and it will be seen that Eqs. (2), (3), and (4), unchanged, will correctly express the equilibrium of

the particle; hence all the equations just found will be correct if the signs of the angular functions are taken with the proper signs.

There are in this problem six unknown quantities, *viz.*, n_x, n_y, s_x, s_y, n and s. There are three statical equations, hence any three of the unknown quantities may be assumed. In what has preceded, n_x, n_y, and s_x were assumed.

8. Principal Stresses.—The plane on which n is a max. is found by differentiating Eqs. (5) or (7), and placing the first derivative equal to zero, thus from Eq. (7),

$$O = (n_x - n_y) \sin 2\alpha + 2s_y \cos 2\alpha. \tag{9}$$

For this plane, comparing with Eq. (6), the shear s is zero; and the value of α from Eq. (9), which may be called α_0, is found from the equation

$$\tan 2\alpha_0 = \frac{2s_y}{n_y - n_x}. \tag{10}$$

This equation is satisfied by two values of α_0, differing by 90°, on one of which n will be a maximum, and on the other a minimum. On either, the shear is zero. The stresses on these planes are called the *principal stresses* at the given point O. The plane of maximum principal stress may be determined algebraically by finding $\frac{d^2n}{dx^2}$, which will be negative when n is a max. and positive when n is a minimum;

$$\frac{d^2n}{d\alpha^2} = -2(n_y - n_x) \cos 2\alpha - 4s_y \sin 2\alpha.$$

If $n_y = 0$, n_x and s_x positive, and therefore s_y negative, $\tan 2\alpha_0$ is positive, hence α_0 lies between 0 and 45°, or between 90 and 135°; hence either both $\sin 2\alpha$ and $\cos 2\alpha$ are positive, or both are negative; hence

$$\frac{d^2n}{d\alpha^2} = 2n_x \cos 2\alpha + 4s_x \sin 2\alpha;$$

and when α_0 lies between 0 and 45°, the plane is that on which the minimum principal stress acts, and when α_0 lies between 90 and 135°, the plane is that on which the maximum principal stress acts. In other words, the maximum principal stress itself makes an angle less than 45° above OX. By Eq. (15) this maximum principal stress is of the same kind as n_x, and it is of course greater than n_x.

These results may be stated thus: *For the case of stress parallel to a plane, at any point in a body there are two planes at right angles to each other, on one of which the intensity of normal stress is greater, and on the other less, than upon any other plane passing through the point, and on each of these planes the shearing stress is zero. These maximum and minimum stresses are called the principal stresses for the point in question.*

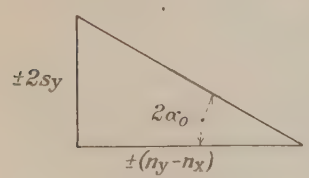


FIG. 83.

Calling the maximum and minimum principal stresses n_1 and n_2 respectively, their values may be found from Eqs. (6), (7), and (10), if the normal and tangential intensities are given on any two planes OX and OY at right angles with each other. From Eq. (10), drawing a right triangle (Fig. 83) it follows that

$$\sin 2\alpha_0 = \frac{\pm 2s_y}{\sqrt{4s_y^2 + (n_y - n_x)^2}} \tag{11}$$

$$\cos 2\alpha_0 = \frac{\pm (n_y - n_x)}{\sqrt{4s_y^2 + (n_y - n_x)^2}}. \tag{12}$$

Substituting these values in Eq. (7), and remembering that there are two values for $2\alpha_0$ differing by 180°, so that in Eqs. (11) and (12) both + signs or both − signs may be used.

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{n_x + n_y}{2} \pm \tfrac{1}{2}\sqrt{(n_y - n_x)^2 + 4s_y^2} \tag{13}$$

n_1 (using + sign) is the maximum principal stress.

n_2 (using − sign) is the minimum principal stress.

The value of s from Eq. (6) reduces to zero for either value of $2\alpha_0$.

From Eq. (13) there results

$$n_1 + n_2 = n_x + n_y, \tag{14}$$

that is,

On any two planes at right angles, the sum of the normal intensities is the same. The sum must of course be taken algebraically.

If the planes of principal stress be taken as the axes in Fig. 84, $n_x = n_2$ and $n_y = n_1, s_x = -s_y = 0$, and Eqs. (5), (6), and (7) become

$$\left.\begin{aligned} n &= n_2 \sin^2 \alpha + n_1 \cos^2 \alpha \\ &= \frac{n_1 + n_2}{2} + \frac{n_1 - n_2}{2} \cos 2\alpha \end{aligned}\right\} \tag{5b}$$

$$s = -\frac{n_1 - n_2}{2} \sin 2\alpha. \tag{6b}$$

If the principal stress intensities are equal, $n_1 = n_2$, and

$$n = n_1 = n_2;\ s = 0$$

or the stress on every plane passing through the given point is normal, and of the same intensity.

The shear will be a maximum when

$$\frac{ds}{d\alpha} = 0 = (n_1 - n_2) \cos 2\alpha$$

or

$$\cos 2\alpha = 0;\ \alpha = 45° \text{ or } 135°.$$

Hence *the planes on which the shearing intensity is numerically a maximum are at* 45° *with the planes of principal stress.* The intensity is of course numerically the same on each of these planes, but opposite in sign. The shearing stress intensity is numerically a minimum (= 0) on the planes of principal stress.

If the principal stresses are equal and opposite, or $n_1 = -n_2$,

$$n = n_1 \cos 2\alpha;\ s = -n_1 \sin 2\alpha$$

and in this case if $\alpha = 45°$, $n = 0$, and $s = -n_1$, which means that on two planes at 45° with the principal stresses there is pure shear, of intensity equal to the principal stresses. This agrees with what was found in Art. 11, Chap. IV.

The resultant of n and s as given by Eqs. (5*b*) and (6*b*) will be the intensity of resultant stress on ab. By replacing, in these equations, α by $90° + \alpha$, the resultant stress intensity on a plane at right angles to ab may be found. Calling these two resultant intensities on any two planes at right angles p_1 and p_2, and performing the simple operations, it will be found that

$$p_1^2 + p_2^2 = n_1^2 + n_2^2 \tag{14a}$$

which tells us that *on any two planes at right angles the sum of the squares of the actual stress intensities is constant and equals the sum of the squares of the principal stresses.*

If n_1 and n_2 are both positive (tension) Eq. (6*b*) shows that s will be negative when $\alpha < 90°$.

9. Ellipse of Stress.—Take as OX and OY (Fig. 84) the planes of principal stress, and let n_1 = max. principal stress act on OX, and n_2 on OY. Since for these planes $s = 0$, the normal and tangential intensities on any other plane making an angle α with OX are, from Eqs. (5), (6), and (7)

$$(n_y = n_1; n_x = n_2; s_y = -s_x = 0);$$

(from Eq. (5)) $$n = n_1 \cos^2 \alpha + n_2 \sin^2 \alpha. \tag{15}$$

(from Eq. (7)) $$n = \frac{n_1 + n_2}{2} + \frac{n_1 - n_2}{2} \cos 2\alpha. \tag{16}$$

(from Eq. (6)) $$s = -\frac{n_1 - n_2}{2} \sin 2\alpha. \tag{17}$$

Equation (17) shows that s is negative when α is less than 90°, since $n_1 > n_2$; also that on two planes making angles of 45° on each side of OX ($\alpha = 45°$ or 135°) the shearing intensity s is a *numerical maximum*,

$$\text{for } \alpha = 45°; s = -\frac{n_1 - n_2}{2} \tag{18}$$

$$\text{for } \alpha = 135°; s = \frac{n_1 - n_2}{2} \tag{19}$$

that is, the maximum intensity of shear is one-half the algebraic difference of the two principal stresses. If the principal stresses are equal, max. $s = 0$; if the principal stresses are equal and opposite, the max. shear is equal to either, numerically.

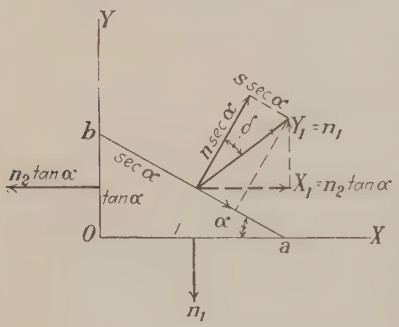


FIG. 84.

These results agree with those of Arts. 10 and 11 of Chap. IV. Let the principal stresses be equal but of opposite kinds ($n_1 = -n_2$); then on planes making angles of 45° with these $s = \pm n_1$, and on these planes the shears must tend to cause rotation in opposite directions, as shown in Art. 10.

Instead of resolving the resultant force on ab (Fig. 80) into normal and tangential components, it may be resolved into components parallel to OX and OY. Let OX and OY be the planes of principal stress (Fig. 84). Then the total horizontal force on ab is $n_2 \tan \alpha$, and the total vertical force is n_1. Dividing these by the area, $\sec \alpha$, the intensity n_1 has horizontal and vertical components, n_x' and n_y' given by

$$n_x' = \frac{n_2 \tan \alpha}{\sec \alpha} = n_2 \sin \alpha. \tag{20}$$

$$n_y' = \frac{n_1}{\sec \alpha} = n_1 \cos \alpha. \tag{21}$$

Now in Fig. 85 let $Oc = n_2$ = intensity of the minimum principal stress, which acts on the plane OY, and $Od = n_1$ = intensity of the maximum principal stress, which acts on the plane OX. Then if the intensity of stress on any plane OD be laid off

as OP in magnitude and direction, the coordinates of P will be n_x' and n_y' just found, and since

$$\frac{{n_x'}^2}{{n_2}^2} + \frac{{n_y'}^2}{{n_1}^2} = 1$$

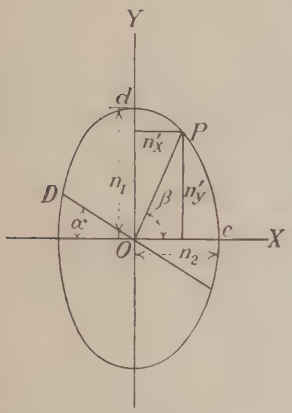


FIG. 85.

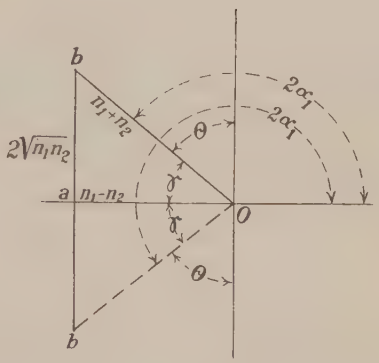


FIG. 86.

it is obvious that the locus of P is an ellipse of which n_1 and n_2 are the semi-axes. This is called the *ellipse of stress*.

The angle that OP makes with OX (in the direction shown) is such that

$$\tan \beta = \frac{n_y'}{n_x'} = \frac{n_1}{n_2} \cot \alpha$$

or

$$\tan \beta \cdot \tan \alpha = \frac{n_1}{n_2}. \qquad (22)$$

Since n_1 and n_2 are independent of α or β, the interchangeability of α and β in this equation proves that OD and OP have reciprocal relations, that is, not only is OP the intensity of stress on the plane OD, in magnitude and direction, but OD is the intensity of stress on the plane OP, in magnitude and direction. Here is once more the principle of *conjugate stresses*.

Exercise.—Prove that OP and OD are not conjugatediameters of the ellipse.

The ellipse of stress shows that on any two planes which are symmetrical with reference to the axes of principal stress, the intensity of stress and its obliquity are the same, and the directions of the stresses are also symmetrical with reference to the principal stresses. Since the obliquity of the stress on OD is the same as that on OP, it follows that there are four planes through O in which the obliquity of the stress is the same, *viz*, OD, OP, and planes symmetrical with each of these.

10. Obliquity of the Stress.—The obliquity, ∂, of the stress on the plane ab (Fig. 84) is such that

$$\tan \partial = \frac{s}{n}. \qquad (23)$$

The obliquity, ∂, is numerically a minimum, namely zero, for the two planes of principal stress, on which there is no shear, and it is a maximum on the planes

for which $\frac{s}{n}$ is greatest. Let α_1 be the angle with OX made by a plane on which the obliquity is greatest. The value of α_1 will be found by inserting in Eq. (23) the values of s and n from (Eqs. (16) and (17), differentiating with respect to α, and placing the first derivative equal to zero. The result is

$$\cos 2\alpha_1 = -\frac{n_1 - n_2}{n_1 + n_2}. \tag{24}$$

This shows (since $n_1 > n_2$) that $\cos 2\alpha_1$ is always negative, and that there are two values for $2\alpha_1$, one $180° - \gamma$ and the other $180° + \gamma$ (Fig. 86). Constructing the right triangle Oab, with $Oa = n_1 - n_2$, and $Ob = n_1 + n_2$, we find

$$ab = \sqrt{(n_1 + n_2)^2 - (n_1 - n_2)^2} = 2\sqrt{n_1 n_2}$$

and thus we find

$$\sin 2\alpha_1 = \pm \frac{2\sqrt{n_1 n_2}}{n_1 + n_2}$$

$$\cos 2\alpha_1 = -\frac{n_1 - n_2}{n_1 + n_2}$$

and from (23)

$$\tan \varphi = \tan \partial_{\max.} = \frac{s}{n} = -\frac{(n_1 - n_2)\sin 2\alpha_1}{(n_1 + n_2) + (n_1 - n_2)\cos 2\alpha_1}$$

$$= \mp \frac{n_1 - n_2}{2\sqrt{n_1 n_2}}. \tag{25}$$

Referring to Fig. 86, the two values of α_1 are

$$\alpha_1 = \left\{ \begin{array}{c} 90° - \frac{\gamma}{2} \left(\text{or } \; 45° + \frac{\theta}{2} \right) \\ \text{and } 90 + \frac{\gamma}{2} \left(\text{or } 135° - \frac{\theta}{2} \right) \end{array} \right.$$

since $\gamma = 90° - \theta$.

Note that the angle α_1 is measured as shown in Fig. 85, by clockwise rotation from OX. The two planes on which the obliquity is greatest will be in all cases symmetrical with reference to OX in Fig. 85. OY being the axis on which the minimum principal stress acts. Two cases may be distinguished:

(1) If n_1 and n_2 are of the same kind, *i.e.*, both tension or both compression Here, on each plane of max. obliquity, the obliquity is numerically the same, namely $\partial_{\max.}$ = the angle Oba in Fig. 86, since the tangent of this angle is $\frac{n_1 - n_2}{2\sqrt{n_1 n_2}}$, as in Eq. (25); but on one of these planes the shear tends to rotate a particle (Fig. 84) in a right-handed direction, while at the other plane of maximum obliquity the shear tends to rotate a particle in a left-handed direction. When $\alpha_1 > 90°$, the particle is similar to Oab in Fig. 82, *i.e.*, to the left of OY.

(2) If n_1 and n_2 are of opposite kinds, *i.e.*, one tension and the other compression, n_1 being always numerically greater than n_2. In this case there are still two planes of maximum obliquity, symmetrical with reference to OY (or OX). The value of $\cos 2\alpha_1$ in Eq. (24) is here greater than unity, and hence impossible, and Eq. (25) is also inapplicable, as the denominator involves the square root of a negative quantity. In this case the maximum obliquity is 90°, that is to say, there are two planes (symmetrical about OY) on which the normal stress is zero and the only stress is shear. The position of these two planes, which may be

designated by calling their angle $\alpha = \alpha_0$, is found by placing $n = 0$ in Eq. (15), or (since n_2 is opposite in sign from n_1), n_1 and n_2 being now numerical,

$$n_1 \cos^2 \alpha_0 - n_2 \sin^2 \alpha_0 = 0$$

$$\tan \alpha_0 = \pm \sqrt{\frac{n_1}{n_2}}. \tag{26}$$

11. Examples:

(1) Let there be no normal stress on either Oa or Ob (Fig. 84), but only shears parallel to the plane of the paper. Let s_x act upward on Ob, as shown in Fig. 87. Then $n_y = n_z = 0$; $s_y = -s_x$ (Eq. (2)); hence s_y causes left-handed rotation, and acts to the right, as shown. From Eqs. (5) and (6),

$$n = -s_x \sin 2\alpha.$$

$$s = -s_x \cos 2\alpha$$

From Eq. (10),

$$\tan 2\alpha_0 = \pm \infty$$

Hence

$$\alpha_0 = 45° \text{ or } 135°.$$

For $\alpha = 45°$, $n = -s_x$; hence on the plane Oa_1 (Fig. 87) there is *compression* of intensity s_x, and no shear. For $\alpha = 135°$, $n = +s_x$, hence on the plane Oa_2 there is

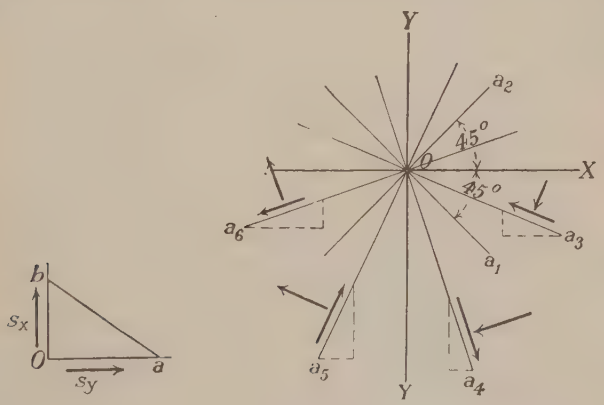


FIG. 87.

tension of intensity s_x, and no shear. This agrees with the results in Art. 10 of Chap. IV.

On any plane Oa_3 between OX and Oa_1, for which $\alpha < 45°$, there is compression, and a shearing force acting to the *left* (left-handed rotation) on the surface of a particle as shown.

On any plane Oa_4 between OY and Oa_1, for which $\alpha > 45° < 90°$ there is compression, and a shearing force acting downward to the *right* as shown (right-handed rotation).

On any plane Oa_5 between OY and Oa_2, for which $\alpha > 90° < 135°$ there is tension, and a shear acting upward to the right as shown, for s is positive, and causes right-handed rotation.

On any plane Oa_6 between Oa_2 and OX, for which $\alpha > 135° < 180°$, there is tension, and a shear acting downward to the left.

(2) Let the actual stresses n_x and s_x be as shown in Fig. 88, in direction and magnitude, and let $n_y = 0$; s_x is positive and n_x is negative. Then from Eq. (2), $s_y = -s_x$, and acts on the plane Oa to the right. From Eqs. (6) and (7).

$$n = \frac{n_x}{2} - \frac{n_x}{2}\cos 2\alpha - s_x \sin 2\alpha$$

$$s = \frac{n_x}{2}\sin 2\alpha - s_x \cos 2\alpha.$$

From Eq. (10), using the numerical values shown in Fig. 88

$$\tan 2\alpha_0 = \frac{2s_x}{n_x} = -\frac{120}{360} = -\frac{1}{3};$$

$$\alpha_0 = 80° 48' \text{ or } 170° 48'.$$

On the principal axis OY_1 (Fig. 89) for which $\alpha_0 = 80° 48'$

$$n_1 = -180\,(1 - \cos 161° 36') - 60 \sin 161° 36'$$
$$= -370, \text{ or compression.}$$

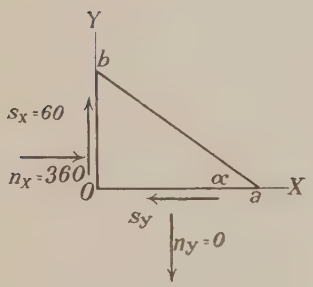


Fig. 88.

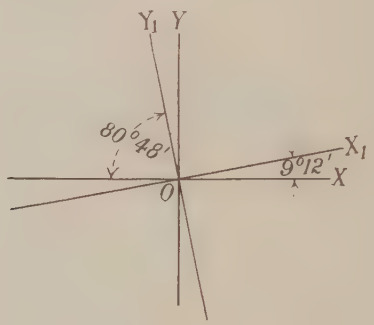


Fig. 89.

On the principal axis OX_1, for which $\alpha_0 = 170° 48'$

$$n_2 = -180(1 - \cos 341° 36') - 60 \sin 341° 36'$$
$$= +10, \text{ or tension.}$$

It is not necessary to perform this last computation, however, for by Eq. (14), having found $n_1 = -370$, it follows that $n_2 = +10$.

12. The Circle of Stress (*Stress Parallel to a Plane*).—The preceding results are somewhat difficult to visualize, and are apt to remain in the mind as mathematical abstractions, if at all. They are, however, capable of graphical representation in a very simple manner, which is commended to the student as a beautiful illustration of graphical methods (Fig. 90).

(1) *Suppose first that both principal stresses are tension*, and lay off from O

$$ON_1 = n_1 = \text{the max. principal stress,}$$
$$ON_2 = n_2 = \text{the min. principal stress.}$$

Draw a circle with N_1N_2 as a diameter, C being the center. Then

$$OC = \frac{n_1 + n_2}{2};\ N_2C = \frac{n_1 - n_2}{2}.$$

Suppose OX to be the plane upon which n_1 acts, and draw through N_2 a line N_2P making an angle α with OX, or parallel to the plane ab in the small figure. Then the angle PCN_1 is 2α, and from Eqs. (16) and (17),

$$n = OA$$
$$s = AP \text{ (below } OX \text{ being negative).}$$

These are the normal and tangential intensities on a plane parallel to N_2P, and the actual intensity is OP and its obliquity AOP.

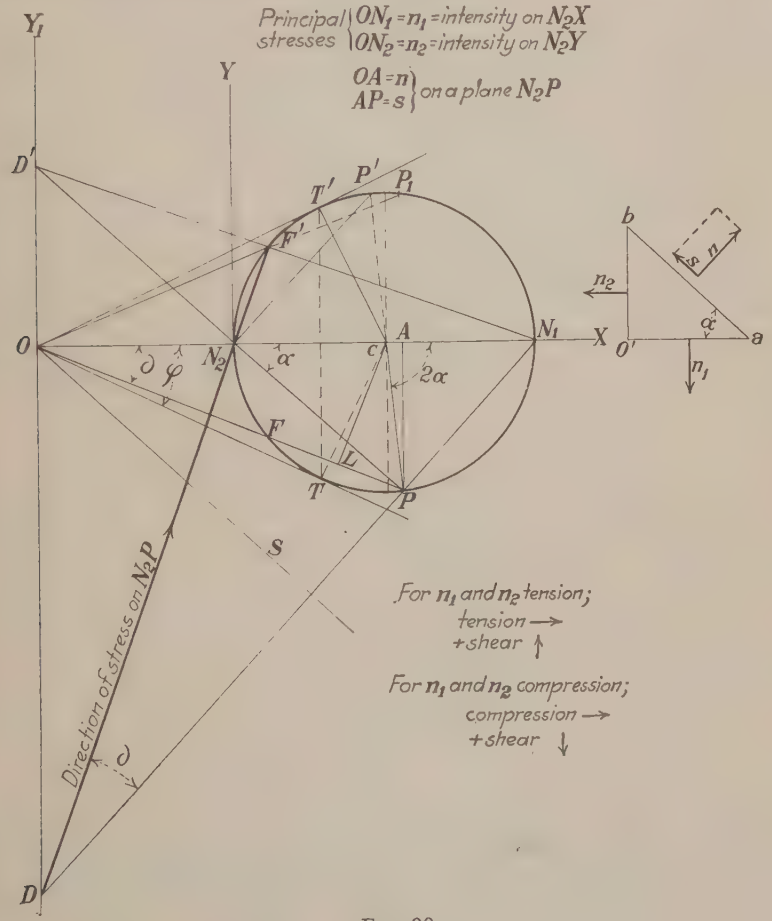


FIG. 90.

Knowing, therefore, both principal stresses at any point in a body, and their planes of action, the maximum principal stress acting on the plane OX and the minimum principal stress acting on the plane OY, both being tension, and the point in question N_2, then the stress intensity at N_2, upon any plane, is found by drawing the plane N_2P to intersect the circle at P; then OP is the intensity of stress on N_2P. Normal stresses are positive to the right from O, and shears positive upward. Then, on N_2P the shear is negative, as it is on any plane for which

$\alpha < 90°$ as was found in Art. 8. For the plane N_2N_1, or OX, the normal intensity is ON_1, and the shear zero; for the plane N_2Y, tangent to the circle at N_2, the normal intensity is ON_2, and the shear zero. The shear is greatest on two planes at 45° with OX, and the max. shear intensity is $\frac{n_1 - n_2}{2}$, agreeing with Eq. (17).

For $\alpha = 45°$ the shear is negative, for $\alpha = 135°$ it is positive.

The direction of the stress on N_2P may also be constructed: draw N_1P, and prolong it to meet OY_1 at D; then DN_2 is the real direction of the stress on the plane N_2P. This follows from the fact that DON_2 and DPN_2 are right angles, and consequently a circle drawn with DN_2 as a diameter will pass through O and P, and the angles PDN_2 and PON_2 are equal, both being measured by half the arc N_2P of this circle, and since $PON_2 = \partial$, $PDN_2 = \partial$ also This stress OP on N_2P (or ab) acts in the direction from D to N_2, or the normal stress is tension (OA) and the shear negative (AP), or left-handed.

All the other relations which have been deduced analytically in previous articles may be obtained at once from this figure. Thus, on any two planes at right angles to each other, such as N_2P and N_2P', the shear has obviously the same intensity but opposite signs, since PP' is a diameter. Also, the max. obliquity φ (max. value of ∂) occurs on the two planes N_2T and N_2T', OT and OT' being drawn tangent to the circle. To find the value of this obliquity, for the plane N_2T, $\alpha_1 = +N_1N_2T = +N_2TC$; hence $OCT = 180° - 2\alpha_1$; $N_1CT = 2\alpha_1$, and

$$\cos 2\alpha_1 = \cos N_1CT = \cos (90° + COT)$$
$$= -\sin COT = -\frac{CT}{OC} = -\frac{n_1 - n_2}{n_1 + n_2}.$$

For the plane N_2T', $\alpha_1 = ON_2T' = 180° - CN_2T'$; $2\alpha_1 = 360° - 2CN_2T' = 180° + OCT'$,

$$\cos 2\alpha_1 = -\cos OCT' = -\sin COT' = -\frac{CT'}{OC} = -\frac{n_1 - n_2}{n_1 + n_2}.$$

These results agree with Eq. (24).

Again, φ being the max. obliquity

$$\tan \varphi = \frac{CT}{OT} = \frac{CT'}{OT'} = \pm \frac{n_1 - n_2}{\sqrt{4n_1n_2}}$$

as in Eq. (25).

(2) *Suppose Both Principal Stresses to be Compression.*—Lay off ON_1 and ON_2 to the left of OY_1. The figure will be the same as Fig. 90, but on the other side. Positive shears will still be upward, compression to the left, and tension to the right. The shear on a plane parallel to N_2P in Fig. 90 will in this case be positive, as shown also by Eq. (17). In order to avoid the necessity of drawing another figure, Fig. 90 may be used, but shear considered positive when below OX and compression positive toward the right. Figure 90, therefore, applies with this understanding to all cases in which the principal stresses are of the same kind.

(3) *Let the Principal Stresses be of Opposite Kinds* (Fig. 91).—From O on OY_1 lay off ON_1 = intensity of principal tensile stress, and ON_2 = intensity of principal compressive stress. On any plane N_2P (N_2 being the point at which the

stresses are being studied), the intensity of the normal stress will be OA, and the intensity of the shearing stress will be AP.

The obliquity will be AOP, and the direction of the resultant stress on N_2P will be DN_2, constructed, as shown, in a manner exactly analogous to that used in Fig. 90. The student should satisfy himself of this by going over the demonstration just as it was developed for the case of principal stresses of the same kind.

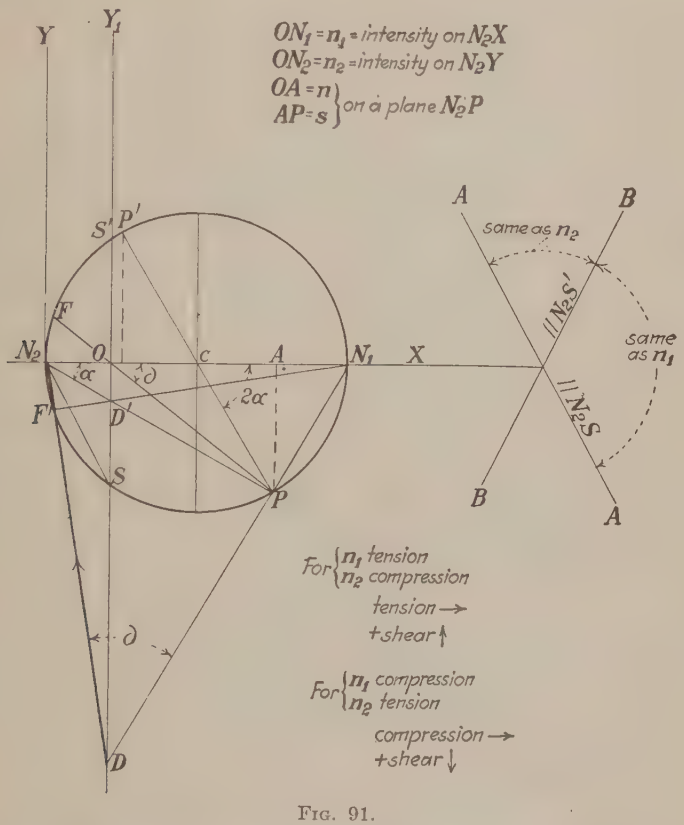


FIG. 91.

13. Conjugate Stresses as Shown by the Circle of Stress.—The circle of stress affords simple relations between conjugate stresses, which are often useful. In Fig. 90,

On the plane N_2P,

the intensity of stress is OP
its direction is DN_2.

Produce DN_2 to meet the circle at F'; draw N_1F' to meet OY_1 at D'. Draw $D'N_2$; then

On the plane N_2F' (parallel to the direction of stress on N_2P),

the intensity of stress is OF'
its direction is $D'N_2$.

In the triangle $DD'N_1$, DF' and ON_1 are the perpendiculars from two of the vertices upon the opposite side, and they intersect at N_2. N_2P is perpendicular to the third side; hence, since the perpendiculars from all three vertices upon opposite sides must meet in a point, PN_2 must pass through D'.

This states the principle of conjugate stresses, *i.e.*, the stress on the plane N_2P is parallel to N_2F' and the stress on the plane N_2F' is parallel to N_2P.

Another important result follows:

The obliquity on the plane N_2P is $PDN_2 = PON_2$.

The obliquity on the plane N_2F' is $F'D'N_2 = F'ON_2$.

But $F'D'N_2 = 90° - D'N_2F' (= DN_2P)$

and $PDN_2 = 90° - DN_2P$

$\therefore PDN_2 = F'D'N_2$, and the obliquities are equal.

Hence $PON_2 = F'ON_2$, and the point F, where OP cuts the circle (other than at P) must be so situated that $N_2F = N_2F'$; $OF = OF'$; and FF' is at right angles to OX. Similarly, if OF' be produced to meet the circle at P_1, $OP = OP_1$ and PP_1 is perpendicular to OX.

Thus we see that if from O any line is drawn, as OP, cutting the circle in F and P, OF and OP represent the intensities of two conjugate stresses, acting respectively on planes N_2F' and N_2P, in directions respectively $D'N_2$ and DN_2, and having the same obliquity N_2OP.

It may now easily be shown that the ratio of the intensities of these two conjugate stresses may be found in terms of their common obliquity ∂, and the maximum obliquity φ.

Let $OP = p_1$ and $OF = p_2$

be the two conjugate intensities.

Draw from C a perpendicular to OP in L.

$$\text{Then } \frac{p_1 + p_2}{2} = OL = OC \cdot \cos \partial$$

$$p_1p_2 = OT^2 = OC^2 \cdot \cos^2 \varphi = n_1n_2. \tag{27}$$

It is easy to perceive the identity[1]

$$p_2 = \frac{p_1 + p_2}{2} - \sqrt{\left(\frac{p_1 + p_2}{2}\right)^2 - p_1p_2}$$

so that in this case

$$p_2 = OC[\cos \partial - \sqrt{\cos^2 \partial - \cos^2 \varphi}] \cdot$$

Similarly

$$p_1 = OC[\cos \partial + \sqrt{\cos^2 \partial - \cos^2 \varphi}] \cdot$$

From which

$$\frac{p_2}{p_1} = \frac{\cos \partial - \sqrt{\cos^2 \partial - \cos^2 \varphi}}{\cos \partial + \sqrt{\cos^2 \partial - \cos^2 \varphi}} \tag{28}$$

This formula gives the ratio of the smaller of two conjugate stresses to the larger, their common obliquity being δ, and the maximum obliquity being φ.

If the principal stresses are of opposite kinds, the principle of conjugate stresses follows as in the case just treated, but the max. obliquity is 90° (on planes N_2S and N_2S' in (Fig. 91), and no formula like Eq. (28) is available. Any line through O, however, cuts the circle in two points F and P (Fig. 91), such that OF and OP

[1] For the expression under the radical, if expanded becomes $\left(\frac{p_1 - p_2}{2}\right)^2$.

are the intensities p_1 and p_2 of two conjugate stresses, and (numerically and algebraically)

$$p_1 p_2 = n_1 n_2, \text{ as before in Eq. (27).}$$

14. From the preceding discussion, some important generalizations may be made, which should be carefully impressed upon the mind.

(1) From Eq. (14), the sum of the normal intensities on any two planes, at right angles to each other, is always the same, and equal to the sum of the two principal stresses.

(2) The product of the intensities on any two conjugate planes is always the same, and equals the product of the principal stresses (Eq. (27)).

(3) If the two principal stresses are of the same kind, the stress on every plane will be of the same kind (from Fig. 90 and Eq. (15)).

(4) If on any plane there is no normal stress, but only shear, then the two principal stresses are of opposite kinds (for Fig. 90 is inapplicable).

(5) If on any plane there is neither normal stress nor shear, it is one of the principal planes, and the case is one of "pure tension" or compression, parallel to that plane.

(6) If on two planes at right angles the normal stresses are of opposite kinds, then the two principal stresses will be of opposite kinds also. (This follows from Fig. 90.)

From Fig 91, it is obvious that on any plane lying between the positions N_2S and N_2S' in the sector containing OX, the stress will be of the same kind as n_1, while in the sector containing N_2Y the stress will be of the same kind as n_2, as indicated on the right of the figure.

(7) If on two planes at right angles the normal stresses are of the same kind, the principal stresses may or may not be of the same kind; but in any case the max. principal stress will be of the same kind as the two given normal stresses. (This follows from Eq. (14).) See the next article for criterion.

(8) If both principal stresses are equal in intensity and of the same kind, the circle of stress reduces to a point, and the stress on all planes is normal and of the same intensity and kind. There are no shearing stresses.

(9) If both principal stresses are tension, the shear is negative on all planes for which α lies between 0 and 90°, and positive on all planes for which α lies between 90 and 180°. The reverse is true if both stresses are compression.

(10) If the two principal stresses are of opposite kinds, there are two planes N_2S and N_2S' in Fig. 91 for which there is only shear. Numerically,

$$ON_2S = ON_2S' = \tan^{-1}\sqrt{\frac{n_1}{n_2}}. \qquad (29)$$

This follows from Eq. (15).

If $n_1 = n_2$ numerically, these planes are at 45° to the planes of principal stress.

The value of the shear intensity on the two planes which have only shear is $s = \sqrt{n_1 n_2}$. This is shown by Fig. 91, and by Eqs. (15) and (17).

(11) If the two principal stresses are of opposite kinds, the max. intensity of shear, which acts on planes making angles of 45° with the planes of principal stress, is *numerically* $\frac{n_1 + n_2}{2}$ (Eqs. (18) and (19)). This is shown by Fig. 91.

(12) If both principal stresses are of the same kind, any two conjugate stresses must be of the same kind. If two conjugate stresses are of opposite kinds, the principal stresses will be of opposite kinds, and the maximum principal stress (numerically) will be of the same kind as the maximum conjugate stress.

It must be remembered that this discussion is of the case of stresses parallel to a plane, assumed as that of the paper.

15. To Find the Principal Stresses by the Circle of Stress.—Suppose the normal and tangential stresses are known on two planes at right angles, and that the principal stresses and the principal planes are to be found. In Figs. 90 and 91, the maximum principal stress ON_1 acts on the plane OX (or N_2X) and the minimum principal stress ON_2 on the plane OY_1 or (N_2Y). Then on any two planes at right angles, N_2P and N_2P', the stresses are respectively OP and OP'. If the stresses are given (not principal stresses) as in Fig. 92, on any two planes at right angles N_2P and N_2P', namely, n and s on N_2P and n' and s' $(= -s)$ on N_2P', then ON_1 and ON_2 would be the principal stresses, acting on OX and OY respectively, and they could be constructed as follows: lay off from any point $OA = n$

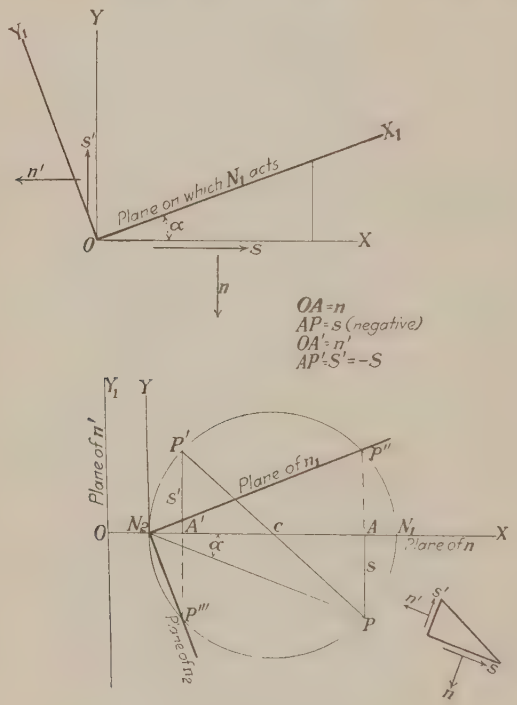


Fig. 92.

and $AP = s$ (negative in the case shown), and $OA' = n'$ and $A'P' = s'$ (positive as shown); take PP' as a diameter, with center at C, and draw the circle. Then the principal stresses will be ON_1 and ON_2 in magnitude and sign. It remains to find the planes on which they act. When the circle is drawn, if N_2P is parallel to the real plane on which n and s act, then ON_1 will act on the real plane ON_1 and ON_2 on OY_1 (or N_2Y); clearly, then, ON_1 acts on a plane making an angle α above the plane on which n and s act.

Consequently, if n and s act on any plane OX as in the upper figure, and n' and s' on OY_1, when the construction is made by laying off OA, AP, OA' and $A'P'$ and drawing the circle, since $AP'' = AP$ and $A'P''' = A'P'$, the plane on which $ON_1 = n_1$, acts is N_2P'', or OX_1 in the upper figure, while the other principal stress $n_2 = ON_2$ acts on N_2P''', or OY_1, in the upper figure.

If the circle drawn with C as a center should enclose O, as in Fig. 93, the principal stress $n_1 = ON_1$ would be of the same kind as n, and would act on a plane parallel to N_2P''', while the other principal stress $n_2 = ON_2$, would be of the opposite kind, and would act on a plane parallel to N_2P'''. This enables a determination of the question raised under (7) of the preceding article.

$$OC = \frac{n + n'}{2}\left(= \frac{n_y + n_x}{2}\right);$$

$$OA = n(= n_y);\ AP = s(= s_y);$$

$$AC = \frac{n - n'}{2}\left(= \frac{n_y - n_x}{2}\right).$$

Hence

$$CP = CP' = \sqrt{s_y^2 + \frac{(n_y - n_x)^2}{4}}.$$

The principal stresses will be of opposite kinds if

$$\sqrt{s_y^2 + \frac{(n_y - n_x)^2}{4}} > \frac{n_x + n_y}{2}$$

or if

$$s_y^2 > n_x n_y.$$

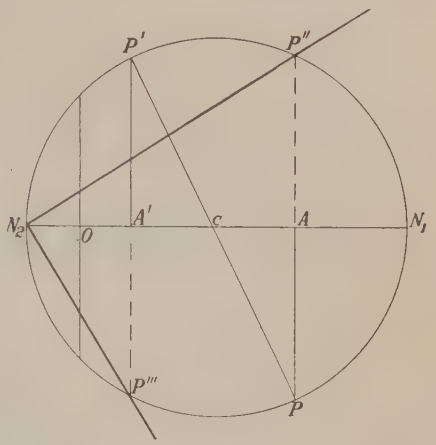


Fig. 93.

If $s_y^2 = n_x n_y$, one principal stress is zero, and the circle with PP' as a diameter goes through O.

As another illustration, take the second example in Art. 11, in which there is on a horizontal plane a negative shear s_y of 60 pounds per square inch and no normal stress, and on a vertical plane a positive shear s_x of 60, and a compression

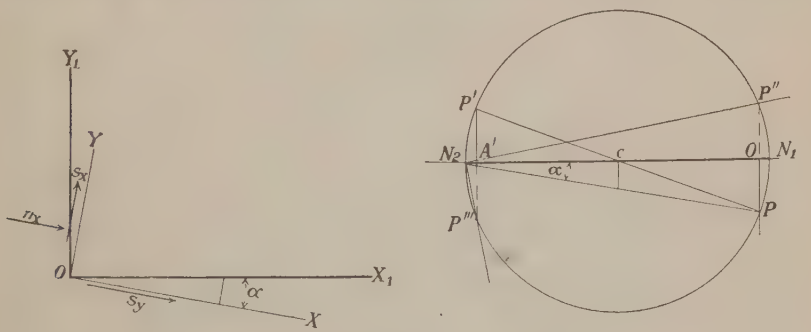


Fig. 94.

of n_x of 360 pounds per square inch. Lay off, in Fig. 94, $OP = s_y = -60$, $OA' = n_x = -360$, $A'P' = 60$, and draw the circle with PP' as a diameter. Then, if N_2N_1 were the plane on which the principal stress ON_1 acted, which is a tension, the plane on which the normal stress is zero would be parallel to N_2P, on which the shear is negative and equal to OP. But the last plane is really horizontal; hence the principal plane on which ON_1 acts is really parallel to N_2P'', or is OX_1 in the upper figure, while OY_1, parallel to N_2P''' is the other principal

plane, on which the compression $ON_2 = 370$ pounds per square inch. The tensile stress on OX_1 is $ON_1 = 10$ pounds per square inch, as found in Art. 15. The angle α is $\tan^{-1} \frac{60}{370} = 0.1622$; $\alpha = 9° 12''$, as previously found.

16. General Case of the Circle of Stress When the Principal Axes Are Not Known.—In the discussion of Art. 12, the planes of principal stress, and the principal stresses themselves, were assumed known. Thus in Fig. 90, Oa and Ob were the planes of principal stress, and the intensities were n_1 and n_2. But a circle may be drawn in the more general case shown in Fig. 95, in which Oa and Ob are any two planes at right angles to each other, and ab is any plane making an angle with Oa. *Suppose the normal stresses all tension.* Then there are six forces in equilibrium on the particle Oab, and, by the equation $\Sigma M = 0$, $s_x = -s_y$. Hence there are five independent unknown intensities, and two remaining equations of equilibrium;

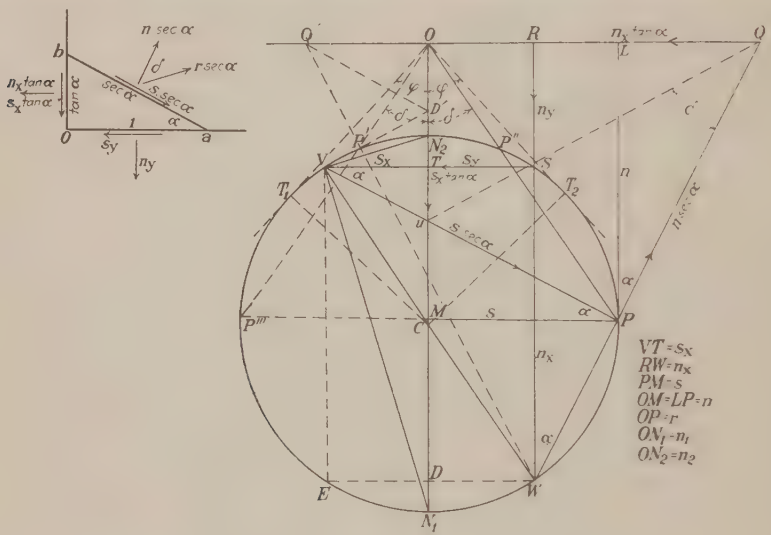


FIG. 95.

and any three of the five unknown intensities may be assumed. In order to study their relations, suppose them laid off as shown in the polygon $TUPQRST$. Drawing the other lines shown, the angle α occurs several times, and it is obvious that taking downward distances to represent positive (tensile) normal intensities, and horizontal distances to represent shearing intensities, positive to the right and negative to the left, the following relations hold:

VT (negative) $= s_x$ (causing left-handed rotation);
$VT = TS$; ($TS = s_y$, acting toward the left);
$RW = n_x$ (since $RQ = n_x \tan \alpha$ and $RWQ = \alpha$);
$PM = s$ (since $UP = s \sec \alpha$ and $UPM = \alpha$);
$OM = LP = n$ (since $PQ = n \sec \alpha$ and $LPQ = \alpha$);
$OP = r$ (the resultant intensity on ab).

Hence, since VSW and VPW are right angles, a circle may be drawn through V, S, P, and W, of which VW will be the diameter. As the angle α is varied, the

point P will move around the circle, but OM, MP, and OP will always represent the normal, tangential, and resultant intensities on ab, to which VP is always parallel. Hence the maximum stress will be ON_1 occurring on a plane parallel to VN_1, and there will be no shear on this plane, and the minimum stress will be ON_2, on a plane parallel to VN_2, or the principal stresses are ON_1 and ON_2. The circle necessarily has somewhat different relations from the circle in Fig. 90, but they are just as simple, more obvious, and less dependent upon equations. The relations here deduced are purely geometrical.

Of the six stress intensities, n_x, n_y, s_x, s_y, n, s (remembering that $s_x = -s_y$), any three may be assumed, hence there are ten possible cases, in each of which we may see how to draw the circle:

(1) *Given* n_x, n_y, s_x; lay off $OT = n_y$, $OD = n_x$, $TV = s_x$, make $TS = TV$, and $DE = DW = TV$. Then VW is the diameter of the circle.

(2) *Given* s_x, n, s; lay off $OM = n$, $MP = s$; draw $PV \parallel ab$ until $TV = s_x$, which fixes V; VP is a chord of the circle, whose center is on OM, hence the center may be found.

(3) *Given* n_y, s_x, s; lay off $OT = n_y$, $TV = s_x$, $TS = s_y\ (= -s_x)$. Draw $VP \parallel ab$, till $MP = s$, which fixes P, and so the circle.

(4) *Given* n_y, s_x, n; proceed as in Case 3, fixing P by the fact that $OM = LP = n$.

(5) *Given* n_x, s_x, n; lay off $OD = n_x$, $DE = s_x$, $DW = DE$, draw $WP \perp ab$, until $PL = n$, thus fixing P; PW is a chord, and the circle may be drawn.

(6) *Given* n_x, s_x, s; proceed as in Case 5, fixing P by the fact that $PM = s$.

(7) *Given* n_x, n, s; lay off $OM = n$, $MP = s$, $OD = n_x$, Draw $PW \perp ab$, fixing W by the line $DW \perp OD$.

(8) *Given* n_y, n, s; lay off $OT = n_y$; $OM = n$; $MP = s$; draw $PV \parallel ab$ to meet $TV \perp OT$, fixing V, and VP is a chord.

(9) *Given* n_x, n_y, s; lay off $OT = n_y$, $OD = n_x$, $OL = s$; draw LP indefinitely, as a line in which P lies somewhere; draw TV and DW indefinitely. The problem is to locate P on LP so that if PV is drawn $\parallel ab$, and $PW \perp ab$, these lines will cut equal intercepts TV and DW. This problem is definite and may be solved by geometry or by algebra,[1] and P is then located. Or, $CT = CD$, fixing C.

(10) *Given* n_x, n_y, n; lay off $OT = n_y$, $OD = n_x$, $OM = n$; draw MP, TV and DW indefinitely: the problem is then to locate P on MP to fulfil the same condition as in Case 9; this is definite and may be solved.[2] Or, $CT = CD$, fixing C.

Any case may thus be solved under the conditions assumed. If s_y acts to the right instead of as shown, the problem may be solved equally well by the same general procedure. Also, it will be found that if any of the normal stresses are assumed to be compression the same rules will apply; it is only necessary to consider normal tensile (positive) forces measured from O *downward*, and compressive forces from O *upward*, just as positive shears are measured from ON, to the right and negative shears to the left.

The maximum obliquity of stress will occur on two planes parallel to VT_1 and VT_2, for which $\varphi = \text{max. } \partial = T_1ON_2$; also

$$OT_1^2 = ON_1 \cdot ON_2 = n_1n_2$$

$$\tan \varphi = \pm \frac{n_1 - n_2}{2\sqrt{n_1n_2}} \text{ as in Eq. (25).}$$

[1] $(s_x + s) \tan \alpha + (s - s_x) \cot \alpha = TD$; from which s_x may be found.

[2] $TM \cdot \cot \alpha - MD \cdot \tan \alpha = 2s_x$, from which s_x may be found.

The angle, α_1, which the plane VT_1 (on which the maximum obliquity occurs) makes with the plane VN_1 (the plane of maximum principal stress) is measured by one-half the arc T_1N_1, or is $\frac{T_1CN_1}{2}$. Hence $2\alpha_1 = T_1CN_1$ and

$$\cos 2\alpha_1 = -\cos T_1CO = -\sin T_1OC$$
$$= -\frac{n_1 - n_2}{n_1 + n_2} \text{ as in Eq. (24),}$$

since $$CT_1 = \frac{n_1 - n_2}{2} \text{ and } OC = \frac{n_1 + n_2}{2}.$$

17. Direction of the Stress on a Plane.—On any plane parallel to VP the intensity is OP, but the direction is that of the resultant of $s \cdot \sec\alpha$ and $n \cdot \sec\alpha$, or QU. Hence, to find the direction, a line must be drawn through W and P to meet the horizontal through O, and the point thus found must be joined to the point U where the line from V parallel to the plane (on which the pressure is sought) crosses ON_1.

18. Conjugate Stresses.—The stress on a plane parallel to VP has the direction QU. Find now the stress on a plane parallel to QU, by drawing $VD' \parallel QU$, meeting the circle in P' and the line ON_1 in D', and drawing WP' to meet OQ at Q'; then the direction of the stress on VD' will be $Q'D'$, and will be parallel to VP. For, the obliquity on the planes VP and VP' must be equal; the former is the angle UQP or UOP, and the latter is $D'Q'P'$ or $D'OP'$. Hence the angles N_2OP and N_2OP' are equal, and the intensities of two conjugate stresses are represented by the intercepts like OP and OP'' on any secant through O. Clearly there are four planes on which the obliquity has a given value ∂, namely planes parallel to VP, VP'', VP' and VP'''; and these planes are symmetrical with reference to the planes of principal stress which are parallel to ON_1 and ON_2, since the angles $P'VN_2$ and N_2VP'' are equal. This is in accordance with the results of Art. 13. The ratio of two conjugate stresses to each other may also be found from this figure, exactly as in Art. 13.

This construction, therefore, affords a complete solution of the problem.

19. Another Construction for Stress on Any Plane, if the Principal Stresses Are Known.—From the circle of stress (Fig. 90) another method or construction may be deduced, by which the stress intensity on any plane may be found, in magnitude and direction. In Fig. 90 the circle of stress was drawn for two principal stresses of the same kind. There it was shown that

max. principal stress $= n_1 = ON_1$ acting on plane OX,
min. principal stress $= n_2 = ON_2$ acting on plane OY_1.

Let O (not N_2) be the point where the stresses are being studied. On any plane OS, parallel to N_2P, the stress is OP, and its direction is found to be DN_2, by drawing N_1P to meet OY_1 at D; or N_2P may be drawn to meet OY_1 at D', then $D'F'N_1$ drawn, and N_2F' will be the direction.

In Fig. 96 the circle is drawn again, as before. Let it be required to find the stress on the plane OS. Lay off, perpendicular to OS, $OC' = OC = \frac{n_1 + n_2}{2}$; and draw the circle with C' as a center and radius $= \frac{n_1 - n_2}{2}$, or the same as the

circle drawn with C as a center. With C' as a center and CO as a radius, draw the circle meeting OX at T, draw TC' and produce it *beyond* C' to M. Draw OM.

Now let the figure be studied geometrically. Calling β the angle between OS and OY_1 or $90° - \alpha$, and γ the angle OTM, it is first obvious that $OP = OM$; for P is the point on the circle with C as a center found by laying off the angle $OCP = 2\beta$, as will be easily seen, since N_2P is parallel to OS. The angle $OC'M = 2\gamma$, and consequently $C'ME = \frac{180 - 2\gamma}{2} = 90 - \gamma = OHT$, that is to say, ME is parallel to OY_1. Now in the triangle OHC' $180° = 90 - \beta + 90° - \gamma + 2\gamma$

or

$$\beta = \gamma.$$

It follows, since $OC = OC'$, $C'M = CP$, and the angles $OC'M$ and OCP are equal, that $OM = OP$, that is, *OM represents the intensity of stress on the plane OS.*

Now as to the direction of this stress, which by our proved construction is DN_2, having an obliquity $\partial = POC = PDN_2$, it is clear from the construction that $C'OM = COP = \partial$, and OM is parallel to DN_2; hence *the stress on* the plane OS *acts along* OM. Hence the construction gives the magnitude and direction of the stress on OS.

It is easily seen that if OS is taken as OX, the construction will give a stress acting normal to OX and equal to ON_1, and that on the plane OY it will give a normal stress equal to ON_2.

The construction will hold equally well if the stresses are of the opposite kind, using the signs correctly. The reader should draw the figure for this case.

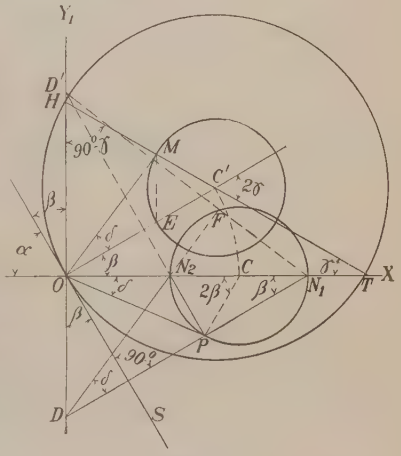


Fig. 96.

20. Ellipsoid of Stress.—If the stresses are not parallel to a plane, as assumed generally in this chapter, the following propositions may be proved by investigating the general case:

(1) At any point in a body, if any two planes pass through that point, the component of the stress *intensity* on the first plane in a direction perpendicular to the second plane equals the component of the stress *intensity* on the second plane in a direction perpendicular to the first plane; and the directions of these stress components are either both away from, or both toward, the line of intersection of the two planes. (The reader should prove this.)

(2) If, at any point in a body, three rectangular axes are assumed, and if the stress intensity on any plane, at the point, be laid off from the point in its true direction, all the points so found will lie on an *ellipsoid of stress*. Any three conjugate radii of this ellipsoid represent the stress intensities on three rectangular planes passing through the point.

(3) The three semi-axes of this ellipsoid represent three *principal stresses*, acting on planes on which there is no shear. In other words, there are three

planes passing through any point, at right angles to each other, to which there is only normal stress; and on one of these planes the normal stress is greater, and on another less, than on any other plane through the point.

(4) At any point, the sum of the squares of the stress intensities on any three planes at right angles will be constant, and hence equal to the sum of the squares of the principal stresses.

(5) At any point, the algebraic sum of the *normal* stress intensities on any three planes at right angles will be constant, and hence equal to the sum of the three principal stresses.

(6) The shearing stress intensities are greatest on six planes, in pairs, each pair containing one of the principal axes and bisecting the angle between the other two, and the intensity is the same on each of the two planes containing one principal axis. If the three principal stresses are n_1, n_2, n_3, then the intensity of the shear on the two planes containing n_1 is

$$s_1 = \pm\frac{n_2 - n_3}{2}.$$

Similarly $$s_2 = \pm\frac{n_3 - n_1}{2}; \quad s_3 = \pm\frac{n_1 - n_2}{2}. \qquad (30)$$

Each of these maximum shears is perpendicular to the axis it contains.

For the case of stress parallel to a plane, all the above propositions will be found to agree with those which have been deduced in this chapter.[1]

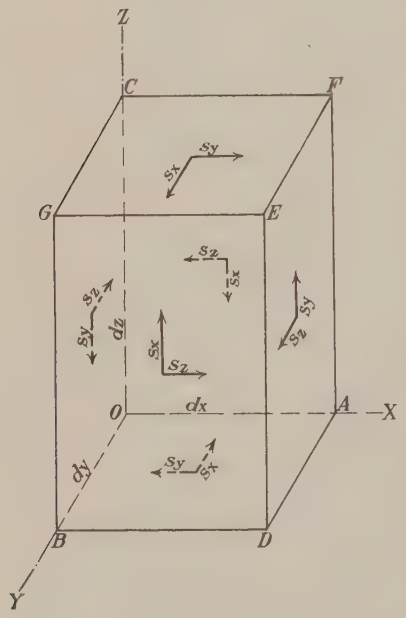


FIG. 97.

21. Laws of Variation of Stress from Point to Point.—In studying the relations between stresses at a point, on different planes, the stresses on the sides of an infinitesimal parallelepipedon are properly considered constant, although the stress in a mathematical sense varies by an infinitesimal amount in a distance dx. If we are studying the variation of stress from point to point, this infinitesimal variation of stress must be considered, and cannot be neglected. This study is of some interest, and leads to three fundamental equations which will be explained.

Let OX, OY, OZ be three rectangular axes, and let Fig. 97 represent a rectangular prism three of whose sides are on those axes. Let the stress on each side be resolved into three components parallel to the axes, one normal stress and two shears. Let normal stresses be positive when tension, and let the normal intensities in planes perpendicular to the X, Y, and Z axes be respectively n_x, n_y, n_z. Then on the plane OG there will be a normal stress $n_x dydz$, and on AE there will be $\left(n_x + \frac{\partial n_x}{\partial x}\, dx\right) dydz$;

[1] A discussion of the general case may be found in GRASHOF, "Elasticität and Festigkeit," Berlin, 1878.

and similarly for the other normal forces (total stresses) on the other planes. On the plane OF there will be a shear parallel to OZ. Let this be of intensity s_{yz} (on plane Y, parallel to Z). Let this be positive when the moment of the two shears, one on OF and the other on BE, is right-handed when looking from X toward O. Then this shear on OF will be $s_{yz}dxdz$ acting down, and on BE there will be a parallel shear acting up of $\left(s_{yz} + \frac{\partial s_{yz}}{\partial y}\,dy\right)dxdz$. By the principle of shears the shear on OD parallel to OY will be $s_{zy}dxdy$, and will act in the direction from Y toward O, and on CE there will be a parallel shear $\left(s_{zy} + \frac{\partial s_{zy}}{\partial z}\,dz\right)dxdy$ acting in the direction from O toward Y. Numerically $s_{yz} = s_{zy}$, and both may be called s_x, indicating that they act parallel to the X plane, *i.e.*, the plane perpendicular to OX. Similarly, on OD there will be a shear s_{zx} parallel to OX and on CE a parallel shear; let these be positive when the couple they form is right-handed looking from Y toward O, so that the total shear in this direction on OD will be $s_{zx}dxdy$, and on CE, $\left(s_{zx} + \frac{\partial s_{zx}}{\partial z}\,dz\right)dxdy$, acting as indicated. On OG there will be a shear parallel to OZ whose intensity will be $s_{xz} = s_{zx} = s_y$ (since both act parallel to the plane OXZ perpendicular to OY). The total shear on OG parallel to OZ will be $s_y dydz$, and on AE, $\left(s_y + \frac{\partial s_y}{\partial x}\,dx\right)dydz$. Again, on OG and AE there will be shears parallel to OY, and if right-handed looking from Z toward O they will act as indicated, the intensity on OG being s_{xy}. On OF and BE there will be shears $s_{yx} = s_{xy} = s_z$ acting as shown. The total forces will be: on OG, $s_z dydz$; on $AE\left(s_z + \frac{\partial s_z}{\partial x}\,dx\right)dydz$; on OF, $s_z dxdz$; on BE, $\left(s_z + \frac{\partial s_z}{\partial y}\,dy\right)dxdz$. Finally, on the prism there may be a force, such as gravity, of X, Y, and Z per unit of volume, parallel to the three axes.[1]

If the sum of all forces parallel to X, Y, and Z be placed equal to zero, we shall have

$$\left.\begin{aligned} \frac{\partial n_x}{\partial x} + \frac{\partial s_y}{\partial z} + \frac{\partial s_z}{\partial y} + X &= 0 \\ \frac{\partial n_y}{\partial y} + \frac{\partial s_z}{\partial x} + \frac{\partial s_x}{\partial z} + Y &= 0 \\ \frac{\partial n_z}{\partial z} + \frac{\partial s_x}{\partial y} + \frac{\partial s_y}{\partial x} + Z &= 0 \end{aligned}\right\} \qquad (31)$$

DEFORMATION OR STRAIN

22. Deformation at a Point.—Let P (Fig. 98) be any point in a body. Consider only the case of stress parallel to the plane of the paper. When the body is stressed, the point P will move. We are not concerned, however, with that movement, but with the unit strains at the point P, due to the fact that points adjoining P do not move exactly the same as P. If a small sphere is supposed to be taken, with P as its center, and if every point in that sphere moved the same distance as P and in the same direction, there would be no deformation or strain in the sphere, but merely a bodily translation. Our object here is to find the unit strain in any direction at P. Let PX, PY be rectangular axes, and let u and v be respectively

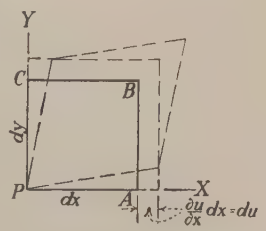


FIG. 98.

[1] Infinitesimal differences are omitted in the figure.

the movement of P along X and Y. The dimensions of the particle (prism) considered are dx and dy, the dimension perpendicular to the paper being unity. The point A will move along X the distance $u + du$, du being a definite quantity. If $\frac{\partial u}{\partial x}$ is the rate of change of u along x,

$$du = \frac{\partial u}{\partial x} dx$$

and clearly $\frac{\partial u}{\partial x} = e_x$ = the unit strain in the direction X at P.

Similarly C will move along Y a distance

$$v + dv = v + \frac{\partial v}{\partial y} \cdot dy$$

and $\frac{\partial v}{\partial y} = e_y$ = the unit strain in the direction Y at P.

In the same way A will move in the direction Y a distance $v + \frac{\partial v}{\partial x} dx$, and C will move in the X direction a distance $u + \frac{\partial u}{\partial y} dy$; and $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \gamma_{xy}$ = change of angle CPA = shearing distortion of the particle, as in Art. 12 of Chap. IV.

Now consider the same particle $PABC$ drawn again in Fig. 99, with an adjoining particle $PCDE$, so shaped that PB and PD are at right angles. Our problem is, to determine the unit strain e in the direction PB, and the shearing distortion of the originally right angle BPD. The strain e is the difference between the new length of PB and its original length PB, divided by PB.

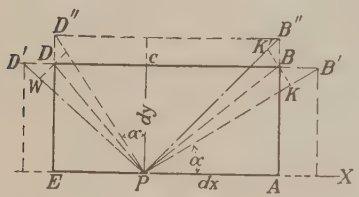


Fig. 99.

The new position of B with reference to P, which is considered as fixed, will be the sum of three parts, e_1, e_2, and e_3.

(1) The movement of B due to the unit strain e_x acting alone, *i.e.*, without e_y or γ.

(2) The movement of B due to the unit strain e_y alone.

(3) The movement of B due to the shearing distortion γ alone. These will now be evaluated, by simple methods.

The effect of e_x alone is to make B move to B', with reference to P, BB' being $e_x dx$. Drawing $BK \perp PB'$, and remembering that the distortions are infinitesimal, and that $PB'B = \alpha$

$$e_1 = \frac{KB'}{PB} = \frac{e_x dx \cos^2 \alpha}{dx} = e_x \cos^2 \alpha.$$

Similarly, the effect of e_y alone is to make B move to B'', and drawing $K'B \perp PB$, BB'' being $e_y dy$,

$$e_2 = \frac{K'B''}{PB} = \frac{e_y dy \sin^2 \alpha}{dy} = e_y \sin^2 \alpha.$$

Again, the effect of the shearing distortion γ_{xy} may be considered (Fig. 100) as though the line EPA is not changed, but $PABC$ is distorted into $PAB'''C'$, and the adjoining particle similarly, and

$$DD''' = CC' = BB''' = \gamma_{xy} dy.$$

Draw $BK'' \perp PB'''$ and $D'''N'' \perp PD$;

then $$e_3 = \frac{K''B'''}{PB} = \frac{\gamma_{xy} dy \cos\alpha \sin\alpha}{dy} = \gamma_{xy} \sin\alpha \cos\alpha.$$

Hence finally

$$e = e_x \cos^2\alpha + e_y \sin^2\alpha + \gamma_{xy} \sin\alpha \cos\alpha. \tag{32}$$

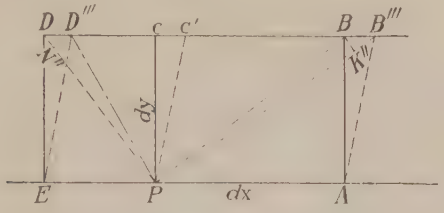


FIG. 100.

The shearing distortion γ of the planes PB and PD may be similarly found, as follows:

The movement of B to B' (Fig. 99) due to e_x *increases* the angle CPB (e_x being positive when an elongation) (Fig. 99) by

$$\frac{BK}{PB} = \frac{e_x dx \sin\alpha \cos\alpha}{dx} = e_x \sin\alpha \cos\alpha.$$

At the same time the elongation of EP, since P is supposed to be fixed, is the same as moving D to D' and the angle CPD is *increased* by

$$\frac{DW}{PD} = \frac{e_x (dx) \cos\alpha \sin\alpha}{(dx)} = e_x \sin\alpha \cos\alpha$$

calling $EP = (dx)$ to distinguish it from $PA = dx$, because they are not the same; hence the angle DPB is *increased* on account of e_x by

$$2e_x \sin\alpha \cos\alpha = e_x \sin 2\alpha.$$

Similarly, the effect of e_y is to *decrease* the angle DPB by $e_y \sin 2\alpha$.

The effect of γ_{xy} (Fig. 100) is to *increase* the angle CPB (Fig. 100) by

$$\frac{BK''}{PB} = \frac{\gamma_{xy}\, dy \sin^2\alpha}{dy} = \gamma_{xy} \sin^2\alpha$$

and to *decrease* the angle CPD by

$$\frac{D'''N''}{DP} = \frac{\gamma_{xy} dy \cos^2\alpha}{dy} = \gamma_{xy} \cos^2\alpha.$$

Hence, finally, if we consider a shearing distortion as positive when it decreases the angle DPB,

$$\gamma = (e_y - e_x) \sin 2\alpha + \gamma_{xy} \cos 2\alpha. \tag{33}$$

Using the nomenclature of the calculus, Eqs. (32) and (33) may be written

$$e = \frac{\partial u}{\partial x} \cos^2\alpha + \frac{\partial v}{\partial y} \sin^2\alpha + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \sin\alpha \cos\alpha \tag{34}$$

$$\gamma = 2\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) \sin\alpha \cos\alpha + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \cos 2\alpha. \tag{35}$$

23. General Case of Strain.—Similar expressions may be deduced for the general case of stress not parallel to a plane. Let there be a Z axis through P perpendicular

to the paper, and let the point B be at the corner of a prism, its coordinates dx, dy, dz. Let the angles BPX, BPY, BPZ be respectively α, β, γ; the coordinates of P, x, y, z; and the movement of P under stress, u, v, and w, along the X, Y, Z axes, respectively. Then the movement of B is given by the following equations, u_1, v_1, and z_1 being its movement along X, Y and Z.

$$\left.\begin{aligned} u &= u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot dy + \frac{\partial u}{\partial z}\cdot dz \\ v_1 &= v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}\cdot dy + \frac{\partial v}{\partial s}\cdot dz \\ w_1 &= w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}\cdot dy + \frac{\partial w}{\partial z}\cdot dz \end{aligned}\right\} \tag{36}$$

$$e = \frac{\partial u}{\partial x}\cos^2\alpha + \frac{\partial v}{\partial y}\cos^2\beta + \frac{\partial w}{\partial z}\cdot\cos^2\gamma + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\cos\beta\cos\gamma + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos\gamma\cos\alpha + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\alpha\cos\beta \tag{37}$$

$$\left.\begin{aligned} e_x &= \frac{\partial u}{\partial x}; \gamma_{yz} = \gamma_x = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ e_y &= \frac{\partial v}{\partial y}; \gamma_{xz} = \gamma_y = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ e_z &= \frac{\partial w}{\partial z}; \gamma_{xy} = \gamma_z = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{38}$$

$$e = e_x\cos^2\alpha + e_y\cos^2\beta + e_z\cos^2\gamma + \gamma_x\cos\beta\cos\gamma + \gamma_y\cos\gamma\cos\alpha + \gamma_z\cos\alpha\cos\beta. \tag{39}$$

e_x, e_y, e_z are the unit strains along X, Y, Z; γ_x is the diminution of the angle between the planes XY and XZ, γ_y is the diminution of the angle between the planes YX and YZ; γ_z is the diminution of the angle between the planes ZX and ZY all at the point P.

24. Principal Strains.—Differentiating Eq. (32) and placing the first differential coefficient equal to zero,

$$\frac{\partial e}{\partial \alpha} = 0 = (e_y - e_x)\sin 2\alpha + \gamma_{xy}\cos 2\alpha, \tag{40}$$

which, by comparison with Eq. (33), shows that $\gamma = 0$ for the direction in which e is a maximum or minimum. From (40), these directions are given by the relation

$$\tan 2\alpha = \frac{\gamma_{xy}}{e_x - e_y}, \tag{41}$$

which corresponds to two directions 90° apart.

It follows that, *for stress parallel to a plane, there are at any point two directions, at right angles, for one of which the unit strain is greater, and for the other less, than for any other directions through the same point. These lines or axes are the axes of principal strain at that point. For these axes the shearing deformation is zero.*

Referred to these axes then, $\gamma_{xy} = 0$, and Eqs. (32) and (33) become

$$e = e_x\cos^2\alpha + e_y\sin^2\alpha \tag{42}$$

$$\gamma = (e_y - e_x)\sin 2\alpha. \tag{43}$$

Equation (43) shows that γ is a maximum when $\alpha = 45°$, for which value

$$\gamma = e_y - e_x. \tag{44}$$

Hence the directions in which the shearing strain is greatest bisect the angles between the directions of principal strain.

For
$$\alpha = 135°,$$
$$\gamma = -(e_y - e_x) = e_x - e_y.$$

This means that if, at a point P (Fig. 101) four adjoining prismatic particles are considered, and if PX and PY (and PZ, perpendicular to the paper) are the directions of the principal strains, e_y being greater than e_x, then along PB, at 45° to PX, the shearing distortion is $\gamma = e_y - e_x$, and is positive; that is the angle between PB and PD is *decreased*. This will obviously be the case, since the stretch along PY is greater than that along PX. For $\alpha = 135°$, or along PD, the shearing distortion is $\gamma = e_x - e_y$, or negative, though the same numerically as for PB; that is, the angle between PD and PF is *increased*. This, too, is obvious.

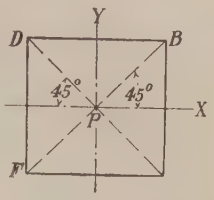


FIG. 101.

When the distortion γ in a direction making an angle α with PX is referred to, it means the change of angle between two lines originally at right angles, one making an angle α with PX measured around to the left as from PX to PB, and the other making an angle $\alpha + 90°$ with PX, measured in the same direction.

For any two directions at right angles with each other, the sum of the values of e is constant, or

$$e + e' = e_x + e_y. \tag{45}$$

25. Relation of Principal Stresses and Principal Strains.—It has been shown that at any point in a body there are three planes on which there is only normal stress, with no shear, and that one of these stresses has a greater intensity and another a smaller, than the intensity on any other plane through the point. For stress parallel to a plane, there are two planes, one with maximum and the other with minimum stress intensity. Lines perpendicular to these planes, through the point, are the axes of *principal stress*.

It has also been shown that at any point there are three lines at right angles, along which there is only direct strain, that is, the strain at the point in the direction of these lines, is along these lines, with no change in the angle between them, or no shearing distortion. These are the axes of *principal strain*.

Since there is no shearing stress on the planes of principal stress, the angles between the axes of principal stress remain unchanged. It follows that at any point *the axes of principal strain coincide with the axes of principal stress*.

CHAPTER VI

PURE TENSION AND COMPRESSION

1. A piece is in pure tension or compression when its axis is and remains straight and vertical, and when it is subjected to two equal and opposite forces, one at each end and uniformly distributed over the section.

If these forces act away from each other, the piece is in tension; if they act toward each other, it is in compression. The resultant force on any cross-section acts at its center of gravity. On any section at right angles to the axis, if the section is constant from end to end (a prismatic piece) there is only tension or compression uniformly distributed over the section. If the section is not uniform, the normal stress may not be uniformly distributed, and there will be shearing stresses. Thus, in the piece shown in Fig. 102, whose axis is straight, and which would be called a piece in pure tension, if a section be taken as shown, at right angles to the axis, but not at the center, it has been shown in Chap. V that at the surface the stress on any plane must be parallel to the surface, hence on the right section it cannot be normal, and there must be shearing stresses at and near the surface, gradually changing to normal at the axis. The applied forces at the ends must of course be parallel to the axis; just what the internal stresses are near the ends, where an adjustment is taking place, cannot be stated.

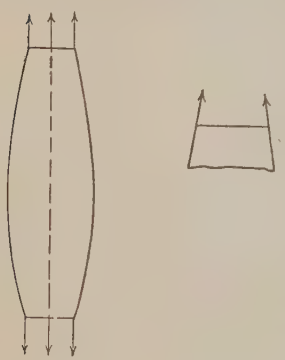


Fig. 102.

In the prismatic piece the stress will be uniformly distributed, and the resultant will be the same on all sections except for the weight of the piece itself.

If the piece is not vertical, but has equal and opposite forces applied at the ends, it is in pure tension or compression as regards those forces, but the weight will cause flexure (if the piece is horizontal) or flexure combined with a direct force (if the piece is inclined). Flexure will be treated in Chap. X and flexure combined with a direct stress will be treated fully in Chap. XIII. For the present the intensity caused by flexure and that caused by direct stress may be added.

If the piece is in compression, its axis will not remain straight, and there will not be pure compression, if the length exceeds a certain ratio to the lateral dimensions. If it bends, the piece becomes a *column*, or a *long column*, and there is flexure combined with compression. Columns will be considered in Chap. XVII.

By the principle of superposition of effects, the dead weight may be considered separately, and the stress which it causes added to that caused by the other forces. It will here be neglected.

Pure tension or compression may also occur when there are other outer forces acting at different points along the piece, in addition to forces at the ends. The

true criterion of pure tension or compression is that at every normal section the resultant acts normal to the section and at its center of gravity; and this condition may be fulfilled even for a curved piece.

2. Piece with Uniform Section. – In this case, with loads only at the ends, the intensity of stress on each section is

$$f_c \text{ or } f_t = \frac{P}{A} \tag{1}$$

this being a particular case of Eq. (8) of Chap. IX. It must not be supposed that in such a piece, however, there is only tension. On any section inclined to the axis there will be a normal force $N = P \cos \alpha$, (Fig. 103) and a tangential force $S = P \sin \alpha$; the intensity of the normal force will be

$$f_n = \frac{P}{A} \cos^2 \alpha = f_t \cos^2 \alpha \tag{2}$$

and the intensity of the shear will be

$$f_s = \frac{P}{A} \sin \alpha \cos \alpha = \tfrac{1}{2} f_t \sin 2\alpha \tag{3}$$

and the same for a piece in pure compression, which does not bend.

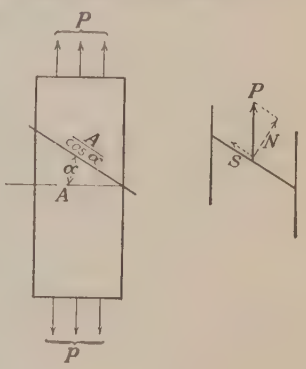


FIG. 103.

The max. value of f_n occurs on the normal section and equals $\frac{P}{A}$; the max. value of f_s occurs for $\alpha = 45°$ and equals $\frac{P}{2A}$, and on this same plane f_n also equals $\frac{P}{2A}$. On a longitudinal plane ($\alpha = 90°$) there is no stress at all.

3. Mode of Fracture in Pure Tension or Compression.—A piece in pure tension, if homogeneous, should fail by breaking across a normal or nearly normal section unless the shearing strength on an oblique plane (at more than 45° with the axis in tension and less than 45° in compression) is so small that it fails there first. A piece of structural steel of rectangular section will often or generally break square across, but the fracture is often beveled along the edges, suggesting that it failed there by shearing first and afterward by tension across the central part of the section. Similarly a piece of circular section will often be cup-shaped on one side, and a truncated cone on the other.

In wood, a clean tension fracture is difficult to obtain, owing to the large tensile strength of the fibers and the small shearing strength. Especially if slightly cross-grained, the fracture will generally be by shear along one or more oblique planes. Difference of homogeneity, irregularities of grain, and differences in adhesion of the cell walls to each other, often cause very irregular fractures, long fibers sometimes pulling out. A pure tension break can only be obtained by making the net section very small and straight grained.

Cement briquettes of standard shape often break cleanly across, and often diagonally or irregularly (see Chap. XXIII).

Short blocks in compression break very differently according to the character of the material. Granular materials, like cement, concrete, brick or stone, generally fail by shearing off of triangular or wedge-shaped pieces at the ends,

leaving a pyramid or wedge at the center. Sometimes the planes of shear slope toward the longitudinal axis, and a single wedge at the end splits the block lengthwise. If the resistance to tensile *strain* is small, the lateral expansion produced by the longitudinal compression sometimes splits the block into several slabs by nearly longitudinal cracks.

Wood fails in compression sometimes by shearing along a diagonal plane or planes, not resulting in a clean break and separation, but in a bending and crumpling of the fibers; and sometimes by a breaking-down and crumpling of the fibers at the outside. Compression across the grain simply makes the cells collapse; there is, strictly speaking, no ultimate strength, but a progressive yielding; the assumed ultimate is at some arbitrary yielding.

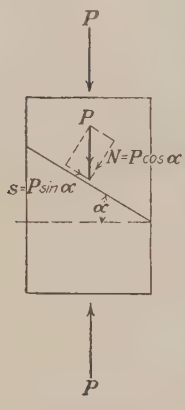


FIG. 104.

Ductile materials, like steel and wrought iron, will compress longitudinally and expand laterally, and the point of failure may be very uncertain. Sometimes the lateral bulging results in radiating cracks from the center toward the outside. Sometimes the fibers will crumple or buckle, especially if there are thin outstanding legs, as of angle irons. A long column will fail by bending as a whole, and then buckling of some part locally; such pieces are treated in Chap. XVII.

Cast iron will fail in compression generally by shearing along an oblique plane, separating into two parts.

Cylinders of ductile material will "wrinkle" on the surface.

In considering the shearing fracture of granular materials in compression, it must be noted that on an oblique plane there is not only shear, but also a frictional resistance depending upon the normal force. Thus (Fig. 104), if f_s is the ultimate strength in shearing, and

μ is the coefficient of friction,

then, in order to fail, the shearing force $S = P \sin \alpha$ must fulfil the equation

$$P \sin \alpha = f_s A \sec \alpha + P \cos \alpha \cdot \mu. \tag{4}$$

From this it follows that

$$P = \frac{f_s A}{\sin \alpha \cos \alpha - \mu \cos^2 \alpha}. \tag{5}$$

The value of α for the plane on which sliding will occur is the value which will make P a minimum; hence by differentiating the above value of P and making $\frac{dP}{d\alpha} = 0$, we find

$$0 = \cos 2\alpha + \mu \cdot \sin 2\alpha$$

or

$$\cot 2\alpha = -\mu$$

$$\mu = -\cot 2\alpha = \tan (2\alpha - 90°) = \tan \varphi$$

if φ is the angle of friction ($\tan \varphi = \mu$).
Hence

$$\varphi = 2\alpha - 90°$$

$$\alpha = 45° + \frac{\varphi}{2}. \tag{6}$$

This means that α *is greater than* 45° *by* $\frac{\varphi}{2}$, while in a tension piece the tendency to shear is greatest on a plane for which $\alpha = 45°$. It is very probable, however, that there is what may be called a negative friction on an oblique plane, in a pure tension piece: that is, a tensile stress diminishes the resistance to shearing, just as a compressive stress increases it, so that the tendency of a tensile piece to fail by shear is greatest on a plane less steep, than 45°, *i.e.* $\alpha < 45°$, or the angle with the axis $> 45°$. Tests of steel show this angle with the axis to be about 55°. In other words, Eq. (5) may hold for both tension and compression, μ being positive for compression and negative for tension. See "Laws of Failure of Solid Bodies due to Stress" by Chido Sunatani, in Technology Reports of the Tohoku Imperial University, Vol. III, No. 1, November, 1922.

From Eq. (5), dividing by A, and calling $\frac{P}{A} = f_c$, which is the ultimate crushing load divided by the area of the cross-section, or the ultimate crushing strength,

$$f_s = f_c(\sin \alpha \cos \alpha - \mu \cos^2 \alpha). \tag{7}$$

$$\text{But } \mu = \tan \varphi = -\cot 2\alpha = -\frac{\cos^2 \alpha - \sin^2 \alpha}{2 \sin \alpha \cos \alpha}. \tag{8}$$

From this it follows that

$$f_s = \frac{f_c \cos \alpha}{2 \sin \alpha}$$

$$f_c = 2f_s \tan \alpha. \tag{9}$$

The following table is given in Johnson's "Materials of Construction" as the results of tests by Charles Bouton at Washington University, in 1891:

Material	Number of experiments	Observed angle of rupture α (degrees)	Observed angle of repose φ (degrees)	Theoretical angle of rupture $45 + \frac{\varphi}{2}$ (degrees)	Difference (degrees)
"F" cast iron	24	54.8±0.2	20.6	55.3	−0.5
"C.W." cast iron	24	55.0±0.2	16.9	53.4	+1.6
Limestone	4	62.2	33.4	61.7	+0.5
Asphalt paving mixture	3	59.7	27.3	58.6	+1.1
Milwaukee brick	4	58.2	27.0	58.5	+0.3

"F" was good foundry iron, with a tensile strength of 22,000 pounds per square inch and a modulus of elasticity of 14,500,000 pounds per square inch. "C.W." was car-wheel iron with values of respectively 20,000 and 6,500,000 pounds per square inch.

4. Tension Piece with Notch (Fig. 105).—In this case the total force on any normal section is P, and if the notch is symmetrical P acts at the c.g. of the section, but is not necessarily uniformly distributed over it. On a section *aa* the total stress is the same as on the section *bb*; but it is unreasonable to suppose that the stress from the outer fibers of *bb* can be transmitted to *aa* and uniformly distributed, if *aa* is very near to *cc*. If the notch is short, tests indicate that the stress on a section *aa* is greatest at the outside and least at the center, due to the

crowding of the stress from the larger section *bb* into the smaller section *aa*.[1]

If the notch is curved (a grooved piece) (Fig. 106), instead of rectangular, the transmission of stress will be easier, and the excess at the edges less. In the rectangular notch the excess at the edges is greatest toward the ends of the notched portion. With a long notch the stress at the center of the length may be distributed with tolerable uniformity. If the notch is triangular, the excess at the edges will be great.

The percentage of increase at the edges of a notch will depend upon the depth, shape, and length of the notch, and hence cannot be stated generally. The subject is very uncertain.

But while a notched specimen shows often a great increase of *stress* at the edges of the notch, it is obvious that the fibers there cannot stretch as freely as they could if there were no notch. In Fig. 105 for instance, the fibers at *d* can stretch but little, because they are restrained by the fibers next outside, in *cd*, which have no stress. Hence the *strength* of the piece at *d* is much increased, perhaps even more relatively than the increase of stress. Hence there may be no actual *weakening* due to the unequal distribution of the stress on *dd*. There may even be a strengthening.

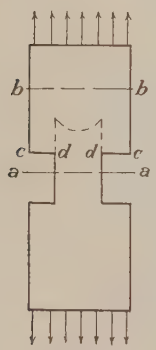


Fig. 105.

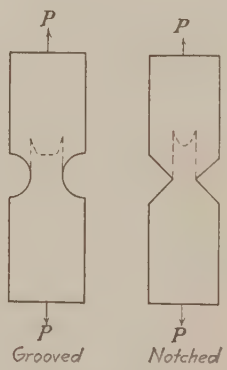


Fig. 106.

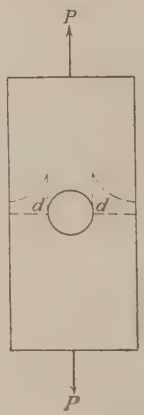


Fig. 107.

5. Punched Sections, or Sections with Holes.—Tension pieces of steel frequently have rivet holes. Sometimes there are holes for other purposes, like hand-holes and manholes in boilers or tanks (Fig. 107). Such a piece will be weakest in the section across the hole or line of rivets, and there will also be an excess of stress on this section at the edges of the hole, *d*. There is no satisfactory way of computing this excess; obviously it will depend on the diameter of hole and the ratio of this diameter to the width of the piece, being greater the greater this ratio and the greater the diameter. In this case also, as in the case of the notch, there will be an increase of *strength* at the edges of the hole, which will make up, in part or in whole, for the excess of stress.

[1] See Preuss, "Zeitsch. des Vereins deutscher Ingenieure," 1913, pt. 1, p. 664; and "Mittheil. über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens," vol. 126, p. 54. These results are quoted, with diagrams, in the 5th ed. of Johnson's "Materials of Construction," p. 663.

The case is evidently the same if the piece is in compression. The stresses in various forms of riveted connections are considered in Chap. XVI. The weakest section, as there shown, is not always the one with the largest number of rivets, because the total force on the sections may not be the same on all sections of a piece which is connected to another piece.

No satisfactory theoretical solution of the case of a piece with a hole in it has ever been made, to the knowledge of the writer. Measurements of stress are also open to doubt, though Mr. Preuss (*loc. cit.*) gives results of such measurements. One theoretical investigation claims to give the stresses in such a case (see Fig. 108), giving the following formulae,[1] which are supposed to hold below the elastic limit:

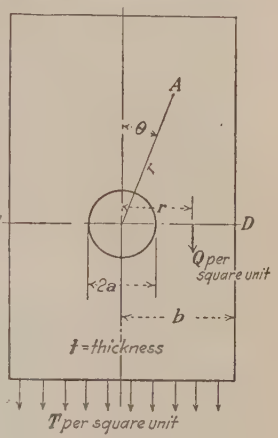


Fig. 108.

$$\left.\begin{aligned} P &= \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T}{2}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta \\ Q &= \frac{T}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos\ 2\theta \\ S &= \frac{T}{2}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta \end{aligned}\right\} \quad (10)$$

in which

P = normal stress intensity at A on a plane at right angles to r.
Q = normal stress intensity at A on a plane in the line of r.
S = shearing stress intensity at A on either of the above planes.
T = uniform stress intensity on the gross section.

The stress on the gross section is assumed uniformly distributed, and the width of the plate infinite. This is a case of stress parallel to a plane, and the thickness makes no difference.

For the section CD, $\theta = 90$, and the formulae become

$$\left.\begin{aligned} P &= \frac{3T}{2}\left(\frac{a^2}{r^2} - \frac{a^4}{r^4}\right) \\ Q &= \frac{T}{2}\left(2 + \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right) \\ S &= 0. \end{aligned}\right\} \quad (11)$$

At the edge of the hole, $a = r$, and

$$\left.\begin{aligned} P &= o \\ Q &= 3T \end{aligned}\right\} \quad (12)$$

or the stress intensity at the edge of the hole is three times the average stress in the gross section.

The formula does not depend upon the size of hole, if the stress on the edge of the hole is being considered. No matter if the hole is microscopic, the stress at its edge is three times that in the gross section (below the elastic limit). At the limit, then, this formula proves that $3 = 1$.

[1] Föppl, "Technische Mechanik," vol. 5, p. 352.

Suyehiro, "The distribution of stress in plates having discontinuities, and some problems connected with it," *Engineering*, Sept. 1, 1911.

Coker and Scoble, "The distribution of stress due to a rivet in a plate," *Trans. Inst. Naval Arch.*, March, 1913.

The writer therefore considers these formulae erroneous, and it is unnecessary to give their derivation.

It is obvious that if Q is the stress intensity at any point in the net section through the center of the hole, and T that in the gross section, the ratio $\frac{Q}{T}$ must be a function of the width of the plate and the diameter of the hole; and that this function must be such that when the diameter of the hole becomes zero, the value of the ratio becomes unity. Any formula which does not give this result *must* be erroneous.

Equation (10) may also be proved to be incorrect by another method. For the section CD the value of Q in Eq. (11) is the stress intensity at any point distant r from the center of the hole. Integrating this over the entire section, we must obtain the total pull on the section; or

$$P = 2\int_a^b Qtdr = Tt\left(2b - \frac{a^2}{b} - \frac{a^4}{b^3}\right).$$

But this *must* obviously equal $2Ttb$. Hence Eq. (11) can only be correct, if at all when $\frac{a^2}{b}$ and $\frac{a^4}{b^3}$ are zero, that is, when either $a = 0$ or $b = \infty$. Hence it only holds, if at all, for a microscopic hole in any finite plate, or really for no hole at all. Hence Eq. (12) means that this theory proves that $3 = 1$; because with a microscopic hole (no hole at all) the stress at the edge of the hole is by this theory $3T$, while we know that in such a case it is T.

The statement that a microscopic or infinitesmal hole is the same as no hole at all is inaccurate in this case, for Eqs. (10) are based on the assumption that there is a *discontinuity*, that is to say, *some hole*. The situation, therefore, appears to be this: those equations require that there must be a hole, but they are not correct, if at all, unless there is no hole at all. And it is undeniably true that they are not correct in any practical case whatever. They are based on the two assumptions: (1) that there is a hole (discontinuity), and (2) that there is no hole. The reader may judge whether such methods merit confidence. They are an abuse of mathematics, and of common sense as well.

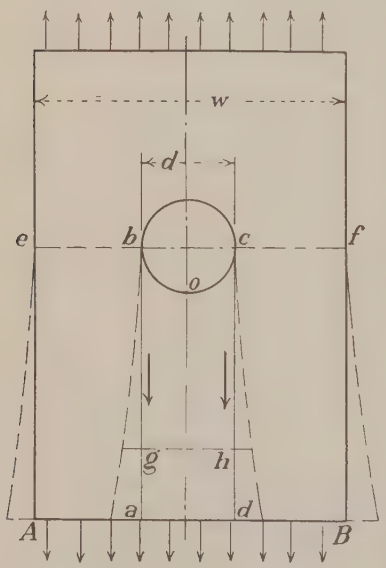


Fig. 109.

Further, for $\theta = 0$, and $a = r$, that is, at the upper and lower edges of the hole, the formula gives $Q = -T$, *i.e.*, there is at those points a *compression* equal to T; and as this only holds for a microscopic hole, it means that in a solid plate subject to longitudinal tension only there is a lateral compression of equal intensity, whereas we know that there is no compression at all. Perhaps in this may be found the fallacy in the theoretical demonstration, but the writer has not gone through with it. He has no time for such illusory mathematical recreations. And yet the writer has seen this theory used and supported by testing experts as a basis for an important legal decision.

The problem may be looked at from still another point of view. In Fig. 109 the part *abcd* tends to pull out, and hence there are shearing stresses on *ab* and

cd. If the stress is uniformly distributed over AB, and its intensity is f_t, and if the hole is in the center of the plate, each of these shearing forces is $f_t \frac{dt}{2}$. On eb there is a direct tension of intensity $\frac{f_t w}{w-d}$, and a bending moment, which, if the shear is the *only* force causing it, is $f_t \frac{dt(w-d)}{8}$. This moment produces a max. intensity of tension at b, and the total intensity there will be found to be, inside the elastic limit,

$$f_{\max.} = f_t \frac{w+3d}{w-d}. \tag{13}$$

This will vary with w and d. If $w = 4d$, $f_{\max.} = 2\frac{1}{3} f_t$. If $w = 10d$, $f_{\max.} = 1\frac{1}{9} f_t$. As w increases, $\frac{f_{\max.}}{f_t}$ decreases.

But this is not correct. For the bending moment will tend to make the part $ebaA$ bend as shown by the dotted lines, and similarly for the part on the other side of the hole. This will bring into action tensile forces on ab and cd, which will tend to counteract the moment caused by the shears. There will, however, still remain some excess stress at b and c, dependent on the fact that $ebaA$ will actually bend as indicated, to such extent that the horizontal movement at any

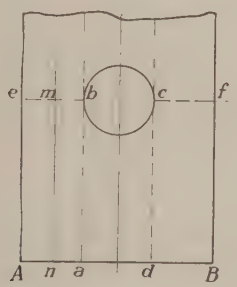


Fig. 110.

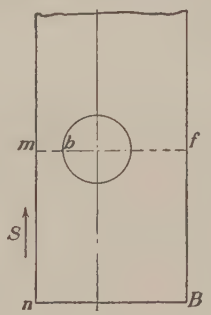


Fig. 111.

point g is one-half the stretch of gh under the tension developed. It is clear, however, since this tension acts, that $f_{\max.}$ is *surely* less than given by Eq. (13). Also, if the plate in question were part of the lower ring of a tank, and the hole were a manhole, the ring would be riveted along say eA to the next ring above, and this connection would resist the tendency of $abeA$ to bend.

A rather interesting question is, supposing a hole in the middle of a plate, with definite stresses on eb, the max. being at b, what would be the effect of removing a strip on the left. This will be clear from the following point of view. The shear on ab (Fig. 110) tends to pass as tension into the next strip; this tends to stretch that strip; this tends to throw part of the shear into the adjacent strip, and so on to the edge of the plate. It is obvious that the effect of the hole will be to increase the tension over the whole of eb, though the increase will be greatest at b. The separate piece $mfBn$ (Fig. 111) will be in exactly the same condition as the same portion of the wider plate (Fig. 110), if along nm is added the shearing force S actually acting along nm in Fig. 110. This force S will be distributed on

mf, causing direct compression and bending, and it is obvious that *S* will cause *compression* on the whole of *mb*. Hence if *S* does not exist, *i.e.*, in the free plate *mfBn*, the stress at *b* will be *larger* than the stress at *b* in the wider plate *efBA*.

Attempts have been made to measure the stresses on a section through a hole, but the writer has seen no experimental results that he considers quantitatively reliable. The experiments of Preuss have been already referred to. Professor Coker has recently succeeded in showing the concentration of stress at the edge of a hole by optical methods,[1] "based on the discovery of Sir David Brewster, in 1816, that when a piece of glass is loaded and viewed in polarized light under suitable conditions it shows brilliant color effects due to the internal stresses produced in the material." Professor Coker uses celluloid (xylonite) as the transparent material to be observed under stress. By his method a beam of light is passed through a Nicol prism, from which it emerges as plane polarized light; it is then passed through a quarter-wave plate of mica, with its axis at 45° to the plane of polarization, which makes the light circularly polarized; it is then passed through the stressed xylonite, and emerges as two polarized components at right angles and in the planes of principal stress. One of these components is retarded more than the other, depending only on the *difference of the principal stresses* for a given thickness, so that the ray is either circularly polarized (when the two principal stresses are equal or when there is no stress), plane polarized (when the difference of retardation results in a difference of phase of $\frac{1}{4}$ wave length) or elliptically polarized (when the difference of phase is an odd eighth wave length). The light is then passed through a second quarter-wave plate having its axis at 90° with the first, and then through a second Nicol prism having its axis at 90° with the first, the result being that if the retardation in passing through the specimen results in a difference of phase of the vibrations, in the planes of the principal axes, of an integral wave length, for a monochromatic color, the light is all cut out, for half wave lengths it is all transmitted, and for intermediate fractions of a wave length it is transmitted with varying intensity. If the principal stresses are equal, or if there is no stress, or if the difference in phase is a wave length, that color is all cut out and black results. But different colors are retarded different amounts by the same stress, so that the result with white light is a series of bands of color; the color depending only on the difference of principal stresses (for a given thickness). The color can therefore give only, if it gives anything, the difference of the principal stresses. Professor Coker finds the sum of the principal stresses by measuring the change of thickness, and using the law given by Poisson's ratio, and thus finds each principal stress. If one principal stress is zero, as it must be at a free edge of a plate, the color observation, he thinks, gives the actual other principal stress, which must be parallel to the edge (see Art. 5, Chap. V). But since the color depends only on the difference of phase of the vibrations parallel to the principal axes, and not upon the actual retardation, there must

[1] Coker, Chakko and Sataka, "Photo-elastic and strain measurements of the effect of circular holes in the distribution of stress in tension members," *Trans. Inst. of Engineers and Shipbuilders in Scotland.*

Coker, "Photo-elasticity for engineers," *General Electric Review*, Nov. and Dec., 1920 and Jan., 1921.

Heymans, "La Photo-élasticimétrie," *Societé Belge des Ingenieurs et Industriels, Bull.*, vol. 11, No. 2, 1921.

The optical examination of materials under stress is not new. In Dec., 1872, Louis Nickerson, M. Am. Soc. C.E., presented a paper (vol. III, p. 31) entitled "A record of experiments showing the character and position of neutral axis, as seen by polarized light," the experiments having been made with glass specimens.

See also, M. Mesnager, "Utilization de la double réfraction accidentelle du verre à l'étude des efforts intérieurs dans les solides, *Ann. d. Ponts et Chaussées*, 1913, p. 137. The same author published an earlier paper in 1901. See bibliography in Heymans, *loc. cit.*

be an indefinite number of values of the difference of principal stresses which would give the same color effect. The writer, therefore, is unable to see that these experiments can give quantitative results, though they may give relative results in some cases. Professor Coker himself says "it is quite possible for a material to be highly stressed at a place, and yet exhibit little or no color effect," this referring, no doubt, to the fact that with equal principal stresses there would be no color. He also says "there is no certain way of deciding whether the part under inspection is in tension or compression." These difficulties he claims to have overcome, by neutralizing the color by a test piece under a given stress placed in front of the specimen under examination, the stress in the test piece being assumed the same as that in the specimen when there is exact neutralization.

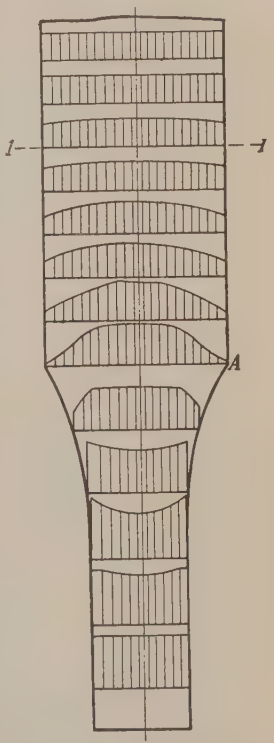
The writer, however, is not yet prepared to accept these results. Many of them are interesting, while some seem absurd. He does not believe that an actual piece under stress can be examined by this method and the stress determined. It may be that in laboratory tests a piece may be gradually stressed and the number of color phases passed through noted, and then the control piece made to pass through the same number of phases, and some reliable results obtained. The field is a good one for research, and the name of Professor Coker is a guarantee that it is promising. Professor Coker's apparatus has been set up by the General Electric Co. in Schenectady, N. Y., for experimental purposes.

Professor Coker finds, for a plate with a hole (Fig. 109) a tension at b about three times the average in the gross section, and a *compression* at o approximately equal to that average. This last result agrees with the second of Eq. (10), which becomes, making $a = r$

$$Q = T(1-2\cos 2\theta)$$

and purports to give the stress around the edge of the hole. For $\theta = 0$ this gives $Q = -T$.

Professor Coker also claims that the tension along the straight edge of the plate with a hole is *less* opposite the hole than anywhere else, and thinks it possible that it might even be compression if the hole is sufficiently large. In one plate 5.2 inches wide with a 5-inch hole, he found, as he states, the stress at the edge zero, and a linear variation up to the hole. For elliptical holes he approves the formula giving the max. stress at the edge of the hole as $T\left(\frac{1+2a}{b}\right)$[1] where a is the axis of the ellipse perpendicular to the line of pull and b is the axis along the line of pull; this formula approaching an infinite value for the stress as b decreases, which would mean that a piece with a very short microscopic crack extending through its thickness at right angles to the pull would carry no load at all. That this is not true is abundantly proved by the experience of bridge and testing engineers. He thinks, however, that in drilling a hole at the end of a crack in steel, as is sometimes done to prevent the crack from extending, it would be better to make an elliptical hole with the long axis perpendicular to the crack, which is perhaps true.



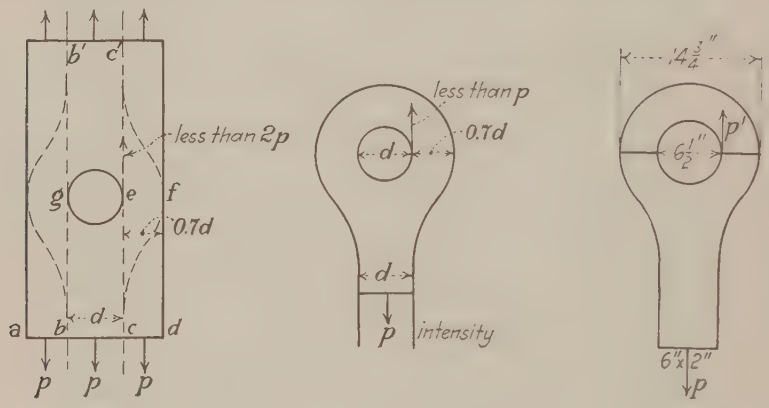
Taken from "Photo Elasticity for Engineers," F. G. Coker, General Elec. Review, Jan. 1921.

FIG. 112.—Distribution of normal stress.

[1] INGLIS, "Stresess in a plate due to the presence of cracks and sharp corners," *Proc. Inst. Naval Arch.*, 1913.

Professor Coker finds, for the standard British tension test piece (Fig. 112), that at a section 1-1 the stress is a max. at the center, that it remains so for some distance below the point A, and becomes a max. at the edges a little farther down.

6. The stress in a plate with a hole may be further examined in the light of experience with eye bars, which are rectangular bars increased in width at the ends, through which pins are passed for connecting to other parts (Fig. 112*a*). An eye bar is therefore a kind of plate with a hole in it. Many tests have shown that the net area through the hole should exceed the cross-section of the bar, and that if the excess width at the hole is 33 to 40 per cent of the width of the bar, such a bar will not break at the pin hole, but will break in the body of the bar, even if the diameter of the pin equals the width of the bar, or somewhat exceeds it. Taking the diameter of the pin equal to the width of the bar, and using 40 per cent excess, the width on each side of the pin will be 0.7 d, and if such a bar does not break at the pin, it means that the stress intensity at the edge of the hole, p, is less than that in the bar.



FIGS. 112*a*, *b* and *c*.

The plate with a hole may be divided into three parts, ab, bc, and cd (Fig. 112*b*). The stress in ab and in cd may be considered to go straight by the hole, causing a uniform stress intensity, p, on the net section ef. The stress in bc must go around the hole, just as the stress in the eye bar goes around the pin; but it does not go clear around, but curves back into the area $b'c'$ above the hole. If ef is not less than 0.7 bc, the additional stress at the edge of the hole, due to the stress in bc being diverted around the hole, must be less than p, so that the total stress at e must be less than $2p$, and must decrease as the width of the plate increases in proportion to the diameter of the hole. This point of view seems perfectly sound. Of course, the stress at e is probably not just the same if there is an empty hole as it is if there is a pin or rivet in the hole; but if the theory by which it is claimed that the stress at the edge of a hole only holds when the hole is empty, then of course it does not apply at all to riveted joints, and its application is of little value, for empty holes are seldom found, and if they occur, as in manholes in boilers, the hole is reinforced by a ring or nozzle, which has an effect similar to a pin or rivet in opposing deformation of the hole by diminishing the diameter ge

and increasing the longitudinal diameter. A hole should always be so reinforced. Certainly, if the theory held for eye-bars and riveted joints, there are many structures which have been standing for years which would have collapsed long ago.

The above results may be corroborated from another point of view. Assume an eye-bar 6×2 inches Fig. 112c; the maximum diameter of pin is $6\frac{1}{2}$ inches (though this may be exceeded if absolutely necessary), and the total width across pin, for this diameter, is $14\frac{3}{4}$ inches, thus leaving a net width of $8\frac{1}{4}$ inches, or $4\frac{1}{8}$ inch on each side of the pin. The average stress on this net section, if p is the ultimate in the bar, is $\frac{6}{8.25}\,p = 0.727\,p$. The excess of net width over that of the bar is $2\frac{1}{4}$ inches, or 37.5 per cent. The maximum stress at the edge of the hole must be less than p, since tests and experience show that such a bar breaks in the body and not at the pin. Call the stress at the edge of the hole p': then $p' < p$; and,

$$p = 1.375 \times \text{(average on net section through pin)}$$
$$p'(<p) < 1.375 \times \text{(average on net section through pin)}.$$

Now suppose, as in a case in the writer's practice, a plate had a hole in it whose diameter was $\frac{1}{4}$ the width. Then if p was the stress on the gross section, the average on the net section was $\frac{4}{3}\,p$; and, according to the above ratio

$$\text{stress at edge of hole} = p' < 1.375 \times \tfrac{4}{3}p < 1.8\,p.$$

In this case it was claimed by the other side that the stress at the edge of the hole was $3p$, or greater.

7. If it is said that the stresses given by eqs. 10, 11, and 12, which are found by the theory of elasticity under the erroneous assumption stated, only hold up to the elastic limit, and that above that limit a redistribution occurs, it may be replied that a stress above the elastic limit indicated by these formulae would at each application raise that limit higher and higher, up to nearly the ultimate, and hence that the formulae would finally apply up to far above the original elastic limit. If this is not true, then the only other conclusion possible is that an indicated or theoretical stress above the elastic limit, as deduced by the theory of elasticity, is of no importance whatever as indicating any danger of failure.

This discussion has been given because the subject is attracting some attention, but it appears to show that the engineer must be cautious in accepting the results of purely mathematical investigations, or even the results of experiment.

8. Screw Threads.—When a round bar has a screw thread cut at the end, it becomes a grooved specimen. The shape of the groove will be different according to the form of thread; it may be a V-shaped triangular notch, or the apex of the triangle may be curved as in the Whitworth thread, or may have the apex cut off as in the Sellers or German threads (Fig. 113). Screw threads have been tested by Prof. Martens of Berlin, and the results are given in Johnson's "Materials of Construction," pp. 663–665. The diameter of bars were $\frac{1}{2}$ and 1 inch.

Owing to the strengthening effect above referred to, it was found that the bars, when subjected to plain tension, were actually about 14 per cent stronger than plain bars with the same net area. There was no marked difference in the strengh of the different forms of thread. When the load was put on by turning the nut, thus introducing torsional stress, the strength was reduced by 15 to 20 per cent.

To allow for applying the load to threaded bars by turning the nut, and also to allow for the effects of repeated applications of the load, incipient cracks, etc. it is common, when cutting threads or tension bars, to upset or enlarge the ends and cut the threads on these enlarged ends; and the enlargement is made such that the diameter at the root of the thread is somewhat greater than that of the body of the bar. The following tables give the usual standard dimensions used for

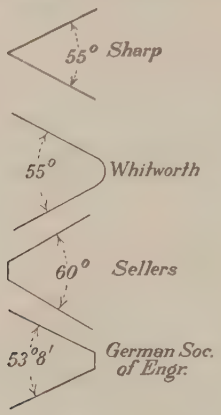
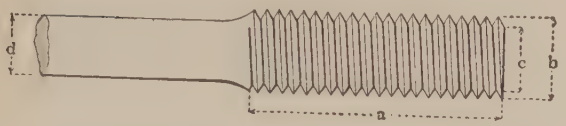


Fig. 113.

round or square bars. When a thread is cut on a round bar whose ends have not been upset, the net or effective size of the bar should correspond to these tables; that is to say, if a thread is cut on a round bar 1 inch in diameter, not upset, look in the table for $b = 1$ inch and take the net size for the corresponding value of d. or ¾ inch diameter, although the diameter at the root of the thread would be 0.838 inch.

Upset Screw Ends for Round Bars
(American Bridge Company standard)

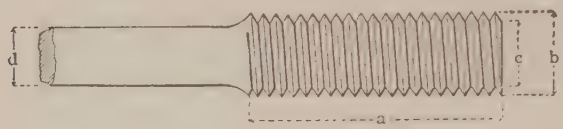


Pitch and shape of thread A. B. Co. standard

Bar			Upset					
							Area	
Diameter *d*, inches	Area, square inches	Weight per foot, pounds	Diameter *b*, inches	Length *a*, inches	Additional length for upset +10 per cent, inches	Diameter at root of thread *c*, inches	At root of thread, square inches	Excess over area of bar, per cent
* ¾	0.442	1.50	1	4	4	0.838	0.551	24.7
* ⅞	0.601	2.04	1¼	4	5	1.064	0.890	48.0
1	0.785	2.67	1⅜	4	4	1.158	1.054	34.2
1⅛	0.994	3.38	1½	4	4	1.283	1.294	30.2
1¼	1.227	4.17	1⅝	4	4	1.389	1.515	23.5
1⅜	1.485	5.05	1¾	4	4	1.490	1.744	17.5
1½	1.767	6.01	2	4½	4½	1.711	2.300	30.2
1⅝	2.074	7.05	2⅛	4½	4	1.836	2.649	27.7
1¾	2.405	8.18	2¼	5	4	1.961	3.021	25.6
1⅞	2.761	9.39	2⅜	5	4	2.086	3.419	23.8
2	3.142	10.68	2½	5½	4	2.175	3.716	18.3
2⅛	3.547	12.06	2⅝	5½	3½	2.300	4.156	17.2
2¼	3.976	13.52	2⅞	6	4½	2.550	5.108	28.4
2⅜	4.430	15.06	3	6	4½	2.629	5.428	22.5
2½	4.909	16.69	3¼	6½	5½	2.879	6.509	32.6
2⅝	5.412	18.40	3¼	6½	4½	2.879	6.509	20.3
2¾	5.940	20.19	3½	7	5½	3.100	7.549	27.1
2⅞	6.492	22.07	3¾	7	6	3.317	8.641	33.1
3	7.069	24.03	3¾	7	5	3.317	8.641	22.2
3⅛	7.670	26.08	4	7½	6	3.567	9.993	30.3
3¼	8.296	28.21	4	7½	5	3.567	9.993	20.5
3⅜	8.946	30.42	4¼	8	5½	3.798	11.330	26.6
3½	9.621	32.71	4¼	8	5	3.798	11.330	17.8
3⅝	10.321	35.09	4½	8½	5½	4.028	12.741	23.4
3¾	11.045	37.55	4¾	8½	6	4.255	14.221	28.8
3⅞	11.793	40.10	4¾	8½	5½	4.255	14.221	20.6

Upsets marked * are special.

Upset Screw Ends for Square Bars
(American Bridge Company standard)



Pitch and shape of thread A. B. Co. standard

Bar			Upset					
							Area	
Side of square *d*, inches	Area, square inches	Weight per foot, pounds	Diameter *b*, inches	Length *a*, inches	Additional length for upset +10 per cent, inches	Diameter at root of thread *c*, inches	At root of thread, square inches	Excess over area of bar, per cent
* ¾	0.563	1.91	1⅛	4	4	0.939	.0693	23.2
* ⅞	0.766	2.60	1¼	4	3½	1.064	0.890	16.2
1	1.000	3.40	1½	4	4	1.283	1.294	29.4
1⅛	1.266	4.30	1⅝	4	3½	1.389	1.515	19.7
1¼	1.563	5.31	1⅞	4½	4½	1.615	2.049	31.1
1⅜	1.891	6.43	2	4½	4	1.711	2.300	21.7
1½	2.250	7.65	2¼	5	5	1.961	3.021	34.3
1⅝	2.641	8.98	2⅜	5	4½	2.086	3.419	29.5
1¾	3.063	10.41	2½	5½	4½	2.175	3.716	21.3
1⅞	3.516	11.95	2¾	5½	5	2.425	4.619	31.4
2	4.000	13.60	2⅞	6	5	2.550	5.108	27.7
2⅛	4.516	15.35	3	6	4½	2.629	5.428	20.2
2¼	5.063	17.21	3¼	6½	5½	2.879	6.509	28.6
2⅜	5.641	19.18	3½	7	6½	3.100	7.549	33.8
2½	6.250	21.25	3¾	7	7	3.317	8.641	38.3
2⅝	6.891	23.43	3¾	7	5½	3.317	8.641	25.4
2¾	7.563	25.71	4	7½	6½	3.567	9.993	32.1
2⅞	8.266	28.10	4¼	8	7½	3.798	11.330	37.1
3	9.000	30.60	4¼	8	6	3.798	11.330	25.9
3⅛	9.766	33.20	4½	8½	7	4.028	12.741	30.5
3¼	10.563	35.91	4¾	8½	7½	4.255	14.221	34.6

Upsets marked * are special.

CHAPTER VII

SHEARING

1. Action of a Shearing Force.—If the resultant force acting on a plane surface be resolved into two components, one normal to the surface and the other parallel to the surface, the latter is the shearing force on the surface. Its *average* intensity will be the force divided by the area of the surface. There is no way of determining its distribution except by considering the normal forces. *Pure shear exists* on a surface when there is no normal force on that surface.

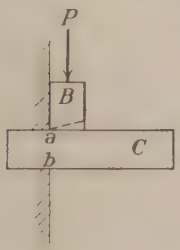


FIG. 114.

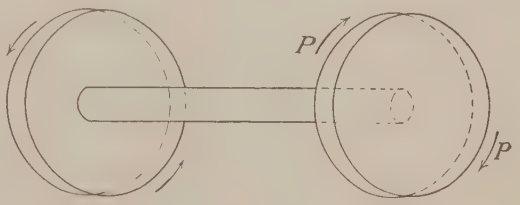


FIG. 115.

Shear alone cannot exist in a body. Pure shear, as shown in Chap. IV is always accompanied by both tension and compression, which are caused by the shear; and it is generally accompanied also by bending, which is brought into action when the shear is brought into action (but not caused by the shear). If a piece C (Fig. 114) is held rigidly at the left end, and a piece B is forced down upon it by a force P, there will be a tendency to develop pure shear on the plane ab. But this requires a compressive force P on the bottom of B, and this force cannot be

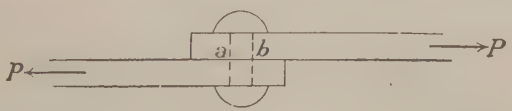


FIG. 116.

concentrated on the left edge of B, for that would mean an infinite compressive stress for square unit; hence the compressive force must be distributed over a certain area, both of B and C. Hence, while the shearing force on ab is P, there is also a bending moment, and C will be bent down. This would tend to throw the compressive force P, between B and C, still more to the left, if B is constrained to remain vertical, without rotation; but P can never be actually concentrated, even if B is bevelled on the bottom as shown by the dotted line, for the compression on the edge of B would deform it and make a small flat surface. The same

thing holds for the reaction of C on the support at b. Hence the shear will be accompanied by some bending, depending on the deformation of the material.

Pure shear can only be produced by torsion. If a rod be subjected to opposite twisting moments at the ends (Fig. 115), then, on any section there will be no normal force, and no resultant shear, but there will be shearing stresses on the section (in different directions) giving a resultant moment about the axis equal to the moment of the two forces P applied at each end. There will be no bending.

The nearest approach to pure shear, with shearing stresses parallel and in the same direction, is probably the action of a pair of scissors. In structures, the

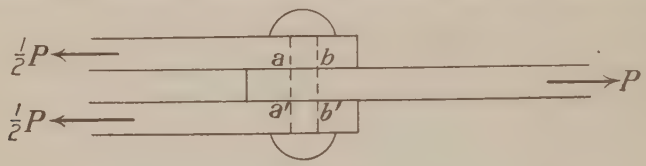


Fig. 117.

nearest approach is where two pieces are connected by a rivet, or rivets, as in Figs. 116 and 117 where the forces P cause shearing on ab. Yet even here, in Fig. 116, the forces P are not in the same line, hence a bending moment exists.

Where a force P is resisted by shearing on a single surface, as in Fig. 116 it is called *single shear;* where it is resisted by equal shearing on two surfaces, as in Fig. 117 it is called *double shear*. There is some bending on the rivet in each case, but it is seldom or never taken account of in computation, in the case of rivets, though it is in the case of pins, as will be seen later.

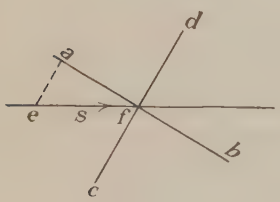


Fig. 118.

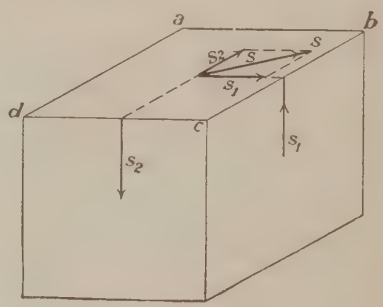


Fig. 119.

2. Pure Shear Cannot Exist without Tension and Compression.—The reader should here review Arts. 9, 10 and 11 of Chap. IV. It may be well to repeat here the fundamental facts.

If there is a shear on any given plane at a point, there must be a shear on every plane through the point, at right angles to the first plane, *except on a plane parallel to the direction of the first shear*. On any two planes at right angles, the component of the shearing stress at right angles to the line of intersection of the two planes, must have the same intensity, and both must act towards that line of intersection, or both away from it. Thus, in Fig. 118, if there is, at f, a shearing force of intensity $s = ef$ on the plane of the paper, that stress may be resolved

into two components, *ea* and *af*, perpendicular to any two planes *ab* and *cd*, at right angles to each other and to the plane of the paper. There must then be a shear of intensity *ea* on plane *ab* and a shear of intensity *af* on *cd*. This will be seen from Fig. 119, where s is shown acting on the top of the cube, and it is resolved into s_1 and s_2, perpendicular to *ad* (or *bc*) and *ab*. There must then be shears s_1 and s_2 acting on the planes shown, otherwise the cube would rotate about one of its rectangular axes. (See also Art. 7, Chap. V.)

Tension alone may exist in a body, at a point, only if the two principal stresses are equal tensions; and similarly for compression. In pure tension or compression, so-called, there is shear of the same intensity as the tension on diagonal planes at 45° to the shear (Chap. VI). If there is pure shear, there must be tension and compression. If there is shear combined with normal stress, on any plane, there may or may not be the opposite kind of normal stress on other planes (Chap. V).

3. An Erroneous Conclusion.—It may at first sight be supposed that by the same methods we may prove that, with stress parallel to the plane of the paper,

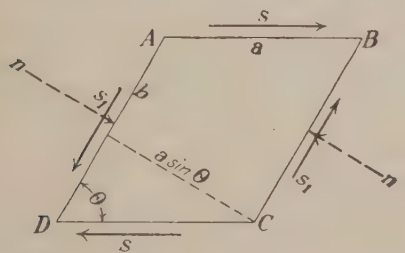


Fig. 120.

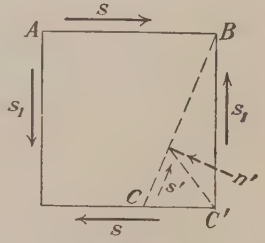


Fig. 121.

the shearing intensities on *any* two planes AB on BC (Fig. 120) are equal; for if there is only a unit shear s on AB and DC, and s_1 on AD and BC, the total shear on AB is sa, and its moment about D is $s \cdot ab \sin\theta$, while the total shear on BC is $s_1 b$ and its moment about D is $s_1 ba \sin\theta$. If these were the only forces acting, $sab \sin\theta = s_1 ba \sin\theta$, or $s = s_1$.

This, however, is fallacious. The error consists in assuming that in a solid body there could be only shears on AB and BC. If the particle is rectangular (Fig. 121), then $s = s_1$, no matter whether or not there are normal forces on AB and BC', because those normal forces act at the center of AB and BC' (since the particle is infinitesimal), and each pair of normal forces is equal and opposite, and balance each other. But on BC there will be a normal stress even if there is none on AB or BC'. The total stress on BC will be the resultant of s_1 times BC' (acting along BC') and s times $C'C$ (acting along $C'C$), and this resultant will be a compression on BC in addition to a shear. Thus in Fig. 120, in addition to s and s_1, there will be equal normal forces on BC and AD, and these will not be in the same line, so that the moments of s and s_1 about D will not be equal and opposite.

If the piece $ABCD$ were a separate block, it would be possible to apply to it the forces s and s_1, and no others, and in this case equilibrium would require the unit forces s and s_1 to be equal, but if the piece $ABCD$ is an imaginary particle in a solid, it would be impossible to have s and s_1 alone, without normal forces and s would not equal s_1.

4. Shearing Deformation.—The reader should review here Art. 12, of Chap. IV, regarding the deformation due to shear, and should clearly understand the nature of the unit deformation produced by shear, and the fact that it is measured, not by a distance, but by the tangent of an angle of distortion. He should remember that there is a relation between the modulus of elasticity for tension or compression and that for shear, and that if Poisson's ratio is 0.25, $E_s = \frac{2}{5}E$.

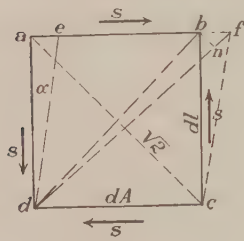


FIG. 122.

5. Work Done by a Shearing Force.—Let s be the intensity of a shearing force acting on a plane section at a given point, and dA the infinitesimal area on which it acts. Consider a prism (Fig. 122) whose base is dA and whose height, perpendicular to s, is dl. The work done by the force $s \cdot dA$, gradually applied, will be, since $\tan \alpha = \frac{s}{E_s}$,

$$dW = \frac{s \cdot dA}{2} \cdot dl \cdot \tan \alpha = \frac{s^2 dA \cdot dl}{2E_s}. \tag{1}$$

This expression, integrated over the cross-section of which dA is a part, will give the work done upon a slice between two sections a distance dl apart, and this again integrated over the length of the piece will give the total work, or

$$W = \frac{1}{2E_s}\int_o^l dl \int s^2 dA. \tag{2}$$

Often it will be close enough to assume the total shearing force S on a section to be uniformly distributed over the section, or $s = \frac{S}{A}$, hence in this case

$$W = \frac{1}{2E_s}\int_o^l \frac{S^2 dl}{A}. \tag{3}$$

If the total work done by tensile and compressive forces in a piece is computed including that done by the tension and compression *caused by shear*, according to §2, the total work done in the piece will be correctly obtained, without paying any attention to the work done by shear, for this will have been already included. This may be proved by considering the prism represented in Fig. 122. Let four sides of this prism be dl in length, the dimension perpendicular to the paper being unity; then dA, the infinitesimal area on which s acts, will be equal to dl, and considering shear alone the work done by shear will be given by Eq. (1). Instead of considering the shear, consider the work done by the tension and compression. The intensity of the tension on ac is s, the area on which it acts is $dA\sqrt{2}$, the total tension is $s \cdot dA\sqrt{2}$, and the average force (for gradual uniformly increasing application) is $\frac{s \cdot dA\sqrt{2}}{2}$. The distance moved through by the force at the center of ac is nf, which is (all sides of the square being dl) $\frac{s}{E} \cdot dl\sqrt{2}(1 + \sigma)$, where σ is $\frac{1}{m}$, Poisson's ratio. But the distance moved through by the tension at a and c is zero, and the average distance moved through is $\frac{s}{E} \cdot \frac{dl\sqrt{2}}{2}(1 + \sigma)$; hence the work of tension is $\frac{s^2 \cdot dA \cdot dl(1 + \sigma)}{2E}$. The work done by compression is the same; hence the total

work done by tension and compression is $\frac{s^2 \cdot dA \cdot dl(1+\sigma)}{E}$. But $E = 2E_s(1+\sigma)$; hence

$$dW = \frac{s^2 \cdot dA \cdot dl}{2E_s}$$

the same as found by considering the shear alone.

In other words, in computing work it is only necessary to consider one kind of stress, neglecting any work done by stresses *caused by the first*. Hence in finding the work done in a piece exposed to direct tension, although shear exists, it should not be considered; and in finding the work done by shear, any stresses *caused by the shear* should be omitted from consideration. In finding work due to flexure, since both tension, compression, and shear are all caused by the flexural action, and not caused by each other, all three should be considered.

6. Professor Mörsch[1] suggests the following theory to show the relation between tensile, compressive, and shearing ultimate strengths. He says that shearing action on a section, at the point of failure, may be considered a combination of tensile and compressive stresses on a tooth-shaped section, the teeth being infinitesimal (see Fig. 123). At the point of failure he assumes the ultimate

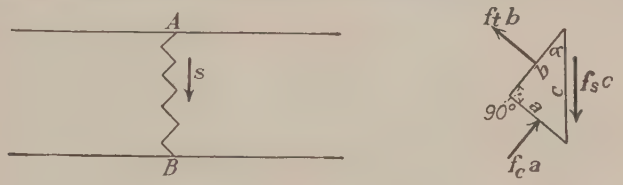


FIG. 123.

values of the shearing, tensile, and compressive stresses to be reached simultaneously, the angle between b and c being such as to produce this result. Applying the fundamental equations with the forces as shown, the following are the results:

From the triangle of forces, the angle between a and b being 90°

$$f_s^2 c^2 = f_t^2 b^2 + f_c^2 a^2.$$

From $\Sigma H = 0$

$$f_t b \cdot \frac{b}{c} = f_c a \cdot \frac{a}{c};$$

$$\therefore f_t b^2 = f_c a^2; f_t = \frac{f_c a^2}{b^2}; f_c = \frac{f_t b^2}{a^2}$$

$$\therefore f_s^2 c^2 = b^2 f_t f_c + a^2 f_t f_c = f_t f_c (a^2 + b^2) = f_t f_c c^2$$

$$\therefore f_s = \sqrt{f_t f_c}.$$

This may seem very simple and reasonable, at first, but a little consideration will show that it is *entirely wrong*. If there is pure shear on c as assumed, the principal stresses of tension and compression (*i.e.*, the max. intensities of tension and compression) are on planes at 45°; and on any other planes, such as a and b in the figure, there are not only direct stresses, but shear also, which has been entirely omitted in the above equations. This shows the necessity of a careful adherence to fundamental principles, and the danger of being led astray by false but specious reasoning.

[1] "Der Eisenbetonbau," 3d ed., pp. 38–39, 1908.

CHAPTER VIII

TORSION

1. Torsion means twisting, and *pure torsion* occurs when a rod or shaft is subjected to two equal and opposite couples lying in parallel planes at right angles to the axis of the rod, which is then in torsion between these two planes. Such a rod is called a shaft. The moment of either couple is the *twisting moment*, generally called *torque*. If the shaft is exposed to more than two parallel couples the torque varies along the rod, but is always constant between adjoining couples, just as the shear in a beam is constant between adjoining loads. The resultant outer force on one side of any section is a couple, lying in a plane parallel to the section.

2. Torsional Stress.—Let the circular shaft in Fig. 124 be subjected to a clockwise couple whose moment is T (meaning torque) at the right end, which is

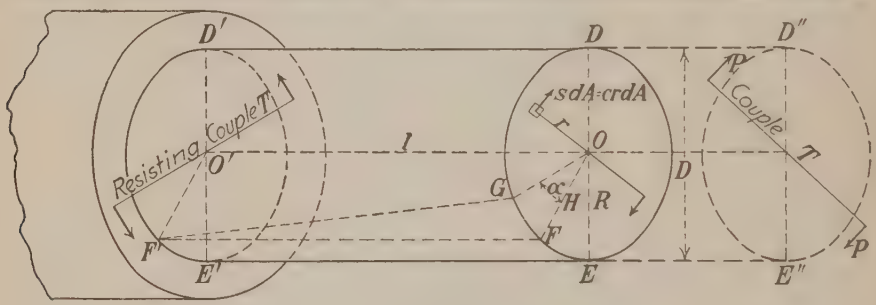


Fig. 124.

resisted by an equal couple at the left end. Take any section between the two ends and consider the part to the right removed, as dotted. Then since the resultant of the outer forces to the right of the section is a couple lying in a plane parallel to the section, there can be no normal stress on the section, but only shear; and the resultant of all the shearing stresses must be a couple with moment T. Obviously, at any point distant r from the center O, the shearing stress is perpendicular to the radius, by the principle of symmetry, and the shears acting on the part of the shaft to the left of the section all act so that their moments about O are positive or right-handed, or clockwise (since T is positive in this case). Also, the intensity of shear must be proportional to r, or equal to cr, if c is the intensity at a unit distance from the center.[1] Hence the total moment about O, which must equal T, will be $T = c\int_o^R r^2 dA$.

[1] This is necessarily true if any radius remains a straight line. If it does not, the strain, and therefore the stress, would not be proportional to r, but would vary according to the distance between the original radius and its position after distortion (under the elastic limit). In this case there would be a shear around the circumference of a smaller cylinder contained within the shaft; or, in other words, such smaller cylinder would be tending to shear apart or separate from the circular ring surrounding it. While the torque might be applied in such a way as to produce this effect over some limited length of the shaft; or while it might occur if a shaft were not of uniform section from end to end, but consisted of cylindrical portions having different diameters; it would not generally occur, and, at all events, is assumed not to occur.

The quantity under the integral sign, being each infinitesimal area multiplied by the square of its distance from O, is the polar moment of inertia J (see Chap. III). Its value is found by taking a ring at a distance r, and having a width dr; then

$$J = \int_o^R r^2 dA = 2\pi \int_o^R r^3 dr = \frac{\pi R^4}{2},$$

$$T = cJ = \frac{\pi c R^4}{2} = \frac{f_s J}{R} \tag{1}$$

if $f_s = cR$ = max. shear intensity, at the circumference. Hence

$$f_s = \frac{TR}{J} = \frac{2T}{\pi R^3} = \frac{16T}{\pi D^3}. \tag{2}$$

This gives the max. shear intensity for a given torque and diameter of shaft.

3. Strain.—Under the twisting moment T, a radius OF is shifted through an angle α to OG, and the original plane $OO'F'F$ becomes a warped surface $OO'F'G$. Any point H, at a distance r from O, is moved through an arc αr (in radians), which represents the shearing strain. If l is the distance OO',

Shearing strain per unit length $= \frac{\alpha r}{l}$. (3)

Shearing stress per square unit at $H = f = \frac{\alpha r E_s}{l}$. (4)

Max. shearing stress per square unit, at $F = f_s = \frac{\alpha R E_s}{l}$. (5)

Hence, from Eq. (1)

$$T = \frac{f_s J}{R} = \frac{\alpha J E_s}{l} = \frac{\alpha E_s}{l} \cdot \frac{\pi R^4}{2}. \tag{6}$$

$$\alpha = \frac{Tl}{JE_s} = \frac{2lT}{\pi E_s R^4} = \frac{f_s l}{E_s R}. \tag{7}$$

4. Deformation of a Circular Shaft.—When a circular shaft is exposed to pure torsion, every plane cross-section at right angles to the axis remains plane, and is merely rotated with reference to an adjoining plane section. If it be imagined that circles are drawn around the surface of the shaft where the cross-sections cut the surface, and also straight lines parallel to the axis, dividing the surface into equal squares wrapped around the shaft when unloaded, then the effect of the torsion is to distort all of these squares into equal and similar rhomboids. The relative rotation of two sections is proportional to the distance between them, that is, the angle α is proportional to l, as shown by Eq. (7). The angle of twist or $\frac{\alpha}{l} =$ is, if D is the diameter of shaft,

$$\frac{\alpha}{l} = \gamma = \frac{T}{JE_s} = \frac{f_s}{E_s R} = \frac{2T}{\pi E_s R^4} = \frac{32T}{\pi E_s D^4}. \tag{7a}$$

5. Power Transmitted.—When a shaft transmits power, the couple whose moment is T does work. Suppose this couple replaced by one whose lever arm is R, one of the forces $\frac{T}{R}$ acting at O and the other at the circumference. The latter will be the only one which does work, and in each revolution of the shaft it will do

work equal to $2\pi R \cdot \frac{T}{R} = 2\pi T$. As all dimensions are to be taken in inches, the torque must be in inch-lbs., and one horsepower will be $12 \times 33{,}000$ inch-lbs. per minute. Hence if the shaft revolves N times a minute, the horsepower transmitted will be

$$\text{Hp.} = \frac{2\pi TN}{396{,}000} = \frac{\pi f_s JN}{198{,}000R} = \frac{\pi^2 f_s NR^3}{396{,}000} \tag{8}$$

$$T = \frac{396{,}000 \text{ hp.}}{2\pi N} = \frac{63{,}024 \text{ hp.}}{N} = \frac{f_s J}{R} = \frac{f_s \pi R^3}{2}. \tag{9}$$

These equations give the relations between torque, horsepower, stress intensity, number of revolutions per minute, and size of shaft.

To find the max. fiber stress in a solid shaft, when a given hp. is being transmitted, from Eq. (9)

$$f_s = \frac{40{,}122 \text{ hp.}}{NR^3} = \frac{320{,}976 \text{ hp.}}{ND^3} = \frac{63{,}024 \text{ (hp.)}R}{NJ}. \tag{10}$$

The diameter of a solid shaft to transmit a given hp. at a given speed, with a given max. shearing intensity, is

$$D = 68.47\sqrt[3]{\frac{\text{hp.}}{Nf_s}}. \tag{11}$$

6. Hollow Shafts.—For a hollow circular shaft, with outside radius R and inside radius R_1,

$$J = \frac{\pi}{2}(R^4 - R_1^4)$$

$$f_s = \frac{2RT}{\pi(R^4 - R_1^4)} \tag{2a}$$

$$\alpha = \frac{f_s l}{E_s R} = \frac{2lT}{\pi E_s (R^4 - R_1^4)} \tag{7b}$$

$$f_s = \frac{126{,}048\text{(hp.)}R}{\pi N(R^4 - R_1^4)}. \tag{10a}$$

7. Torsional Rigidity.—The rigidity of a shaft exposed to torsion is measured by the ratio of the twisting moment T to the twist per unit length $\frac{\alpha}{l}$, or by

$$\frac{Tl}{\alpha} = JE_s.$$

The greater this quantity, the less the twist of a given shaft under a given torque. On the other hand, the less this quantity, or the greater the torsional ductility (as it may be termed) the larger the number of twists in a given length under a given T, or when twisted to fracture.

8. Stresses in Pure Torsion.—On any cross-section of a circular shaft in pure torsion there is, as shown, only shearing stress perpendicular to the radius, varying in intensity with the distance from the center. There is no normal stress on this section. Owing to the transverse shear, however, there is a shear of equal intensity on a longitudinal section containing the axis of the shaft, acting parallel to that axis. Hence on two planes containing the radius, and making angles of 45° with the cross-section there is pure normal stress of the same intensity as the shear

—tension on one plane, and compression on the other—and these are the principal stresses at any point on that radius.

In a wooden shaft the shear on a longitudinal axial section would be along the grain, and the strength to resist it would be much less than the strength to resist the shear on the cross-section. The same would be true in wrought iron, which consists of fibers welded together, but sometimes with slag between. Such shafts, therefore, would sometimes fail by the formation of longitudinal cracks.

9. Combined Stresses.—The principal function of a shaft is to transmit power. The outer forces applied are seldom couples applied to the circumference of the shaft, as in Fig. 124, but more commonly single forces, as where a weight is suspended by a rope wound around the shaft or the pull of a belt passing around a pulley attached to the shaft. Let P be any force applied to the shaft, not passing through the axis, and let this force be resolved into two components, one (a) parallel to the axis, the other (b) in a plane perpendicular to the axis. The former (a) is the same as an equal direct force (1) acting at the axis, and causing direct tension or compression, together with a bending couple (2); the latter (b) is the same as an equal transverse load (3) applied on the axis, and a twisting couple (4) whose moment is PR, if R is the perpendicular distance of the force from the axis. The case is therefore not one of pure torsion, but a combination of direct stress, flexure, and torsion. The bending couple (2) must be combined with the bending moment due to the transverse force (3). This will be treated in Chap. XIII. The present chapter is concerned with pure torsion, or the effect of torsion alone, which can be added to the effects of other loads. Pure torsion does not often occur in practice.

It is desirable at this point, however, to form a clear conception of all the stresses existing in a shaft exposed to torsion and bending, as follows:

On any cross-section there is at every point, except at the center, a shear due to torsion, in a direction at right angles to the radius and proportional in intensity to the distance from the center. There is also, at every point except the extreme points in the plane of flexure, a shear due to bending, parallel to the plane of flexure. There is also a normal stress due to the bending.

On any longitudinal plane containing the axis of the shaft there is a shear parallel to the axis, equal in intensity to the torsional shear on the cross-section; if the longitudinal plane is the plane of flexure, there is no other stress on it; if it is not the plane of flexure, there is also a longitudinal shear parallel to the axis, equal to the component of the flexural shear at right angles to this plane. There is no normal stress on this plane.

On a longitudinal plane at right angles to the two planes (cross-section and longitudinal plane through the axis) there is no normal stress, but there is a shear equal to the other component of the flexural shear. (See Chap. VII, Art. 2.)

If there is no flexure, it follows, as already stated, that on two planes containing a radius and making angles of 45° with the cross-section, there is tension on one and compression on the other, equal in intensity to the torsional shear.

10. The total internal work of deformation, or resilience, expended in deforming the shaft shown in Fig. 124 is that done by the forces P in twisting the shaft through an angle α, which in this case is the rotation of one end of the shaft with reference to the other, or of one end of the section of the shaft which has a constant torque T with reference to the other. The average force, gradually applied,

is $\frac{P}{2}$, and the angle α is given by Eq. (7), if l is the total length of the section; hence the distance travelled by P (in radians) is $R\alpha$; hence for a solid shaft

$$\text{Work} = \frac{P}{2} \cdot R\alpha = \frac{T\alpha}{2} = \frac{T^2 l}{2E_s J} = \frac{T^2 l}{\pi E_s R^4} = \frac{f_s^2 J l}{2E_s R^2} = \frac{\pi f_s^2 l R^2}{4E_s}$$
$$= \frac{f_s^2}{4E_s} \times (\text{volume of section of shaft}). \tag{12}$$

This gives the work of deformation, or resilience, of any portion of the shaft for which the max. stress is f_s. If there are several sections of the shaft, each with a different torque and f_s, the total resilience is found by taking the sum. The resilience per unit volume is $\frac{f_s^2}{4E_s}$.

Fig. 125.—Helical spring.

11. Helical Spring.—Perhaps the nearest approach to pure torsion that commonly occurs is a helical spring (Fig. 125). This is made by winding a wire around a cylinder. If a weight is hung at the bottom, R being the radius to the center of the spring, there is a uniform torque on every section equal to PR except at the bottom and top, between the points of loading and suspension and the ends of the helix. If this were all, the case would be one of pure torsion; but there is also a total shear P on each section of the helix, and also some tension because, however closely wound, the coils are not exactly horizontal. These two elements are generally neglected, and the case considered as one of pure torsion.

The torque on each section of the spring being PR, each length ds of the spring is rotated an angle $d\alpha$ which, by Eq. (7), making $l = ds$, and $R = r$ = radius of the wire, is

$$d\alpha = \frac{2PR\,ds}{\pi E_s r^4}.$$

This rotation causes an elongation de along the axis of the spring equal to $R \cdot d\alpha$, so that

$$de = \frac{2PR^2}{\pi E_s r^4} \cdot ds; \text{ and the total elongation is}$$
$$e = \frac{2PR^2 l}{\pi E_s r^4} \tag{13}$$

where l is the total length of the spring, or $2\pi Rn$, if n is the number of turns; hence

$$e = \frac{4nPR^3}{E_s r^4}. \tag{14}$$

If the effect of the shear P is to be considered, the average shearing intensity is $\frac{P}{\pi r^2}$, the shearing deformation per unit length $\frac{P}{\pi E_s r^2}$, and the shearing deformation in the entire length of the spring is

$$e' = \frac{2nPR}{r^2 E_s}. \tag{15}$$

Hence the total elongation is

$$e + e' = \frac{2nPR}{E_s r^2}\left(1 + \frac{2R^2}{r^2}\right). \tag{16}$$

These formulae afford a means of determining the modulus of elasticity for shear, by measuring the elongation (or compression) of a spring of known dimensions. Thus, in three tests[1] a steel rod 1.24 inches in diameter and 241 inches long was made into a spring in which R was 3.2 inches. A load of 5,000 pounds shortened this spring 4.64 inches. By Eq. (13) with $2\pi Rn = 241$ inches,

$$e = \frac{2 \times 241 \times 5{,}000 \times 3.2 \times 3.2}{3.1416 \times \overline{0.62}^4 \times E_s} = 4.64$$

whence

$$E_s = 11{,}460{,}000 \text{ pounds per square inch.}$$

Using Eq. (16), which takes account of direct shearing deformation in addition to torsional shear

$$4.64 = \frac{5{,}000 \times (\overline{0.62}^2 + 2 \times \overline{3.2}^2) \times 241}{3.1416 \times E_s \times \overline{0.62}^4}$$

whence $E_s = 11{,}670{,}000$ pounds per square inch.

The closeness of these two results shows that it is admissible here to neglect the direct shearing. These results also afford a means of estimating Poisson's ratio (σ) approximately. Using the last value of E_s, as probably the more accurate, and knowing that $E_s = \frac{E}{2(1+\sigma)}$, and with $E = 30{,}000{,}000$, it will be found that $\sigma = 0.285$. The effect of the small direct compression in the spring would be to increase somewhat the longitudinal compression of the spring above what would be due to torsion and direct shear alone, so that to compute E_s accurately a somewhat smaller value than 4.64″ should be used. This would slightly increase E_s. With $\sigma = \frac{1}{4}$, and $E = 30{,}000{,}000$, the theoretical formula $E_s = \frac{E}{2(1+\sigma)}$ would give $E_s = 12{,}000{,}000$. [Taking $E = 29{,}000{,}000$ the above formula would give $\sigma = 0.242$.] Hence it appears that for structural purposes, so far as these tests go, they show that σ may be taken at $\frac{1}{4}$, as is usually done.

12. Behaviour of a Material When Tested in Torsion.—When a material is tested in torsion, the angular deformation increases with the torque, in a manner similar to the relation between shearing or tension deformation and stress. Up to a certain point, in a steel shaft, the angle of twist will increase directly as the torque; this point will be the *elastic limit in torsion*. Above this point there will be a more rapid increase in twist, and often, as in tension, a yield point at which a rather sudden increase will occur, after which the phenomena will resemble those for tension. The stress at the elastic limit will be the *elastic limit in torsion*, and above this point there will be a permanent set. Tests in torsion show that there is a yield point for shear as well as for tension. But tests are not needed to prove this; for since a pure shear means equal tension and equal compression on planes at 45°, it is obvious, without tests, that if there is a yield point for tension there must be one for shear. The two may not occur at the same unit stress, because in pure tension there is no compression at right angles, while in pure shear there is compression at right angles to the tension, which modifies the strain, and perhaps the yield point. Tests of twist in torsion plotted against the unit shear

[1] "Tests of Metals at Watertown Arsenal," U. S. Ordnance Dept., p. 151, 1905.

are always, of course, against the unit shear as calculated by the formula, not the unknown true unit shear. The load-deformation diagrams will, for any material, be approximately similar in torsion and in tension, but in torsion the phenomena will not be so definite, particularly as regards rapid or sudden increases of deformation (*i.e.*, above the elastic limit) for the reason that in torsion it is only the outside fibers which are exposed to the maximum stress, while in tension the entire area is so exposed. In a pure tension piece, the yielding of one fiber tends to throw the stress on adjacent fibers, but the latter are stressed as much as the one which yields, and there is no relief. In a torsion piece, the yielding of an outside fiber (in shear of course) tends to throw the stress on adjacent fibers which are less stressed. Hence we should expect the phenomena above the yield point to be somewhat retarded as compared with tension tests. With a thin hollow shaft in torsion this effect would be less, but such shafts are not often used.

The resisting moment for a given torque becomes modified when the elastic limit is exceeded. We have seen that within the elastic limit, if f_s is the maximum shear intensity, at the outside, the intensity at a radius r will be $f_s \frac{r}{R}$, and on a thin circular surface of area $2\pi r dr$ the total shear will be $2\pi r\ dr \cdot f_s \frac{r}{R}$, and its moment will be $\frac{2\pi r^3 f_s dr}{R}$ about the center. The torque T will be the integral of this between the limits o and R, or

$$T = \frac{\pi R^3 f_s}{2}. \tag{2}$$

When the elastic limit is exceeded, instead of the stress varying uniformly from the center to the circumference, as in Fig. 126 according to the line cb, since ab tends to exceed the elastic limit, the outer fibers will yield and throw the stress upon those nearer the center, so that the stress will be constant over a distance af, and equal to the elastic limit of shear fe. The greatest possible value of the torque would be when uniform distribution occurred to the center, if this were possible. If f_s' is the elastic limit, the value of T when f_s' is just reached at the circumference will be

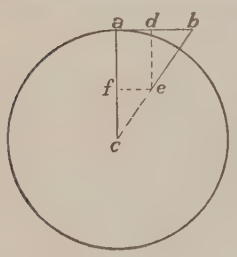


FIG. 126.

$$T = \frac{\pi R^3 f_s'}{2}.$$

The maximum possible value, with f_s' over the entire radius, would be

$$T = \int_o^R 2\pi r^2 dr f_s' = \tfrac{2}{3}\pi R^3 f'_s \tag{2a}$$

or one-third greater. This distribution, however, could never be reached, and Eq. (2*a*) is never used.

In Art. 33, of Chap. X, there is some further discussion of this subject.

13. Torsion in Sections Which Are Not Circular.—Shafts for the transmission of power are circular in section; but torsion often occurs in pieces which are rectangular in section, or have other shapes. It has been common to use for such sections the same general formulae as for circular sections, J being the polar moment of inertia and r the distance from the center of gravity. This is incorrect, as may easily be shown.

When a plane transverse section of a circular shaft deforms in pure torsion, it remains plane, and at each point the shear is perpendicular to the radius through that point. At the circumference the shear is parallel to the surface, as it must be in all sections by Art. 5 of Chap. V. Hence in any section that is not circular, the shear cannot at each point be perpendicular to the line joining that point to the center of gravity; it cannot be so at the surface, and as the direction cannot change suddenly, it will not be so at many other points. If there is an angle in the section, as in a rectangular section, in order to be parallel to the surface the shear at the corner would have to be parallel to both the surfaces meeting at the corner, which is of course impossible; hence at such a corner there can be no shear, and the stress, if any, must be direct tension or compression. The edge of the body through the corner will of course twist under the torsion, and become a spiral, so that, if the stress there is only direct, the section at the corner must be perpendicular to the edge, and the originally plane section will not remain plane. If we imagine, for instance, a rectangular shaft to be twisted as in Fig. 127, the moment exerted *on* the part of the shaft behind the paper being right-handed, the originally straight longitudinal line through A will become a spiral; and since

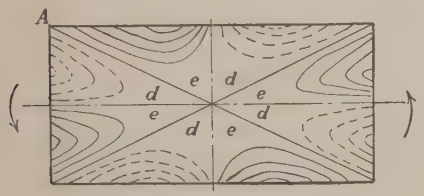


Fig. 127.

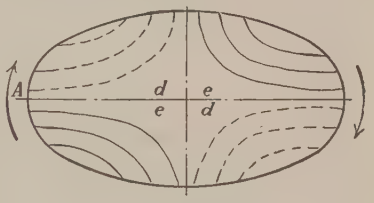


Fig. 128.

there is no shear there on a cross-section, and only direct stress, if any, the section at A will be perpendicular to the helical line, certain parts of the section will be depressed and others elevated, as indicated. Similar results occur in an elliptical shaft, as indicated in Fig. 128.

The theory of torsion has been exhaustively investigated mathematically by St. Venant. His results are extremely complex, and not generally intelligible to engineers. They are founded on assumptions, which may not be correct, and probably are not strictly correct in practice. They are only interesting to the theoretical mathematician. The engineer should give due consideration to such theoretical investigations, but should not use them nor rely on them without understanding the assumptions on which they are based, and satisfying himself that those assumptions are reasonably correct for the case in hand. Such investigations are discussed exhaustively in Todhunter's "History of the Theory of Elasticity and of the Strength of Materials;"[1] but even the editor of that work, Karl Pearson, remarks (vol. 1, p. 626) "We find even in mathematicians of the standing of Lamé not infrequently an omission to state clearly the physical principle upon which they base their calculations of a physical phenomenon. The history of mathematical elasticity gives many examples of this divorce between theory and physical fact; the mathematician has too often identified elasticity with the solution of certain differential equations, the constants of

[1] Cambridge University Press, 1886.

which are to be determined by a purely fanciful and often practically idle, if not impossible distribution of load."

14. General Theory of Torsion.—The general theory of torsion, for any shape of cross-section of a prismatic shaft, may be stated as follows:

The twisting moment at any cross-section is the algebraic sum of the twisting moments applied to the shaft, between the section in question and either end of the shaft; hence the twisting moment will be constant in the axial distance between the sections where twisting moments are applied. Taking any two sections a distance dx apart, within the distance where the torque is constant, and taking a point in each section lying on a line parallel to the axis, these two points will be displaced with reference to each other. The shearing strain per unit length will be the displacement of one relative to the other, in the plane of the section, divided by the distance dx between the sections. This unit strain will vary for different points in the section, and will be zero for one point, namely the axis of the shaft, which passes through the c.g. of the section. If the section has two axes of symmetry, at right angles to each other, these axes of symmetry of any section will remain straight lines at right angles to each other and to the axis of the shaft, during the twisting; for it cannot be conceived that any deviation from this condition could occur in one direction rather than in the opposite direction. The rotation of one section with reference to another after twisting is measured by the angle between similar axes of symmetry in the two sections.

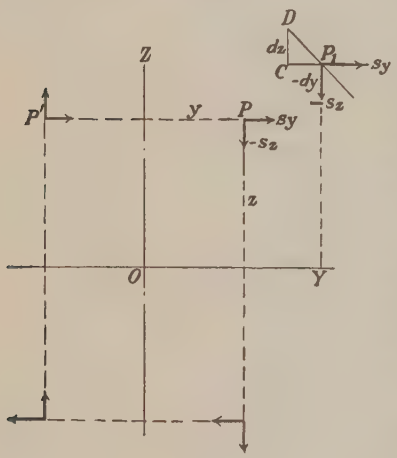


FIG. 129.

Shearing stresses correspond to the shearing strains, and for isotropic bodies[1] are proportional to the strains. The problem of torsion is to find the angle of rotation between two sections, the shearing strain per unit length, and the max. shearing stress in a section of any form and size, for a given torque M.

Let O be the c.g. of a section lying in the plane of the paper (Fig. 129), and let OY and OZ be the axes of symmetry. The axis OX is the axis of the bar, at right angles to the section and to the page. At any point P, whose coordinates are x (along the axis), y, and z, there are shearing stresses having intensities

$$s_y \text{ parallel to } OY$$
$$s_z \text{ parallel to } OZ.$$

These will be considered positive when they act outward along the axis (on the portion of the shaft back of the page), hence the stresses in the directions shown in the figure will be s_y and $-s_z$. These are functions of y and z; and, on account of symmetry, at a point P' symmetrical with P on the other side of OZ, s_y will have the same direction and s_z the opposite direction. Hence:

[1] An isotropic body is one which has equal elasticity in all directions. This is generally assumed in the theory in this book.

if y changes sign, s_y does not but s_z does;
if z changes sign, s_z does not but s_y does.

Hence if the expressions for s_y and s_z are expressed as a series of terms with ascending integral powers of y and z,

s_y must contain only even powers of y and odd powers of z;
s_z must contain only even powers of z and odd powers of y.

Restricting the terms to those of less than the fourth power, the expressions for s_y and s_z must therefore be

$$\left.\begin{aligned} s_y &= nz + n_1zy^2 + n_2z^3 \\ s_z &= my + m_1yz^2 + m_2y^3 \end{aligned}\right\} \tag{17}$$

in which n, n_1, n_2, m, m_1, m_2 are constants, to be determined by the following conditions:

(1) The stresses on the section must equal the outer moment M exerted by the portion of the shaft toward the reader *upon* the part behind the paper.

(2) The stresses on the surfaces of any particle $dx \cdot dy \cdot dz$ must be in equilibrium.

(3) The stress at the edge of the section must be parallel to that edge (see Art. 5, Chap. V).

If the six constants cannot be determined by these conditions, the number of terms in the equations for s_y and s_z must be increased (if there are too many conditions) or decreased (if there are too many constants).

With reference to the first of the above conditions, the stresses on the section will *balance* the outer moment M, or will be *equal* to it, depending on which side of the section is considered. If the part of the shaft on one side of the section be called A and that on the other side B, and if the resultant outer twisting moment applied to A is M, then, for equilibrium, the resultant outer twisting moment applied to B will be $-M$. If now the stresses s_y and s_z in the plane of the section are the stresses exerted *by* the part A, *on* the part B, then these stresses *equal* M; but if they are the stresses exerted *by* B *on* A, they balance M or are equal and opposite to M. We shall consider them as exerted *by* A *on* B. A may be considered as the part of the shaft toward the reader and B the part back of the paper.

15. Use of the Conditions.—(1) The first of the above conditions, since the resultant outer force on the shaft, on either side of the section, is a twisting moment M, leads to the equation

$$\int s_y dA = 0;\ \int s_z dA = 0;\ \int (s_y z - s_z y) dA = M. \tag{18}$$

The integrals extend over the entire section, and the values of s_y and s_z in Eq. (17) are to be substituted. If this is done, the first two of Eqs. (18) will be fulfilled in any case, on account of symmetry, no matter what the constants are. The third equation, if I_y and I_z are the moments of inertia of the section about OY and OZ respectively, becomes

$$nI_y - mI_z + (n_1 - m_1)\int y^2z^2 dA + n_2\int z^4 dA - m_2\int y^4 dA = M. \tag{19}$$

(2) The second of the above conditions gives the Eq. (31) in Art. 21, Chap. V, and remembering that in this case n_x, n_y, n_z, s_x, X, Y, and Z are all zero.[1]

$$\frac{ds_y}{dy} + \frac{ds_z}{dz} = 0;\ \frac{ds_y}{dx} = 0;\ \frac{ds_z}{dx} = 0.$$

[1] Note that in the chapter referred to, as shown in Fig. 97 s_y and s_z are shearing stresses on the plane perpendicular to the axis OX, as in the present case, but that s_z acts parallel to OX and s_y parallel to OZ, while here they are interchanged.

The first of these, from the values of s_y and s_z in (17), gives

$$2n_1 zy + 2m_1 yz = 0$$

or

$$n_1 + m_1 = 0. \tag{20}$$

The last two, remembering that the particle is taken in a portion of the shaft where M is constant, and that therefore the constants are not only independent of y and z, but also of x, are fulfilled in any case.

(3) The third condition is illustrated in Fig. 129. The straight diagonal line through P_1 represents the direction of the surface of the body, supposing P_1 to be at the surface. $P_1C = -dy; CD = dz$. Hence clearly, at the surface,

$$\frac{s_y}{-s_z} = \frac{-dy}{dz}; \text{ or } \frac{s_z}{s_y} = \frac{dz}{dy}. \tag{21}$$

This condition leads to different values for the constants, depending on the shape of the section, so that s_y and s_z are different functions of y and z depending on the shape of the section. The three Eqs. (19), (20), and (21), must be fulfilled in all cases.

16. Elliptical Section.—If the section is an ellipse with semi-axes b and c along Y and Z respectively (Fig. 130), its equation is

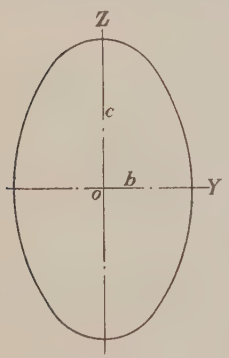


Fig. 130.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

From this

$$\frac{ydy}{b^2} + \frac{zdz}{c^2} = 0$$

$$\frac{dz}{dy} = -\frac{c^2}{b^2} \cdot \frac{y}{z}.$$

Hence from Eqs. (17) and (21), at the surface,

$$\frac{m + m_1 z^2 + m_2 y^2}{n + n_1 y^2 + n_2 z^2} = -\frac{c^2}{b^2}. \tag{22}$$

This equation, as well as (19) and (20) must be fulfilled. If we assume

$$s_y = nz; s_z = my, \text{ or}$$

$$m_1 = m_2 = n_1 = n_2 = 0;$$

then from Eq. (22), $mb^2 + nc^2 = 0$

or, since

$$I_y = \frac{\pi}{4} bc^3; I_z = \frac{\pi}{4} cb^3; \tag{23}$$

$$mI_z + nI_y = 0.$$

This assumption also fulfils Eq. (20) since both m_1 and n_1 are assumed zero, and Eq. (19) becomes

$$nI_y - mI_z = M. \tag{24}$$

Equations (23) and (24), therefore, contain the solution of this case. They are based on the assumption that $s_y = nz$ and $s_z = my$, combined with the surface conditions, and the equality of outer and inner moments. From these equations it follows that

$$n = \frac{M}{2I_y} \tag{25}$$

$$s_y = nz = \frac{Mz}{2I_y} \tag{26}$$

$$m = -\frac{M}{2I_z} \tag{27}$$

$$s_z = my = -\frac{My}{2I_z} \tag{28}$$

$$s = \sqrt{s_y^2 + s_z^2} = \frac{M}{2}\sqrt{\frac{z^2}{I_y^2} + \frac{y^2}{I_z^2}} \tag{29}$$

$$= \frac{2M}{\pi bc}\sqrt{\frac{z^2}{c^4} + \frac{y^2}{b^4}}. \tag{29a}$$

The resultant shear s increases with y and z, and is greatest at some point in the circumference. Putting Eq. (29a) in the form

$$s = \frac{2M}{\pi b^2 c}\sqrt{\frac{y^2}{b^2} + \frac{z^2}{c^2}\cdot\frac{b^2}{c^2}}$$

and assuming $b < c$, it is clear, since $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, that the radical is less than 1. At the end of the short axis, for which $y = b$; $z = 0$; and at the end of the long axis, for which $z = c$; $y = 0$; hence we have

at end of short axis, $$s = \text{max. } s = \frac{2M}{\pi b^2 c};\ y = \pm b,\ z = 0 \tag{30}$$

at end of long axis, $$s = \frac{2M}{\pi bc^2};\ y = 0;\ z = \pm c \tag{31}$$

$$= \text{max. } s\cdot\frac{b}{c}.$$

For a circle $b = c = r$, and $I_y = I_z$; $I_y + I_z = 2I_y = J = \frac{\pi r^4}{2}$;

hence $$s = \frac{M}{J}\sqrt{y^2 + z^2}$$

$$\text{max. } s = \frac{Mr}{J} = \frac{2M}{\pi r^3} = \frac{16M}{\pi d^3}$$

as in Eq. (2), M being of course the same thing as T.

17. Rectangular Section.—For a rectangular section with sides $2b$ and $2c$ (Fig: 131)

$$I_y = \tfrac{4}{3}bc^3;\ I_z = \tfrac{4}{3}b^3c.$$

The surface conditions require that at all points along the vertical sides $s_y = 0$, and at all points along the horizontal sides $s_z = 0$; or

$$s_y = 0, \text{ for } y = \pm b, \text{ for all values of } z$$
$$s_z = 0, \text{ for } z = \pm c, \text{ for all values of } y.$$

Hence, from Eq. (17),

$$0 = nz + n_1 zb^2 + n_2 z^3 \text{ for all values of } z$$

or $$n_2 = 0;\ n_1 = -\frac{n}{b^2}.$$

Similarly $m_2 = 0; m_1 = -\frac{m}{c^2}$

whence
$$\left.\begin{aligned} s_y &= nz\left(1 - \frac{y^2}{b^2}\right) \\ s_z &= my\left(1 - \frac{z^2}{c^2}\right) \end{aligned}\right\} \tag{17a}$$

This equation, with Eqs. (19) and (20), determine the case.

Since $\int y^2 z^2 dA = \frac{4}{9}c^3b^3 = \frac{1}{3}I_y b^2 = \frac{1}{3}I_z c^2$

Equation (19) becomes

$$nI_y - mI_z = \tfrac{3}{2}M. \tag{19a}$$

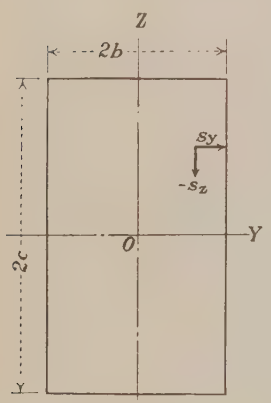


Fig. 131.

Equation (20) becomes

$$\frac{n}{b^2} + \frac{m}{c^2} = 0$$

or dividing by $\frac{4}{3}bc = \frac{I_y}{c^2} = \frac{I_z}{b^2}$

$$nI_y + mI_z = 0. \tag{20a}$$

From these two Eqs. (19a) and (20a) we find

$$\left.\begin{aligned} n &= \tfrac{3}{4}\frac{M}{I_y} \\ m &= -\tfrac{3}{4}\frac{M}{I_z} \\ s_y &= \tfrac{3}{4}\frac{M}{I_y} z\left(1 - \frac{y^2}{b^2}\right) \\ s_z &= -\tfrac{3}{4}\frac{M}{I_z} y\left(1 - \frac{z^2}{c^2}\right) \end{aligned}\right\} \tag{32}$$

The resultant shear intensity is

$$\begin{aligned} s &= \tfrac{3}{4}M\sqrt{\frac{y^2}{I_z^2}\left(1 - \frac{z^2}{c^2}\right)^2 + \frac{z^2}{I_y^2}\left(1 - \frac{y^2}{b^2}\right)^2} \\ &= \tfrac{9}{16}\frac{M}{bc}\sqrt{\frac{y^2}{b^4}\left(1 - \frac{z^2}{c^2}\right)^2 + \frac{z^2}{c^4}\left(1 - \frac{y^2}{b^2}\right)^2} \\ &= \tfrac{9}{16}\frac{M}{b^2c}\sqrt{\frac{y^2}{b^2}\left(1 - \frac{z^2}{c^2}\right)^2 + \frac{z^2}{c^2}\cdot\frac{b^2}{c^2}\left(1 - \frac{y^2}{b^2}\right)^2}. \end{aligned} \tag{33}$$

The max. value of the quantity under the radical is 1, for $z = 0$, $y = \pm b$, so that

$$\text{max. } s = \tfrac{9}{16}\frac{M}{b^2c} \tag{34}$$

at the middle of the long side.

At the middle of the short side,

$$y = 0, z = \pm c,$$

and

$$s = \tfrac{9}{16}\frac{M}{bc^2} = \frac{b}{c}\cdot \text{max. } s. \tag{35}$$

At the corners,

$$y = \pm b,\ z = \pm c,$$
$$s = 0,$$

as stated in Art. 13.

18. Relative Economy of Different Shapes.—If A is the area of cross-section; then for the ellipse $A = \pi bc$, and hence,

$$\text{for the ellipse, max. } s = \frac{2M}{Ab} \ (b = \text{small semi-diameter})$$
$$\text{for the circle, max. } s = \frac{4M}{Ad} = \frac{16M}{\pi d^3} \ (d = \text{diameter})$$
$$\text{for the square, max. } s = 9/2\ \frac{M}{Al} \ (l = \text{side}).$$

For given values of M and A, max. s for the ellipse will be smaller the greater b is, or when the ellipse becomes a circle. For the rectangle, max. s will be smaller the greater b is $\left(bc \text{ being constant } = \frac{A}{4}\right)$, or when the rectangle becomes a square. If a circle and a square have equal areas

$$\frac{\pi d^2}{4} = l^2;\ l = \frac{d}{2}\sqrt{\pi}$$
$$\text{max. } s \text{ for the square} = 9/2 \cdot \frac{M}{Al} = \frac{9}{\sqrt{\pi}} \cdot \frac{M}{Ad}.$$

Hence, for equal areas,

$$\text{max. } s \text{ for square} = \frac{9}{4\sqrt{\pi}} (= 1.269) \times \text{max. } s \text{ for circle.}$$

This is because some parts of the square have small stress, with none at the corners.

Hence *the circle is the most economical form for a solid shaft.*

The hollow circular section would be even more economical theoretically,[1] but not practically.

As shown above, in the rectangular and elliptical sections, the greatest stress is at the point nearest the axis of the shaft. The same is true of irregular sections. A re-entrant angle is especially bad in a shaft. Ribs do not strengthen a shaft, but weaken it, contrary to their action in the case of beams and in some columns.

19. In Chap. IX the question of the distribution of a normal stress over a surface is discussed, when the resultant does not act through the center of gravity, and when therefore the stress is not uniformly distributed. In Art. 36 of Chap. X, some remarks are made about torsion, which should be considered in reference to the present chapter. At this stage the student is not quite prepared to appreciate them.

[1] Prove that if the outside and inside radii of a hollow circular shaft are r and $0.8r$, the max. shear, for a given torque, of a solid shaft of equal cross-section, would be 2.733 that for the hollow shaft.

CHAPTER IX

THE INNER FORCES—DISTRIBUTION OF A NORMAL FORCE OVER A PLANE AREA, THE STRESSES BEING BELOW THE ELASTIC LIMIT

1. Method.—The outer forces acting upon a body having been determined, the next step is to ascertain the inner forces, that is to say, the direction and intensity of stress existing at any point upon any plane. This is the fundamental thing in the design of a structure, for it is clear that such a design requires that at no point in the structure shall the stress exceed the safe limit.

Consider the subject broadly for a moment. The engineer has first to decide what loads the structure shall be designed to resist; this requires the exercise of careful judgment, remembering the properties of materials, and the fact that a load which will be very seldom applied may not injure the structure even if it is much larger than the usual or ordinary load. While the structure must be *safe* for any load which could possibly come upon it, it need not be designed as though that load were the usual load.

Having determined upon the load system to be assumed, the engineer must be able to place that system upon the structure in such a position that the maximum stress at any point on any plane may be determined. If the load system is fixed in position, this question will not arise, but if, as generally happens, the load system (aside from the dead load) is movable, the determination of its most unfavorable position is essential. The position having been determined, the reactions may be found, and the resultant force which acts upon the plane in question. From this, if the distribution of that resultant force over the plane is known, the stress at any given point may be found, which will be the maximum at that point.

The order of procedure is therefore as follows:

(1) Determine live load system.

(2) Determine position of that system which will cause maximum intensity of stress at any point on any assumed plane.

(3) For that position of the live loads, determine the reactions for both live and dead loads.

(4) Determine the resultant outer force acting upon the plane in question.

(5) Knowing the law of distribution of that outer force over the section (which is equal and opposite to the resultant inner force or stress), the maximum intensity of stress at the given point may be found.

It is clear, therefore, that the determination of the distribution of the resultant force over a section is fundamental. To study this there is *one universal method.* This consists in conceiving the body separated into two entirely distinct parts by the plane upon which we wish to find the distribution of stress, and considering the equilibrium of one of these parts, supposing the other to be removed. *Either part* may be the part which is considered. This part is then acted upon by cer-

tain known outer forces, and if upon the cut section we replace the stresses actually exerted by the part which has been removed *upon* the part under consideration, these stresses, together with the known outer forces just mentioned, will constitute a system of forces in equilibrium, and by means of the statical conditions of equilibrium the relation between these forces may be studied.

2. Relations to Be Studied.—If the particle or portion of the body separated out is bounded by several planes and is *entirely within* the original body, then the imposition, upon the forces which act upon it, of the conditions of equilibrium, leads to results expressing the relations between the inner stresses on the bounding planes, and the weight of the particle, the latter being the only outer force. The stress on any of the bounding surfaces is the product of two linear dimensions by the average intensity of the stress upon it, the latter being always a finite quantity; the weight is the product of three linear dimensions by the weight per cubic unit, the latter being also always a finite quantity. It follows that by taking a particle of infinitesimal size its weight disappears, being a differential of a higher order than the stresses acting on the bounding surfaces, and the investigation therefore gives *the relations between the intensities of stress existing simultaneously on different planes passing through a given point.* This, then, is the universal method of studying the relations *of internal stresses to each other, and has been treated in Chap. V.*

If, on the other hand, the particle of the body separated out is bounded partly by external surfaces of the body, upon which outer forces act, then the investigation of the equilibrium of this particle gives the *relations between the external forces (including the weight) and the internal stresses, or,* if the body is cut entirely in two by a single plane, between the internal stresses acting upon a plane and the resultant (outer or inner) force upon that plane.

It is thus clear that there is one universal method of studying stress in a body, and that it consists in separating out from the rest of the body some particular portion, bounded by plane or curved surfaces, and either entirely within the body or bounded in part by its exterior surfaces, and imposing upon this portion, so separated out, the conditions of equilibrium. When we can choose the bounding surfaces, they are always assumed as planes for simplicity. In this chapter we are concerned only with the relations that exist between the outer and inner forces. We wish to find what stresses are produced by given loads. We therefore consider a portion of the body bounded partly by exterior surfaces upon which known outer forces act. For simplicity we assume the body cut entirely in two by a plane, and the surface which this plane cuts we term the *section.* The problem is to determine the distribution of the internal force or stress over this section, that is to say, the intensity of the internal stress at every point of the section, and also its direction. The plane which cuts the section may be taken arbitrarily. We shall see later how it is generally taken. The point is, to choose the plane intelligently. If we can choose the one upon which the maximum intensity of stress will be greater than the maximum upon any other plane, we avoid the necessity of trying any other plane.

The outer forces acting upon the portion of the body on one side of the section being known, their resultant can be found by the simplest principles of statics. From Chap. II it appears that if all the outer forces lie in one plane, which is the usual case, the resultant outer force on one side of the section is either a single

force or a couple. If it is a single force, it will cut the plane of the section somewhere in the line of intersection of the plane of the outer forces and the plane of the section, either within or without the section itself, and at that point it may be resolved into two components, one acting in the plane of the section, and the other perpendicular to it. If it is a couple, it may be considered as two equal and opposite forces at right angles to the plane of the section at a fixed distance apart; or as a single force, of infinitesimal magnitude, acting at an infinite distance in the line of intersection referred to. Practically, then we may say that in this case the resultant is always a single force.

If the outer forces do not lie in a plane, their resultant is always, as shown in Chap. II, either (1) a single force perpendicular to the plane of the section, together with a force lying in the plane of the section, or (2) a couple composed of forces at right angles to the plane of the section, together with another couple lying in that plane. Whatever the resultant is, the student should be able to find it without difficulty. The effect of a force or couple lying in the plane of the section is to produce shearing stresses, and these are considered in Chaps. V and VII.

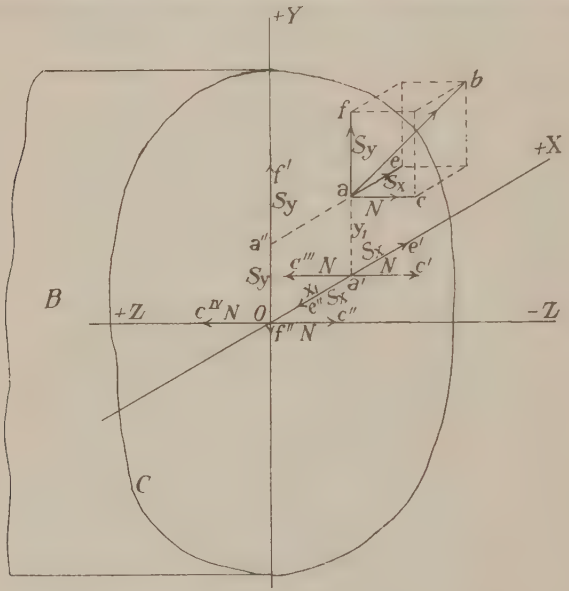


Fig. 132.

3. General Procedure.—In order, then, to study the stresses caused by any loads, let the body be cut entirely in two by a plane, giving a section C, and let the part of the body on one side of this section be called A and that on the other side B. Let A be removed, and let the equilibrium of B be considered. Since the entire body is in equilibrium, the resultant outer force on B is equal and opposite to that on A, and equal and opposite to the resultant of the internal stresses on the section, exerted by A on B, which we suppose applied to the section to balance the outer forces on B.

Confining ourselves to the simple case where the outer forces lie in a plane, Fig. 132 shows a portion of a body which has been cut in two by a plane section

and the right hand portion (A) removed, leaving the portion B. The resultant of all the outer forces on B is the single force ab, cutting the plane of the section at a. This is the resultant of the stresses on the section, exerted by B on A. Let ab be resolved into three rectangular components, parallel to the axes OX, OY, OZ, passing through the center of gravity of the section, O, OZ being perpendicular to the section; ac is the normal component, which we call N, and ae ($= S_x$) and af ($= S_y$) are tangential components parallel respectively to OX and OY. Let the coordinates of a be x_1 and y_1.[1]

Let two equal and opposite forces ($a'c'$ and $a'c'''$) be applied at a' and two others (Oc'' and Oc^{IV}) at O, all parallel to N and equal to it in magnitude; also at a' let two equal and opposite forces ($a'e'$ and $a'e''$) be applied, parallel and equal to S_x, and at a'' two similar forces ($a''f'$ and $a''f''$) parallel and equal to S_y. The force ab, then, which B exerts on A, is equivalent to 11 forces, consisting of the following three forces and four couples, viz.:

(1) A normal force $Oc'' = N$, applied at O, which, as we shall see, causes only a uniformly distributed stress over the section (compression as shown).

(2) A tangential force $a'e' = S_x$ applied at O, tending to cause B to slide on A in the direction OX.

(3) A tangential force $a''f' = S_y$ applied at O, tending to cause B to slide on A in the direction OY.

These two forces ($a'e'$ and $a''f'$) are shearing forces, and cause only shearing stresses in the plane of the section.

(4) A couple tending to rotate B about the axis OZ, being the resultant of two couples, one composed of the forces ae and $a'e''$ (whose moment is numerically equal to $S_x y_1$), and the other composed of the forces af and $a''f''$(whose moment is numerically equal to $S_y x_1$).

(5) A couple tending to rotate B about the axis OX, composed of the forces ac and $a'c'''$ (whose moment is numerically Ny_1).

(6) A couple tending to rotate B about the axis OY, composed of the forces $a'c'$ and Oc^{IV} (whose moment is numerically Nx_1).

The problem before us is to find the inner stresses distributed over the section which will have the same resultant components and moments as the above. But as these inner stresses will have normal and tangential forces, we may say that our problem is to find

(a) The distribution of the normal force N.

(b) The distribution of the tangential forces S_x and S_y.

4. Representation of Normal Stress.—In Chap. II reference has already been made to the fact that it is convenient to represent the normal stress upon a plane section C by conceiving at each point of the section a line perpendicular to the plane, whose length represents (is proportional to) the intensity of the normal stress

[1] The student must clearly grasp the fact that a stress upon a plane is always due to the fact that the outer forces acting on the portion of the body on *one side* of that plane have a resultant, which is balanced by the stresses on the plane coming from the portion of the body on the *other* side of the plane. It is important to remember always that the portion of the body on one side of the plane is considered removed, and to clearly see *which side is removed*. In finding the resultants N, S_x and S_y we may deal with the portion of the body lying on *either* side of the plane, choosing the side which is simplest to deal with (generally the one having the smaller number of outer forces). The forces which are obtained by taking one side of the section will simply be equal and opposite to those which are obtained by taking the *other* side.

at that point. The other ends of all these lines will lie on some surface, V, plane or curved. If V is a plane which is parallel to the section, the resultant N is uniformly distributed over the section, and the stress per square unit at any point is $\frac{N}{A}$, if A is the area of the section. If V is a plane which is not parallel to the section, this plane will intersect the plane of the section in a straight line (which may or may not cut the section itself). In this case the line of intersection is a line along which there is no normal stress, and which is called the *neutral axis;* (which we designate N.A.) while the intensity of stress will vary uniformly along any line in the section (except one parallel to the N.A., along which it will not vary), and at any point will be proportional to the perpendicular distance of that point from the neutral axis. This condition is called *planar distribution of stress.* If V is a curved surface, then the stress is neither uniform nor uniformly varying.

In any case, the resultant normal force N is *proportional to the volume included between C and V; and the resultant will act through the center of gravity of this volume;* for each little normal element of the volume represents a portion of the total normal stress, and the resultant force is the resultant of all the infinitesimal components, and must act through the center of gravity of the volume which represents them.

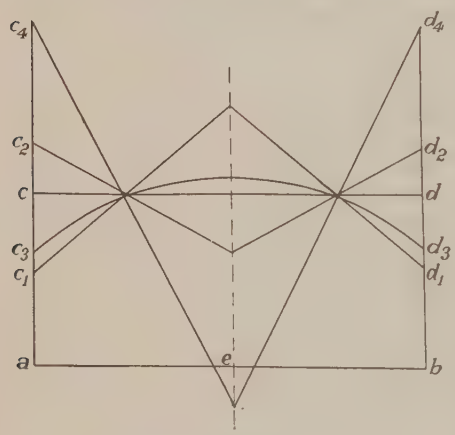


Fig. 133.

Any distribution of the normal stress which fulfills these two conditions, is, so far as the laws of statics are concerned, a possible distribution. That is to say, any distribution which will give a resultant equal to N and acting at a (Fig. 132) will fulfill the statical conditions. From the strictly mathematical point of view, the stress might be tension on any one small area and compression on the adjacent areas all around it, so long as the above conditions were fulfilled, although such a distribution could not really occur. The same may be said of the distribution of S_x and S_y.

Thus, let ab (Fig. 133) represent a rectangular surface one unit in width perpendicular to the paper. Suppose the stress to vary only along the direction ab, not perpendicular to it, and let N act at the center of ab, or e. Then, *statically*, N might be distributed as represented by cd (uniform distribution), c_1d_1, c_2d_2, c_3d_3, or c_4d_4, or in fact any distribution so long as the volume ac_nd_nb represents (algebraically) N, and has its center of gravity in the normal at e. The distribution need not even be symmetrical about e, though any such distribution would satisfy the conditions if the magnitude satisfied the first condition. There might even be a stress of an opposite kind to N on some portions of the section, as in c_4d_4.

The problem, therefore, cannot be solved by statics, and resort must be had to assumptions, unless other considerations, such as physical conditions, or experimental tests, indicate the true law of distribution. The assumptions will be considered later.

DISTRIBUTION OF THE NORMAL FORCE

5. General Equations.—If, on any infinitesimal area da $(= dy \cdot dx)$ whose coordinates are x and y, the shearing stresses per unit area parallel to OX and OY be called respectively s_x and s_y, and the normal stress per unit area be called f, then the total stresses on the area are numerically $s_x da$, $s_y da$, and $f \cdot da$. These are now assumed as the stresses exerted by A on B, and the sum of all such forces balances ab, so that the only statical relations which exist are those which express this fact. Before writing them in algebraic form, we must adopt a convention regarding signs. Let us therefore consider shearing forces S_x and S_y and stresses s_x and s_y positive when they act in the direction from O in the positive direction along their respective axes, normal stresses f positive when they are tension, the normal force N positive when it acts toward positive Z or away from the part A (thus tending to cause tension), and the moment about any axis positive when it is right-handed (clockwise) when viewed from the positive extremity of its axis looking toward O. Then the six equations of equilibrium are

$$\Sigma X = 0; \quad \Sigma Y = 0; \Sigma Z = 0$$
$$\Sigma M_x = 0; \Sigma M_y = 0; \Sigma M_z = 0$$

and in this case they become (x_1 and y_1 being the coordinates of a)

$$\int s_x da + S_x = 0 \quad (1); \int s_y da + S_y = 0; \quad (2) \int f\, da - N = 0; \tag{3}$$
$$-\int fy da + Ny_1 = 0 \quad (4); \quad \int fx da - Nx_1 = 0; \tag{5}$$
$$-\int s_x y da + \int s_y x da - S_x y_1 + S_y x_1 = 0. \tag{6}$$

If we denote by M_x, M_y, M_z the moments of the outer forces (acting on B) about the axes, with the above understanding as to signs, these equations may be written

$$\int s_x da = -S_x \tag{1}$$
$$\int s_y da = -S_y \tag{2}$$
$$\int f da = N \tag{3}$$
$$\int fy da = Ny_1 = +M_x \tag{4}$$
$$\int fx da = Nx_1 = -M_y \tag{5}$$
$$\int s_x y da - \int s_y x da = -S_x y_1 + S_y x_1 = + M_z. \tag{6}$$

So far as signs are concerned, these equations simply mean that a force S_x acting on B from O toward X causes a shearing stress s_x acting in the opposite direction from A on B; that a force S_y acting on B from O toward Y causes a shearing stress s_y acting in the opposite direction from A on B; that a normal force N acting on B away from the section causes a tensile normal stress; that a right-handed M_x on B causes tension above OX; that a right-handed M_y on B causes compression for positive values of x; and that a right-handed M_z on B causes positive s_x and negative s_y (from A on B) for positive values of x and y, all of which results should be consciously seen to be correct by the student, by visualizing the conditions.

Here we have six equations from which to determine the three unknown quantities s_x, s_y, and f, which are themselves functions of (depend upon) x and y. Of these Equations (1), (2), and (6) concern only the shearing stresses, while (3), (4), and (5) concern only the normal stress. If the shearing force were

considered uniformly distributed over the section, we should obtain from Eqs. (1) and (2)

$$s_x = -\frac{S_x}{A}, \quad s_y = -\frac{S_y}{A}$$

and the resultant shearing stresses along OX and OY would act through the center of gravity of the section. This would not satisfy Eq. (6) unless the point a (Fig. 132) should coincide with O (center of gravity of section). Indeed, we shall find later that as soon as the distribution of the normal forces is found, the distribution of the shearing forces is fixed, so that we shall not further consider Eqs. (1), (2) and (6), but shall confine ourselves to the equations which deal with the normal stress. These equations, three in number, cannot be used until we have ascertained or assumed the manner in which f varies with x and y. The equations being three in number, the relation between f and the coordinates x and y of a point, must involve three unknown quantities, in order that the number of unknowns may equal the number of equations.

There are numberless relations which might be assumed. For instance, we might assume

$$f = a + bx + cx^2.$$

This would make f independent of y, that is, for any given value of x, f would be the same at all points along a line parallel to OY. Such an assumption would be unreasonable, for there is no more reason to make f independent of y than there is to make it independent of x. We must, therefore, consider carefully whether any *physical* (as distinguished from mathematical) considerations indicate what the assumption should be. It is at once evident that f should vary with x and y in the same general manner. This leads at once to the result that, since f need not be zero at O, the law of variation must be (excluding possible more complicated expressions)

$$f = a + bx + cy \tag{7}$$

or

$$f = a + bx^2 + cy^2$$

or in general,

$$f = a + bx^n + cy^n.$$

Any law but the first would involve a curved surface of distribution (Fig. 132). The first equation, representing planar distribution, is clearly the simplest.

Making the usual assumption, then, of planar distribution (Eq. (7)) the fundamental Eqs. (3), (4), (5), assume the following form:[1]

$$\begin{aligned} N &= a\int\int dxdy + b\int\int xdxdy + c\int\int ydxdy \\ Ny_1 &= \quad M_x = a\int\int ydxdy + b\int\int xydxdy + c\int\int y^2dxdy \\ Nx_1 &= -M_y = a\int\int xdxdy + b\int\int x^2dxdy + c\int\int xydxdy. \end{aligned}$$

Assuming the point O as the center of gravity of the section, the following relations are evident:

[1] Note that in this chapter it is assumed that the elastic limit is nowhere exceeded. The effect of exceeding the elastic limit will be discussed in Chap. X, Art. 36. In Arts. 29 and 30 of Chap. X, after the consideration of flexure, it is shown why this distribution is correct inside the elastic limit, for normal stress alone, which is all that is being here considered; and why, if there is shearing stress which is not uniformly distributed over the section, it may still hold for the normal stress.

$$\int\int dxdy = A;\ \int\int xdxdy = \int\int ydxdy = 0$$
$$\int\int x^2dxdy = I_y;\ \int\int y^2dxdy = I_x;\ \int\int xydxdy = K$$

I_y and I_x being respectively the "moments of inertia" of the section around the axes OY and OX, and K being the "product of inertia" for these axes. The three equations therefore assume the forms

$$N = aA$$
$$Ny_1 = bK + cI_x$$
$$Nx_1 = bI_y + cK$$

from which

$$a = \frac{N}{A}$$
$$b = N\frac{I_x x_1 - Ky_1}{I_x I_y - K^2}$$
$$c = N\frac{I_y y_1 - Kx_1}{I_x I_y - K^2}$$

and finally

$$f = \frac{N}{A} + \frac{Nx(I_x x_1 - Ky_1)}{I_x I_y - K^2} + \frac{Ny(I_y y_1 - Kx_1)}{I_x I_y - K^2}.^{1} \tag{8}$$

This is the general expression for the intensity of normal stress at any point of a section given by its coordinates x and y referred to any pair of rectangular axes passing through the center of gravity of the section, under the action of a resultant normal force N applied at the point x_1,y_1, assuming planar distribution of stress.

In this equation $a = \frac{N}{A}$ is the intensity of stress at the center of gravity O, where x and y are zero. Hence we see that

The intensity of normal stress at the center of gravity of a section, assuming planar distribution of stress, is always equal to the average intensity.

Moreover, b is the change of intensity of stress in every unit distance along any line parallel to OX; and c is the change of intensity in every unit distance along any line parallel to OY.

If the axes OX and OY not only pass through the center of gravity of the section, but are the *principal axes* of the section, $K = 0$ (see Chap. III), and Eq. (8) becomes

$$f = \frac{N}{A} + \frac{Nx x_1}{I_y} + \frac{Nyy_1}{I_x} \tag{9}$$

or, since numerically

$$Nx_1 = M_y;\ Ny_1 = M_x;$$
$$f = \frac{N}{A} + \frac{M_y x}{I_y} + \frac{M_x y}{I_x} \text{ (numerically).} \tag{10}$$

Under the conventions as to signs which have been adopted, this would become algebraically

$$f = \frac{N}{A} - \frac{M_y x}{I_y} + \frac{M_x y}{I_x}. \tag{10a}$$

[1] This formula was first demonstrated, it is believed, by the writer in an article in Van Nostrand's *Engineering Magazine* for 1880, entitled "A general formula for the normal stress in beams of any shape."

In any case, observation will show the proper signs of the terms. However, in order to avoid confusion as to the conventions, and to indicate that the proper signs are to be determined by consideration of each special case, it is well to write Eqs. (9) and (10) with double signs for the last two terms, thus

$$f = \frac{N}{A} \pm \frac{Nxx_1}{I_y} \pm \frac{Nyy_1}{I_x} \tag{11}$$

or

$$f = \frac{N}{A} \pm \frac{M_y x}{I_y} \pm \frac{M_x y}{I_x}. \tag{12}$$

These, then, taking the principal axes as axes of coordinates, are the most general formulae, assuming planar distribution of stress.

If the plane of the outer forces passes through one of the principal axes, as OY, the expression reduces to the familiar form

$$f = \frac{N}{A} \pm \frac{M_x y}{I_x}. \tag{13}$$

Since the axes of coordinates may be chosen arbitrarily, it is convenient to choose the principal axes, so that in what follows Eq. (9) will be used as a basis of the investigation.

The assumption of planar distribution necessarily means that if the resultant force on a section acts at its center of gravity, the stress is uniformly distributed over the section. *It is important to note that this is not always true, as, for instance in a straight piece exposed to tension, in which there is a sudden reduction of area by a symmetrical notch* (see Chap. VI). *The assumption, then, is certainly untrue in some cases, and should be limited to pieces in which the section is uniform, or at all events in which there are no sudden changes of section. The distribution of stress must always conform to any other physical necessities which may exist.*

NORMAL STRESS IN SPECIAL CASES; PURE FLEXURE

6. The preceding formulae afford a complete solution of the problem of finding the normal stress at any point upon any section of a body, acted upon by outer forces in any manner whatsoever, under the assumption of planar distribution, *and assuming that the section is of homogeneous material.* Obviously, if the section taken should cut material of different degrees of hardness, as, for instance, part steel and part wood or concrete, the law of distribution would be different. Such cases will be treated later. For the present we assume the entire body to be homogeneous.

The fact has already been mentioned that the bodies generally occurring in engineering constructions are approximately prismatic in form, and their length is considerable in comparison with their cross-section. Often they are true prisms, having uniform area and shape of cross-section and a straight axis, perpendicular to which the sections are taken. Frequently, however, the cross-section varies in area and shape, and in such cases the axis may remain straight, or may be a succession of parallel straight lines, though not exactly in the same line. Every one possesses a tolerably distinct conception of what may be called a *straight piece*, such as shown in Fig. 134, in which (*a*) represents a prismatic piece with straight axis, (*b*) a straight piece with suddenly varying section but with straight axis, and (*c*) a straight piece whose axis consists of two straight parallel lines not

coincident, while (*d*) is a curved piece of constant section, and (*e*) is a piece with gradually varying section but straight axis.

A straight piece may best be defined as a body generated by the movement of a plane surface of constant or varying area, which remains always parallel to the same plane, and whose center of gravity moves along one straight line, or along a succession of several parallel and nearly coincident straight lines, to which the surface remains always perpendicular. According to this definition, a straight piece may have its exterior surfaces curved (as in Fig. 134*e*).

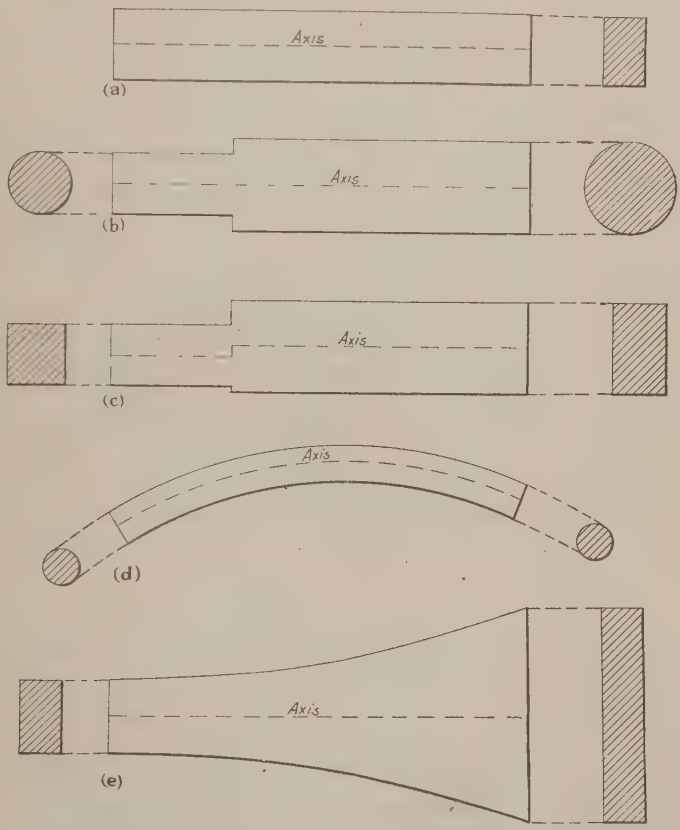


FIG. 134.

A curved piece, on the other hand, is one *whose axis is curved.* Its section may be constant or varying. If singly curved, its axis lies in a plane. *Such a piece may be defined as a body generated by the revolution of a plane surface of constant or varying area about a line or a succession of parallel lines, each lying in the plane of the surface while it serves as the axis of revolution.*

It will be seen in a later chapter that the distribution of a force on the section of a curved piece may be different from that on the section of a straight piece; in other words, that in the curved piece, it may be reasonable not to assume planar

distribution of stress. If such distribution is assumed, the value of f is found from Eqs. (8) or (9), whether the piece be straight or curved.

In the application of the preceding formulae to a straight piece, however, several special cases may occur, which should be here considered.

Case I. Force applied at the center of gravity; S_x and S_y both zero. Here x_1 and y_1 are zero, or in other words M_x and M_y are zero. The normal force is then uniformly distributed over the area, and the intensity at each point is $\frac{N}{A}$. This case is called that of central direct stress.

Case II. S_x and S_y both zero, but N not applied at the center of gravity. In this case Eqs. (11) and (12) apply directly, as in any other case, but as there are no shearing forces, the case is a special one, and is designated as a case of *eccentric direct stress*, as distinguished from Case I of *central direct stress*.

Case III. Normal force $N = 0$; M_x and $M_y = 0$. This is pure shearing by a shearing outer force applied in a plane of the cross-section; or rather, two equal and opposite shearing outer forces an infinitesimal distance apart. This case will not be treated here.

Case IV. Normal force $N = 0$, but M_x and M_y not equal to zero. In other words, the outer force on B in fig. 132 is parallel to the section but not in it, so that it has a moment about OX or OY or both. This case is that of *pure flexure or bending*. In order to understand it clearly, and to see its connection with the general case, the following considerations are useful.

In the general case, represented by Eqs. (11) and (12), there is a combination of a direct force N, applied at the center of gravity of the section with a couple whose moment is numerically $M = N\sqrt{x_1^2 + y_1^2}$, the effect of which is to remove the point of application of the resultant force on the section to the point a (Fig. 132). This couple may be resolved into two couples, acting about OX and OY respectively whose values are, numerically,

$$M_x = Ny_1;\ M_y = Nx_1,$$

as already explained.

If it were not for the couple M, there would be direct stress only, and a straight piece would remain straight under the loads. The couple M causes flexure or bending.

Now if the resultant R of all the outer forces on one side of the section is parallel to the section, there is no component N, but there is a couple M unless R acts *in the section itself*, in which case there is *pure shearing*.

Pure flexure will occur, for instance, if the normal component N is conceived to become smaller and smaller, while at the same time its point of application moves farther and farther from O along the line Oa always preserving the same moment M. When finally the value of N becomes zero, its point of application is at an infinite distance from O on the line Oa, or, in other words, R is parallel to the section and to the line Oa. Pure flexure will therefore occur when a straight piece is acted on by forces all of which are at right angles to its axis, the resultant force on each section being then parallel to that section, though the direction of the line Oa may be different on different sections.

The simplest case of pure flexure occurs when a straight piece is acted upon by forces all of which lie in a plane which contains the axis of the piece, cutting the section

in the line Oa, and when all these outer forces are at right angles to the axis. This is the usual case with which we have to deal.

7. Definition of "Beam."—The word beam, strictly speaking, means any piece in which there is flexure, that is, on any section of which the resultant force is not normal and applied at the center of gravity. The word is, however, seldom used in such a strict sense. Sometimes it is used to denote a straight piece in which any of the outer forces or components are transverse; that is to say, if all the outer forces are parallel to the axis (whether coincident with it or not) the piece is not a beam in this sense, although there would, of course, be flexure owing to the eccentric position of the resultants on the several sections; while if any single outer force, or several of them should make an angle with the axis, the piece would be a beam. Sometimes the term is used in a still more restricted sense, to denote a straight piece in which all the outer forces are at right angles to the axis.

In this book the word beam will mean a piece, whether straight or curved, acted upon by outer forces *any of which are transverse to the axis.* A beam, therefore, may be exposed to pure flexure, or may in addition be exposed to axial forces. A piece subjected to axial forces alone (forces parallel to the axis) will not be termed a beam, although it may be subjected to flexure, as already explained. The theory of beams, or flexure, will be treated in Chap. X, and the theory of pieces exposed to axial loads has been given in Chap. VI.

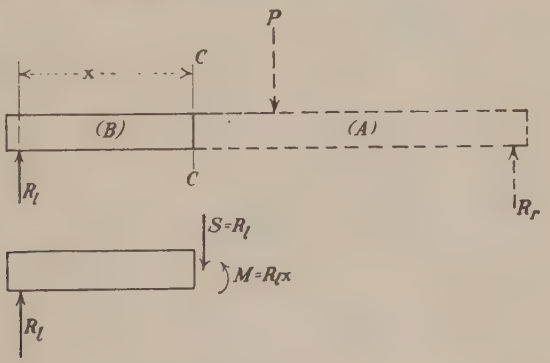


FIG. 135.

8. Representation of Forces.—In studying the stress on a section or in solving problems involving it, a figure should generally be drawn showing the condition existing. If the outer forces lie in a plane, as in pure flexure, the plane of the paper is taken as that plane, and the outer forces (loads and reactions) indicated by arrows. If the section to be studied is CC (at right angles to the axis), the portion of the beam to either side (in Fig. 135 the right side) is considered removed, and the portion on the other side is considered. In pure flexure, with a horizontal beam, the resultant outer force on the portion considered will be a vertical force in the plane of the paper, acting in the line of the resultant outer force R_o on this portion. In the figure this resultant is R_l, the left-hand reaction on the beam, since this is the only outer force acting on the portion B. A force equal and opposite to this is the force acting on the section (from A on B), and it balances R_o

($= R_l$ in this case). The resultant normal force on the section is zero, the resultant shearing force is $S = R_v$ ($= R_l$ in this case), and the point of application of the resultant is at the infinitely distant point in the section; in other words, there is a couple acting on the section which produces equilibrium by balancing the couple formed by R_l and S, and whose moment therefore equals $R_l x$. This is indicated by a line curved in the proper direction and labelled M (in this case a left-handed moment).

In any case of flexure, therefore, the forces acting upon a plane section may be represented, as in Fig. 36, by, (1) a normal force N acting at the center of gravity; (2), a shearing force S acting along the section; (3), a moment M (strictly speaking a couple whose moment is M); and these forces, with the outer forces on the portion considered, are in equilibrium. If the outer forces are all in the plane of one principal axis, that plane is the plane of the paper; if they are not, then for the section in question the plane of the paper may be considered to be the plane of one principal axis, OY, and the condition indicated as in Fig. 137. Figure 137*a* shows the cross-section, and the point a is indicated, at which the resultant force (exerted by A on B) cuts the section. Supposing this to be a tensile force, Fig. 137*b* shows the projection on the plane of Y, with N acting at O, the shear S_y in the proper direction, and a right-handed moment $M_x = Ny_1$; while Fig. 6*c* shows the plane or projection on the plane of X, with N as before, S_x in the proper direction, and again a right-handed moment $M_y = Nx_1$. This will enable the student to apply Eq. (11) intelligently with the proper signs.

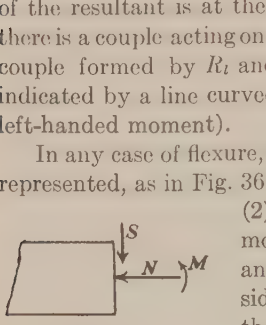


FIG. 136.

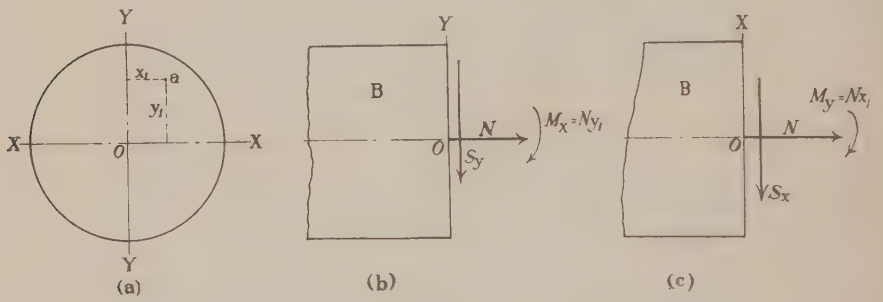


FIG. 137.

9. The Neutral Axis (*abbreviated to N.A.*).—Resuming now the consideration of the general equations developed in Art. 5, it is clear that, if planar distribution exists, the condition of stress on a section will be as represented in Fig. 138 (using a rectangular section for illustration). There will be one line in the plane of the section along which the stress intensity will be zero, and along any other line the stress intensity will vary uniformly from zero at the intersection with the line of no stress.

This line of no stress is called the *neutral axis* (abbreviated N.A.). It may cut the area of the section, as in Fig. 138, in which case there will be tension on one side of the N.A. and compression on the other; or it may lie entirely outside the

area of the section, in which case there will be the same kind of stress over the entire area, the kind depending upon the direction of N.

The equation of the N.A. is obtained by placing the value of f equal to zero; that is, its equation is (referred to the principal axes as axes of coordinates)

$$\frac{1}{A} + \frac{xx_1}{I_y} + \frac{yy_1}{I_x} = 0 \tag{14}$$

or

$$1 + \frac{xx_1}{r_y^2} + \frac{yy_1}{r_x^2} = 0.^1$$

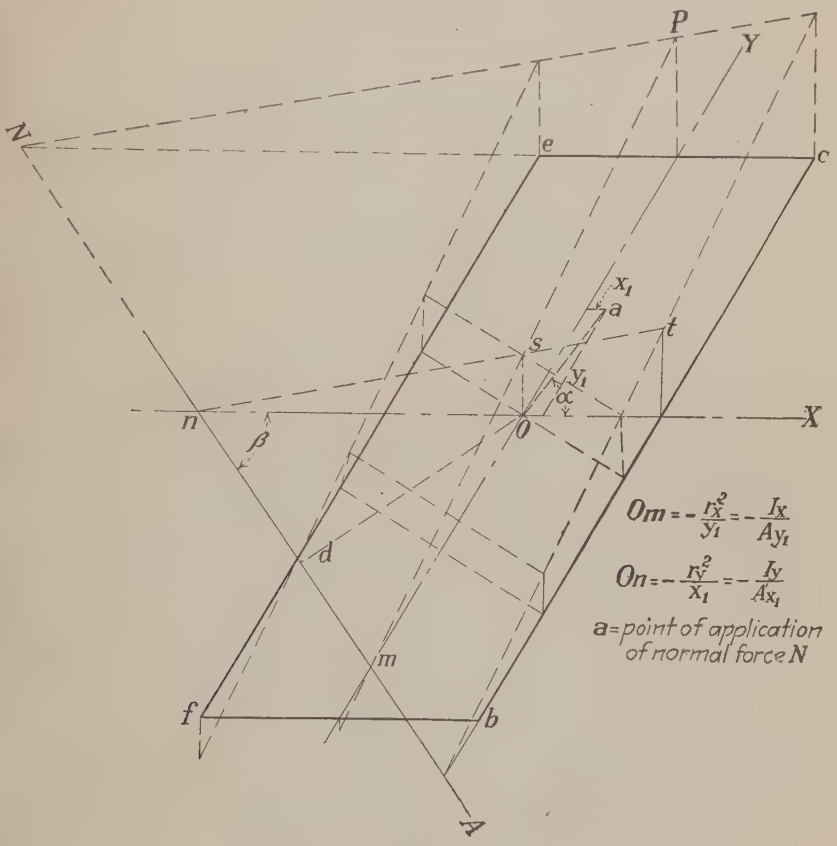


FIG. 138.

hence it cuts the principal axis OY at a point m whose distance from O is (Fig. 138)

$$Om = -\frac{I_x}{Ay_1} = -\frac{r_x^2}{y_1}. \tag{15}$$

[1] The reader who is familiar with analytic geometry will recognize this equation as that of the *antipolar* of the point (x_1y_1) with reference to the central ellipse. If through the point a, whose coordinates are x_1 and y_1, tangents are drawn to the central ellipse, and the points of tangency connected by a chord, that chord is the *polar* of the point a, with reference to the central ellipse. The *antipolar* is the corresponding line on the other side of O, symmetrical to the polar with reference to O; or it is the polar for a point whose coordinates are $-x_1$ and $-y_1$. The property is interesting to know, though perhaps not useful.

Similarly $$On = -\frac{I_y}{Ax_1} = -\frac{r_y^2}{x_1} \tag{16}$$

r_x being the radius of gyration of the section about the axis OX, or the axis of the central ellipse along OY, and r_y its radius of gyration about the axis OY, or the axis of the central ellipse along OX.

The angle which the N.A. makes with the axis OX is given by the equation

$$\tan \beta = \frac{r_x^2 x_1}{r_y^2 y_1} = \frac{r_x^2}{r_y^2} \cot \alpha \tag{17}$$

or $$\tan \alpha \tan \beta = \frac{r_x^2}{r_y^2} \tag{17a}$$

which gives the numerical value of tan β. The actual location of the N.A. can always be perceived by observing that *it crosses the quadrant opposite to that in which the point a (Fig. 132) is situated.*

The following important results have therefore been arrived at:

(1) *If the N.A. does not cut the section, the stress at every point of that section is of the same kind.*

(2) *If the N.A. does cut the section, the stress on the side of the N.A. toward the point of application of N is of the same kind as N, and of the opposite kind on the other side.*

(3) *The maximum intensity of stress occurs at the point which is at the greatest distance perpendicularly from the N.A.*

(4) *If the N.A. is at an infinite distance from O (and only then) the stress is uniform over the entire section.*

(5) *The location of the N.A. does not depend upon the magnitude of N, but only upon the position of the point (a) where it cuts the plane of the section.*

(6) *From Eq. (17) it is obvious that as the point of application of N (the point a) moves along a line passing through O (the c.g.) the corresponding N.A. will move parallel to itself. In other words, if the outer forces lie in a plane cutting the section at the axis (passing through c.g.) then no matter what the loads, the N.A. will always have the same inclination.*

10. Relation between Neutral Axis and Point of Application of N.—It remains to investigate more fully the relation between the location of the N.A. and the position of a, and some further relations between the location of the N.A., the value of N, and the magnitude of the stresses.

The perpendicular distance d of the N.A. from O is given by the equation

$$\frac{1}{d} = A\sqrt{\frac{x_1^2}{I_y^2} + \frac{y_1^2}{I_x^2}} = \sqrt{\frac{x_1^2}{r_y^4} + \frac{y_1^2}{r_x^4}}. \tag{18}$$

From the equation of the N.A., since x is interchangeable with x_1 and y with y_1, it is at once obvious that there are the following reciprocal relations between the point a and the N.A.:

(1) *If N_1A_1 is the N.A. corresponding to the point a_1 (where the resultant N cuts the section), then if, under other forces, the new point of application a_2 (of the new resultant force lies on N_1A_1, the new N.A. (N_2A_2) will pass through a_1.*

(2) *If the point of application a moves along a straight line l, the corresponding N.A. will rotate about a point p, and l will be the N.A. corresponding to p as the point of application.*

(3) (*Conversely*) *If the N.A. rotate about a point p, the corresponding position of the point of application (a) will move along a straight line which will be the N.A. if p is the point of application.*

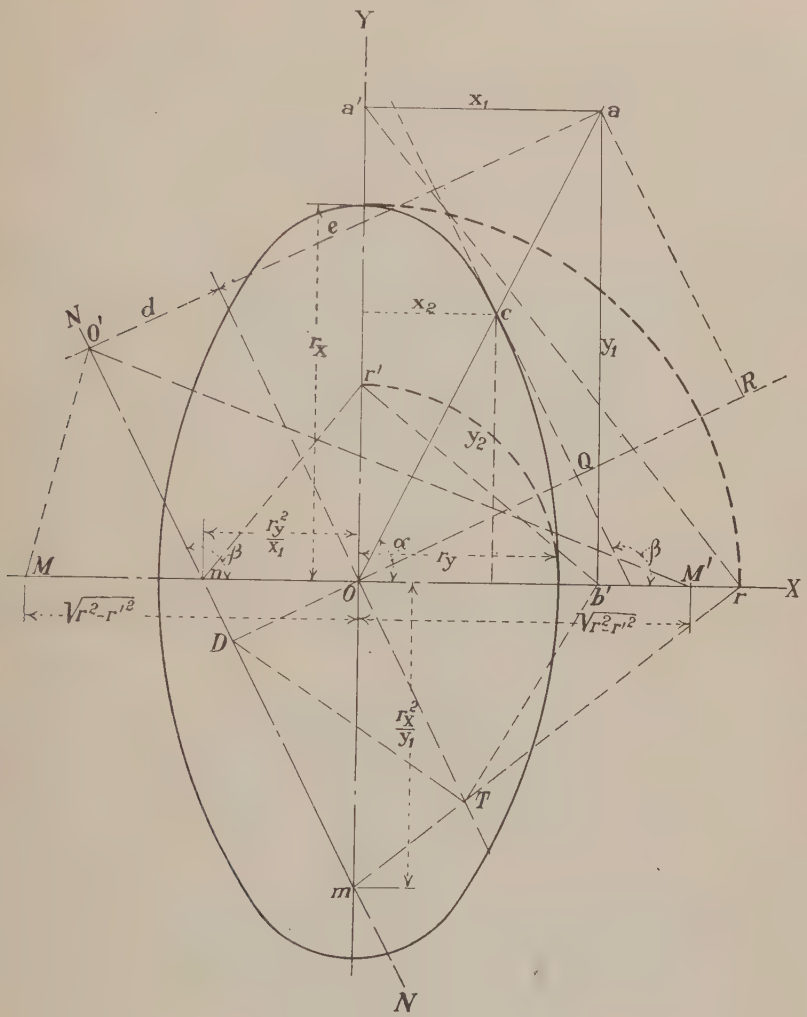


FIG. 139.

The equation of the central ellipse of inertia (or the ellipse of which r_y and r_x are the semi-axes along OX and OY respectively) is (Fig. 139)

$$\frac{x^2}{r_y{}^2} + \frac{y^2}{r_x{}^2} = 1$$

from which follows

$$\frac{dy}{dx} = -\frac{xr_x{}^2}{yr_y{}^2}.$$

Draw the line Oa, and let it cut the ellipse at c, whose coordinates are x_2 and y_2; then the tangent at c will make an angle β with OX defined by the equation

$$\tan \beta = -\frac{x_2 r_x^2}{y_2 r_y^2} = -\frac{x_1 r_x^2}{y_1 r_y^2}.$$

But this is the same angle which the N.A. makes with OX (see Eq. (17)). Hence the following principles:

The N.A. is parallel to the tangent to the central ellipse at the point where it is cut by Oa, *or*,

The N.A. is parallel to the diameter of the central ellipse which is conjugate to Oa.

In the case of pure flexure, *i.e.*, when there is no resultant normal force, the angle (β) which the neutral axis makes with the major principal axis OX, measured in a clockwise direction, will be given by Eq. (17); in which α is the angle which the plane of the loads makes with OX, measured in the opposite direction. The neutral axis in this case passes, of course, through the center of gravity.

In previous equations, f has been expressed in terms of coordinates referred to OX and OY. Draw from a the perpendicular to the neutral axis meeting the latter at O' (Fig. 139) and let similar relations be expressed referred to the rectangular axes NN and $O'a$. Clearly if a distance perpendicular to the neutral axis be termed v, and a distance along $O'N$ be w, f varies directly as v, or if c is a constant

$$f = cv$$

also if da is an elementary area,

$$\int f da = c \int v da = N \tag{19}$$

$$\int f v da = c \int v^2 da = N(e + d) \tag{20}$$

$$\int f w da = c \int v w da = O. \tag{21}$$

But since O is the center of gravity of the area,

$$c \int v da = cAd = N \tag{22}$$

and if I'_1 be the moment of inertia of the section about the neutral axis

$$c \int v^2 da = cI_1' = N(e + d) \tag{23}$$

while the product of inertia for the axes NN_1 and O^1a which is $\int vwda$, is equal to zero, as above.

Therefore, since for these axes the product of inertia is zero, *$O'a$ and the neutral axis NN are the axes of the ellipse of inertia for the point O'.* But it has previously been shown (Chap. III) that the axes of the ellipse of inertia for any point bisect the angle between lines drawn from that point to the fixed points M and M', so that in this case *the neutral axis bisects the angle $MO'M'$.*

Further, from Eqs. (22) and (23) we find

$$c = \frac{N}{Ad}$$

$$e + d = \frac{cI_1'}{N} = \frac{I_1'}{Ad}.$$

But c is the intensity of stress at a perpendicular distance of unity from the N.A., and $N(e + d)$ is the moment of the outer forces about the N.A., which we may call M_1, hence

$$M_1 = cI_1'$$

$$c = \frac{M_1}{I_1'}$$

and the maximum fiber stress, on an extreme fiber situated a distance v_1 from the N.A. is

$$f_{max.} = \frac{M_1 v_1}{I_1'} \tag{24}$$

which is the same formula that applies to the simple beam in pure flexure with all forces in the plane of one principal axis.

The same result may be seen by following the usual demonstration given for beams. If c is the intensity of stress at a unit distance from the N.A. the intensity at a distance v is cv, and the total moment M_1 about the N.A. is

$$M_1 = N(e + d) = \int cv^2 da = cI_1'$$

$$\therefore c = \frac{M_1}{I_1'}$$

$$\text{and } f_{max.} = \frac{M_1 v_1}{I_1'}.$$

It may, therefore, be said that, assuming planar distribution, the formula (24) is the general formula for the normal stress in beams, as well as Eqs. (8), (9), or (11), provided that M_1, v_1, and I_1' are all referred to the neutral axis.

11. Construction of the Neutral Axis.—The point of application of N being given (a), the N.A. is most easily constructed by laying off the distances $Om = -\frac{r_x^2}{y_1}$ and $On = -\frac{r_y^2}{x_1}$. The N.A. goes through m and n, as above shown.

The distances Om and On may easily be constructed. Draw (Fig. 139) with O as a center and r_x as a radius, a quadrant, so that $Or = r_x$; join r with a' and from r draw a perpendicular to $a'r$; this will meet OY in m. Similarly, revolve r_y so that it is laid off on OY at Or', join r' with b', and from r' draw a perpendicular to $b'r'$; this will meet OX in n. This construction renders any computation unnecessary.

The neutral axis may also be constructed by means of the central ellipse, but as this ellipse is not easily drawn, such construction is not as simple as the previous one, which only requires a knowledge of the axes r_x and r_y that is, of the radii of gyration about the principal axes of the central ellipse.

It has been shown that if I_1' is the moment of inertia about the neutral axis (Fig. 139)

$$d + e = \frac{I_1'}{Ad}.$$

If r is the radius of gyration about an axis parallel to the neutral axis but through the center of gravity O (*i.e.*, the perpendicular distance OQ from O to the tangent to the central ellipse where it is cut by Oa at c),

$$I_1' = Ar^2 + Ad^2 = A(r^2 + d^2)$$

$$\therefore d + e = \frac{r^2 + d^2}{d}$$

$$\therefore d = \frac{r^2}{e}. \tag{25}$$

We therefore have the following construction: Draw Oa and the tangent to the central ellipse where this line cuts it at C; Draw aR parallel to this tangent and OQR perpendicular to it. With O as a center and OQ as a radius draw the quadrant arc QT, or lay off $OT = OQ$: draw RT,[1] and from T a perpendicular

[1] The line RT is not drawn in the figure, as it would confuse it.

to RT, meeting OQR produced at D: then $OD = d$, and the neutral axis passes through D and is parallel to the tangent at C. For by this construction $d = \frac{r^2}{e}$.

12. The Kernel of the Section.—The kernel of a section is an area, in the section, such that if the resultant force on the section acts at any point within the kernel the stress on the section will be everywhere of the same kind. To find the kernel it is only necessary to let the N.A. move around the section, always just touching it, but never cutting it, and for each position of the N.A. to find the corresponding point a (Fig. 139). The outline of the kernel will be the locus of the points a. Bearing in mind the principles thus far ascertained, the following theorem will be clear:

For every straight line in the outline of the section, as the N.A. moves around it, there is a corresponding point and angle in the outline of the kernel; and for every angle in the successive position of the N.A. there is a corresponding straight line in the outline of the kernel. If the section is bounded by a curved line, the outline of the kernel will be a curved line.

The problem of drawing the outline of the kernel is to find the point a for a given position of the N.A. To do this, find the intercepts Om and On (Fig. 139) from O to the points where the N.A. cuts the principal axes OY and OX respectively. Then since $Om = -\frac{r_x^2}{y_1}$, it follows that $y_1 = -\frac{r_x^2}{Om}$; similarly $x_1 = -\frac{r_y^2}{On}$, and thus the coordinates of a are found.

13. Examples of the Kernel for Different Sections.

(1) *Rectangle* (Fig. 140).—Let the N.A. first be cd, and let us find the point of application of the corresponding force. Since the intercept of the N.A. on OY is $Oe = -\frac{h}{2}$ it follows that $y_1 = \frac{h}{6} = Og$, and g is the point of application of the force corresponding to cd as N.A. As the N.A. rotates about d the point of application of the force describes a straight line, and is at $h\left(Oh = \frac{b}{6}\right)$ when the N.A. is db. The kernel is the lozenge-shaped area $ghkn$, in which $gk = \frac{h}{3}$ and $kn = \frac{b}{3}$. This is the principle of the "middle third" of a rectangular section, which is generally expressed by saying that if the resultant cuts the section within the "middle third," the stress will be all of one kind. This, however, is inaccurate, and does not mean that the resultant may be anywhere within the area $ll'l'''l''$, or even within the area $tt't'''t''$. It must be within the smaller area $ghkn$ in order that the stress may be all of one kind.

(2) *I-Beam Section* (Fig. 141).—Assume a 20-inch beam, 81.4 pounds per foot for which A = 23.74 square inches (width of flange = 7 inches), $I_x = 1{,}466.3$ inches^4, $I_y = 45.8$ inches^4. Then when cd is the N.A. the point of application of the force is above O a distance $y_1 = \frac{r_x^2}{10} = \frac{1{,}466.3}{237.4} = 6.18$ inches. As the N.A. revolves about d, the next position which it takes tangent to the section is db, and for this position the force acts to the left of O a distance $\frac{r_y^2}{3.5} = \frac{45.8}{23.74 \times 3.5} = 0.55$ inch. The kernel is thus a parallelogram whose vertical diameter is 12.36 inches and whose horizontal diameter is 1.1 inches.

(3) *Angle-iron Section* (Fig. 142).—Assume the dimensions as shown, and the corners square, which is quite admissible; from the tables in the book of shapes, we find $I_x = 38.8$; $I_y = 80.8$; $A = 13.0$. Here the axes OX_1 and OY_1 are not the

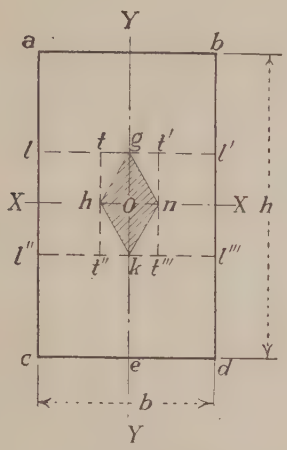


Fig. 140.

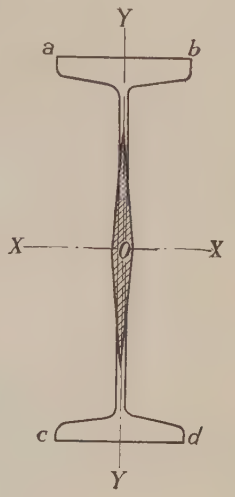


Fig. 141.

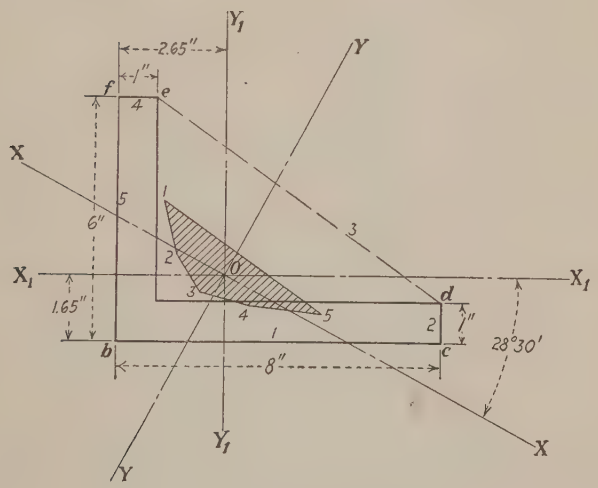


Fig. 142.

principal axes, and these must first be found. By Eq. (44) of Chap. III, the angle α, which a principal axis makes with OX_1, is given by the equation

$$\tan 2\alpha = \frac{2K}{I_y - I_x}$$

in which K is the product of inertia. This must therefore be found before we can proceed. The product of inertia (K_o) for a rectangle about its axes passing

through the c.g. parallel to the sides is zero, and for parallel axes passing through a point whose coordinates are x_o y_o, by Eq. (38) of Chap. III it is

$$K_1 = K_o + x_o y_o A.$$

Hence for the axes OX_1 and OY_1 of the angle iron the product of inertia is

$$K = K_1 = 8(-1.35)(1.15) + 5(-1.85)(2.15) = 32.31.$$

Hence $\alpha = 28°30'$ and the principal axes OX and OY are as indicated. From this, either by drawing the figure and scaling distances, or by computation, the intercepts on OX and OY by any position of the N.A. may easily be found. The Pocket Companion gives the minimum radius of gyration, about OX, as 1.28, which may be checked and found correct. Hence

$$I_x = 1.28 \times 1.28 \times 13 = 21.3$$

$$r_x^2 = 1.64$$

By Eq. 40, Chap. III

$$I_y = 80.8 + 38.8 - 21.3 = 98.3$$

$$r_y^2 = 7.56$$

From Fig. 139, the coordinates, x_1, y_1, of the point a for any neutral axis whose intercepts on the axes are known, are

$$x_1 = -\frac{r_y^2}{\text{intercept on } X} = -\frac{7.56}{\text{intercept on } X}$$

$$y_1 = -\frac{r_x^2}{\text{intercept on } Y} = -\frac{1.64}{\text{intercept on } Y}.$$

Drawing the figure, the results are:

bc as N.A.; intercept on X = 3.4; x_1 = −2.22
intercept on Y = −1.84; y_1 = 0.89
cd as N.A.; intercept on X = 6.05; x_1 = −1.25
intercept on Y = 11.1; y_1 = −0.148
de as N.A.; intercept on X = 20.8; x_1 = −0.36
intercept on Y = 2.6 ; y_1 = −0.63
ef as N.A.; intercept on X = −9.18; x_1 = 0.82
intercept on Y = 5.0; y_1 = −0.33
fb as N.A.; intercept on X = −3.05; x_1 = 2.48
intercept on Y = −5.65; y_1 = 0.29

The outline of the L as the N.A. passes around it, namely $bcdefb$, and the kernel, are a good example of *reciprocal figures;* to each line in one corresponds a point in the other, and to each point in one a line in the other. Thus, to the line bc in the L corresponds the point 1 in the kernel, and to the line 12 in the kernel corresponds the point c in the L. The lines in the L have been numbered to correspond with the points in the kernel.

14. Use of the Kernel in Finding Stress.—By the aid of the kernel, the expressions for max. stress intensity on a section may be simplified. Referring to Fig. 143 let the resultant force act at a, let the outer line represent the section, the next the ellipse of inertia for O (the c.g.) and the inner the kernel of the section. The neutral axis is NN, and let lines be drawn through O and a parallel to NN

(lines N_1 and N_4) and also a line tangent to the section parallel to NN (line N_3), touching the section at C *between* O and a. Let the kernel point corresponding to this tangent be K. Also let a line be drawn parallel to NN tangent to the central ellipse (line N_2). Then the perpendicular distance from a to N_1 is e, and between N and N_1 the distance is d. The distance from N_1 to N_2 is r, the radius of gyration of the section about N_1 (see paragraph 21 of Chap. III). The extreme distance of a fiber from N_1, which we call z, is the distance between N_1 and N_3.

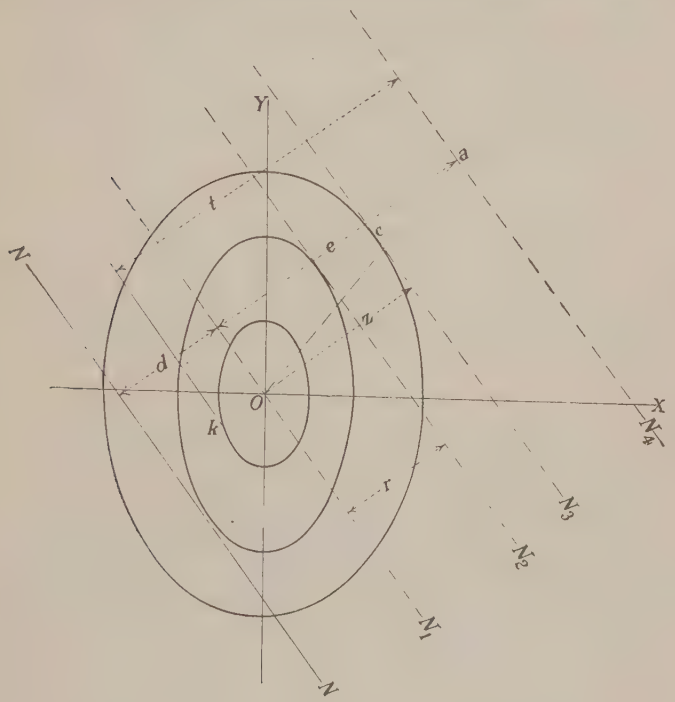


FIG. 143.

It has been shown in §10 that the stress intensity at C is given by the equation, I_1 being the moment of inertia about NN:

$$f_{\max.} = \frac{M(z+d)}{I_1} = \frac{P(e+d)(z+d)}{A(r^2+d^2)}. \tag{24}$$

Another expression for f is

$$f_{\max.} = \frac{P}{A} + \frac{Pe \cdot z}{Ar^2}. \tag{26}$$

These two expressions may be shown to be in agreement if it is remembered that, by Eq. (25), $d = \frac{r^2}{e}$. But since K is the point of application of a force for which the neutral axis is N_3 (Eq. (25) gives also

$$z = \frac{r^2}{l - e}$$

t being the perpendicular distance between K and N_4.

Hence $$tz - ez = r^2;\ r^2 + ez = tz;$$

and from (26)

$$f_{max.} = \frac{P}{A} \cdot \frac{r^2 + ez}{r^2} = \frac{Ptz}{Ar^2} = \frac{M'z}{I} \tag{27}$$

in which M' is the moment of the force applied at a, taken about the axis through K parallel to NN, and I is the moment of inertia about N_1.

Hence the max. stress, which is at C is given by Eq. (27), which has the same form as the formula for pure flexure, though the case may not be one of pure flexure.

There are thus three different formulae for the max. stress in the general case, namely, Eqs. (24), (26) and (27). Putting these together,

$$\text{max. } f = \frac{Mv}{I} \tag{24}$$

in which M, v, and I are all about or from the neutral axis;

$$\text{max. } f = \frac{P}{A} + \frac{M_1 z}{I} \tag{26}$$

in which M_1, z, and I all refer to an axis through the center of gravity parallel to the neutral axis;

$$\text{max. } f = \frac{M'z}{I} \tag{27}$$

in which z and I are as in Eq. 26, M' about a line through K parallel to the neutral axis.

All of these formulae are useful at times.

15. Construction for Maximum and Minimum Pressures.—If the total normal force acting on an area, and its line of action, are known, the maximum and minimum intensities may be found by a very simple construction if the force meets the section in one principal axis.

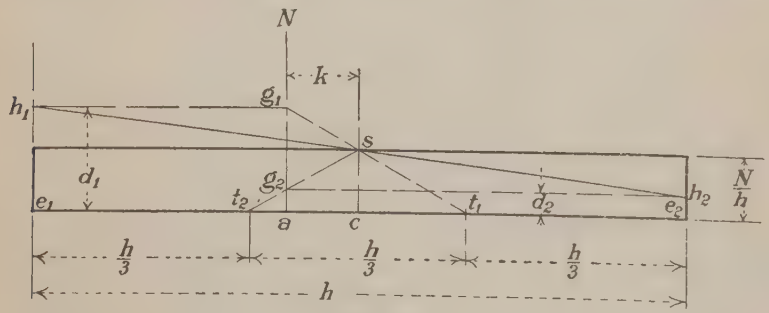


Fig. 144.

Rectangular Section.—In Fig. 144 let h be the depth of a rectangular section, whose breadth, perpendicular to the paper, is unity. Let the total normal force on the section be N acting at a. The plane of the outer forces is supposed to contain the principal axis of the section e_1e_2. Lay off normal to the section, at its center of gravity, c, the average stress intensity $cs = \frac{N}{h}$. Through the middle

third points t_1 and t_2 draw t_1s and t_2s_1 and produce to intersections with N at g_1 and g_2. Then

$$\text{max. intensity} = ag_1$$
$$\text{min. intensity} = ag_2$$

and the line h_1h_2 represents the real stress distribution.

For, if N acts to the left of c, in Eq. (11), $y_1 = k$, $x_1 = 0$, and

$$\text{max. stress, at } e_1 = \frac{N}{h} + \frac{6Nk}{h^2} \left(\text{since } y = \frac{h}{2},\ I_x = \frac{h^3}{12}\right)$$
$$\text{min. stress, at } e_2 = \frac{N}{h} - \frac{6Nk}{h^2}.$$

But $d_1 : \frac{N}{h} :: \frac{h}{6} + k : \frac{h}{6}$

$$d_1 = \frac{N}{h} + \frac{6kN}{h^2} = \text{max. intensity}$$
$$d_2 : \frac{N}{h} :: \frac{h}{6} - k : \frac{h}{6}$$
$$d_2 = \frac{N}{6} - \frac{6kN}{h^2} = \text{min. intensity.}$$

This is only a particular case of a general construction, shown in Fig. 145. Let e_1e_2 be one principal axis of any section which lies perpendicular to the paper.

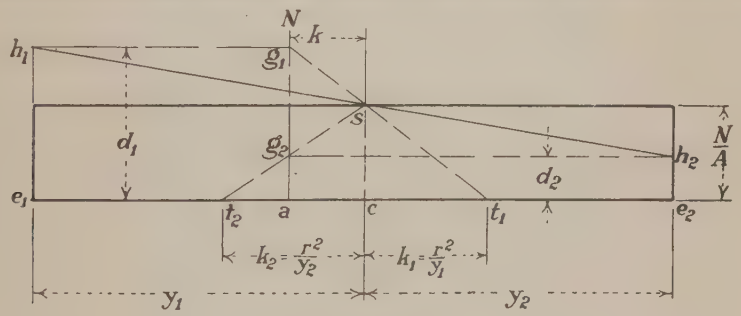


FIG. 145.

Let the plane of the outer forces be the plane of the paper—or if this is not so let the resultant of normal force on the section, N, cut the section in the principal axis, at a distance k from the center of gravity c, and to the left of it. Let t_1 and t_2 be the kernel points of the section, distant respectively k_1 and k_2 from c. The point t_1 is the point where N must act to cause zero stress at e_1; or

$$0 = \frac{N}{A} - \frac{Nk_1y_1}{I} = \frac{N}{A} - \frac{Nk_1y_1}{Ar^2}$$
$$k_1 = \frac{r^2}{y_1}.$$

Similarly,
$$k_2 = \frac{r^2}{y_2}.$$

With N acting at a,

$$\text{max. intensity} = \frac{N}{A} + \frac{Nky_1}{Ar^2}$$
$$\text{min. intensity} = \frac{N}{A} - \frac{Nky_2}{Ar^2}.$$

Lay off cs = average intensity $= \frac{N}{A}$.

Draw $t_1 s$ and $t_2 s$, meeting N at g_1 and g_2.

Then

$$d_1 : \frac{N}{A} :: \frac{r^2}{y_1} + k : \frac{r^2}{y_1}$$

$$d_1 = \frac{N}{A} + \frac{Nky_1}{Ar^2} = \text{max. intensity}$$

$$d_2 : \frac{N}{A} :: \frac{r^2}{y_2} - k : \frac{r^2}{y_2}$$

$$d_2 = \frac{N}{A} - \frac{Nky_2}{Ar^2} = \text{min. intensity.}$$

The real distribution is shown by the line $h_1 h_2$.

16. A neat graphical construction follows from the general equation for intensity of normal stress, *for pure flexure*,

$$f = \frac{M_x y}{I_x} + \frac{M_y x}{I_y}.$$

This equation holds only with reference to the principal axes. In the case where these axes are known, as an I-beam (Fig. 146), if M is the total moment acting on the

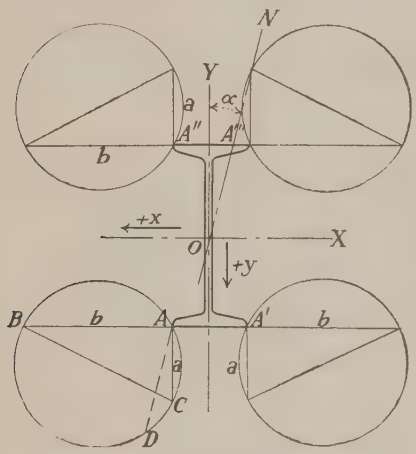


Fig. 146.

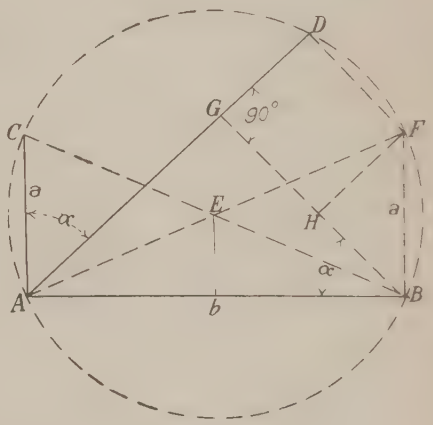


Fig. 147.

section, in the plane ON making an angle α with the Y axis, or in any plane parallel thereto (since to move the plane parallel to itself only introduces torsion, and therefore shearing stress, without affecting normal stress), then

$$M_x = M \cos \alpha;\ M_y = M \sin \alpha$$

$$f = \frac{My \cos \alpha}{I_x} + \frac{Mx \sin \alpha}{I_y} = M(a \cos \alpha + b \sin \alpha) \qquad (28)$$

where a and b are constants for any given point in the section. This equation shows how the normal stress at any given point varies, as the plane of the loads changes with reference to the axes.

Signs must be carefully observed. If M is + when it causes compression in fibers above OX, or to the right of OY, and if a tensile stress is +, x must be + to the left and y must be + downward, and α must be measured from OY in the direction as shown. If it is desired to measure x to the right and y upward when +, a compressive stress must be taken as +, or else M must be + when it causes tension above OX and to the right of OY.

Equation (28) is the polar equation of a circle with reference to a point on its circumference. For, in Fig. 147 lay off $AB = b$ and $AC = a$, at right angles, draw a circle with CB as a diameter, passing through A, draw AD making the angle α with AC and AF through the center E. Then $FB = a$, and is perpendicular to AB. Draw BG perpendicular to AD and FH perpendicular to BG. The angle $ABG = \alpha$; $AG = b \sin \alpha$; $GD = HF = a \cos \alpha$. Hence $AD = b \sin \alpha + a \cos \alpha$.

The same relation holds if b is laid off to the left of A and a downward.

In the I-beam section (Fig. 146), let the point considered be the lower left-hand corner, A, and lay off b and a as shown, and draw the circle with BC as a diameter. Draw AD parallel to ON, the plane of the loads, and AD = stress intensity at A for moment unity. AD must be scaled to the same scale for which a and b were laid off; multiply by M and the real stress at A is obtained.

For the point A', a is $+$ and b $-$, so that a is laid off downward and b to the right; for A'', b is $+$ and a $-$, so that b is laid off to the left and a upward; for A''', both a and b are negative.

The character of the stress itself can best be seen by observing the character of the loads and moment.

17. Distribution over a Section Not Homogeneous.—Considering always a solid body, which must deform as a whole, and in which one part cannot separate from or slide by another, though there may be tensile or shearing stresses between them, a body may nevertheless not be homogeneous. We are not here considering slight differences in homogeneity, as between one particle and another, for few bodies are probably entirely homogeneous; but suppose, for instance, that a steel piece extends longitudinally through a beam of concrete. Here a cross-section of the body, originally plane, though it need not remain plane, must yet remain a continuous surface, with no holes or cavities in it, for that would mean a separation between the parts. Hence two *adjacent* fibers, in the body, one in one material (say the concrete) and the other in the steel, must have the same change of length in a common distance ds; and therefore the stress intensities in the two materials must be directly as the moduli of elasticity. The material having the larger modulus must, at adjoining points, have a proportionately larger stress per unit, in order to change in length the same as the other material, per unit length.

It follows that the distribution of a force over the cross-section of such a piece may be found very simply by the methods already explained. Let one part of the cross-section have a modulus of elasticity E, and the rest the larger modulus E_1. Imagine each infinitesimal area of the latter material to be magnified in the ratio $\frac{E_1}{E}$, while remaining where it is—a purely imaginary conception, it is true, yet easily conceived and utilized—and consider the section to be of the same uniform material as the one whose modulus is E, and the formulae given in this chapter will apply. For example, suppose that the greater part of the cross-section is concrete, for which $E = 2{,}000{,}000$, but that 1 square inch is the section of a steel rod, for which $E = 30{,}000{,}000$. Imagine that the square inch of steel is replaced by 15 square inches of concrete, all concentrated at the single square inch. Find the total area and moment of inertia on this basis, and the formulae of this chapter will apply. If in finding I the area is divided into smaller or even into differential areas, multiply each smaller or differential area of the square inch of steel by 15, leaving the position unchanged. To find the actual stress intensity in

the steel, multiply the intensity at the steel, as found above (which is the intensity on the equivalent section entirely of concrete) by 15.

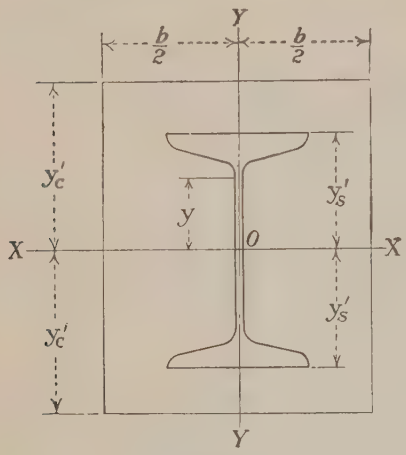


FIG. 148.

Example.—Suppose a rectangular concrete beam has imbedded in it (Fig. 148) an I-beam. Let I be the moment of inertia and A the area of the concrete, and I_1 and A_1 corresponding values for the steel beam. Let $15A_1 = A'$, and $15I_1 = I'$, be the the equivalent values for the I-beam, reduced to concrete. Suppose a compressive force P applied at the center of gravity. Then if f_c and f_s are the fiber stresses in concrete and steel, these will be constant over the respective areas, and

$$f_c = \frac{P}{A + A'}; f_s = 15\frac{P}{A + A'}. \quad (29)$$

If the force P acts at a distance y above the center, its moment about the axis OX through the center perpendicular to the I-beam web being Py; then

$$\text{max. } f_c = \frac{P}{A + A'} + \frac{Py \cdot y_c'}{I + I'} \quad (30)$$

$$\text{min. } f_c = \frac{P}{A + A'} - \frac{Py \cdot y_c'}{I + I'} \quad (31)$$

$$\text{max. } f_s = 15\left[\frac{P}{A + A'} + \frac{Py \cdot y_s'}{I + I'}\right] \quad (32)$$

$$\text{min. } f_s = 15\left[\frac{P}{A + A'} - \frac{Py \cdot y_s'}{I + I'}\right] \quad (33)$$

If there is a resultant tension on the concrete, that is to say, if

$$\frac{P}{A + A'} < \frac{Py \cdot y_c'}{I + I'},$$

then, according to usual practice, the concrete will be considered to carry no tension; but under this assumption the position of the neutral axis for pure flexure will be changed, and the above formulae also. This subject will be pursued farther in the chapter dealing with reinforced concrete beams.

18. Some further discussion of the question of the actual law of distribution of stress over a surface on which the stress varies is given at the end of the next chapter. It is very important, and should not be omitted by the reader who wishes to understand the subject thoroughly. It is deferred till the end of the next chapter because it is necessary first that the reader should have mastered the subject of flexure, and because it has important results bearing on flexure.

19. Historical Notes.—The stress in beams and the position of the neutral axis has been the subject of a long and gradual development, simple though it is.

Galileo supposed that a beam tended to rotate about the extreme fiber, as shown in Fig. 149, so that every fiber was in tension. Clearly he had not realized that

$\Sigma H = 0$. It may be a consolation to some readers to realize that this great man had not grasped the most elementary principles of statics, but it must be remembered that he lived from 1564 to 1642, before Newton, the discoverer of gravitation and of the laws of motion, was born.

Mariotte (1620–1684) supposed that the neutral axis lay at mid-height.

The first correct determination of the position of the neutral axis, by making the sum of the stresses equal to zero for the case of pure flexure, was by *Parent* (about 1710). *Coulomb* used the same principle and found that in a symmetrical section the axis was at mid-height. *Navier* (1785–1836) first showed that the neutral axis, for pure flexure, passed through the center of gravity, and first deduced the equation $M = \frac{fI}{y}$. He, however, wrongly assumed that the neutral axis was always perpendicular to the plane of the forces. *Persy* first showed that this is not correct except in special cases. *Bresse* (1859) first gave the determination of the position of the neutral axis by the aid of the ellipse of inertia, and he was also the first to introduce the kernel of the cross-section, in his "Résistance des Materiaux" (1854).

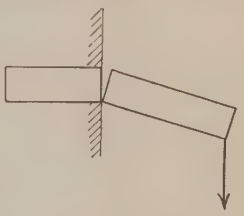


FIG. 149.

20. References.—*Bresse*, in the above-mentioned work (3d edition, 1880), discusses this subject, finds the equation for the neutral axis, and explains the kernel, but does not show the application of the kernel in finding max. stress. W. Ritter, in his "Anwendungen der Graphischen Statik," part I, Zurich, 1888, also discusses the subject, and shows the use of the kernel in finding max. stress. Müller-Breslau, in his "Graphische Statik der Baukonstructionen," vol. 1, 4th ed. Stuttgart, 1905, goes into the subject in detail. Professors Mohr, Land, and others have written on the subject in Germany. Finally, the writer's colleague, Prof. L. J. Johnson, has published two papers on the subject (1) "The determination of unit stresses in the general case of flexure," in the *Journ. Assoc. Eng. Societies*, for May, 1902; and (2) "An analysis of general flexure in a straight bar of uniform cross-section," in *Trans. Am. Soc. C.E.*, 1906, p. 169. The reader who desires to pursue the subject will be interested in these papers. The writer's treatment, in the present chapter, has been different, and he believes he has presented the subject in the simplest and most comprehensible way, requiring the least tax on the memory.

CHAPTER X

FLEXURE, OR THE THEORY OF STRESS IN BEAMS

1. By *pure flexure*, or simply *flexure*, is meant the case of a piece, or a portion of a piece, all of whose sections are exposed only to a bending moment, and to no direct force; in other words, in which the resultant force acting upon any section is parallel to the section but not in the plane of the section, cutting the section itself in an infinitely distant point.

By *section*, or *cross-section*, is meant the area cut from the piece by a plane at right angles to the axis.

By *axis* is meant the locus of the centers of gravity of all the sections. The axis is also known as the *elastic line* when deformed by a load.

The sections and the axis are therefore interdependent; to find one, the other must first have been found. Either, therefore, must be found by a process of successive approximation, except in the case of a prismatic piece, or one in which inspection suffices to indicate the axis. All the forms of members in common use come within this category—the axis and the sections are obvious on inspection.

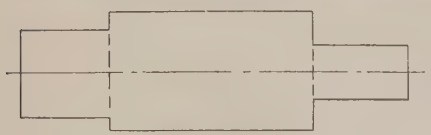


FIG. 150.

FIG. 150*a*.

If a piece varies in cross-section gradually, that is, if its outlines are continuous curved surfaces without angles, the axis is also a continuous curve, having no angles. If, however, the outlines of the piece do not fulfil the above condition—if, for instance, the cross-section varies suddenly as in Fig. 150*a*—the axis *may* have offsets, as shown, and *may* not be continuous. It is possible, however, for the outlines to have sudden changes, and yet for the axis to be continuous, as in Fig. 150. In any practical case, there will be no difficulty in determining the axis and the sections.

2. The principles in Chap. IX enable the distribution of the inner forces, upon any section of a piece exposed to pure flexure, to be determined if the resultant force S, acting upon the portion of the beam to one side of the section, and therefore resisted by the section, is first found. This force, acting parallel to but not in the plane of the section (at a distance d from it) is equivalent to (may be replaced by) an equal force *in* the plane of the section, and a couple composed of two forces also parallel to the section and in a plane at right angles to it. This couple, however, can be rotated in its own plane, and may therefore be considered as consisting of two forces at *right angles to the section*. If the force

S is not in the plane of the paper, which contains the axis of the piece, it causes a torsion.

In pure flexure, therefore, the resultant outer force acting upon any section is equivalent to a shearing force equal to S in the plane of the section, and two equal and opposite forces at right angles to the section whose moment is Sd (Fig. 151). The magnitude of the shearing force is fixed, being equal to the resultant (S) of all the outer forces acting on that portion of the body on one side (either side) of the section, while the magnitude of the forces N at right angles to the section is not fixed, and those forces may have any magnitude provided their distance apart gives to the couple the required moment, Sd.

The location of these forces may be easily perceived. Pass a plane through the resultant outer force S, and perpendicular to the section; then the shearing force acts in the intersection of this plane with the section, that is, in the line

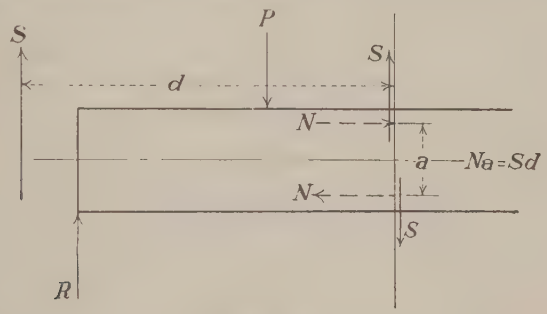


FIG. 151.

which we may call L, while the forces N act at points in L and perpendicular to it.

The line L will not in general pass through the center of gravity of the section O, but may be distant from it a distance c. If it does not pass through O, the shearing force S may be resolved into an equal force S acting through O, and a couple whose moment is Sc in the plane of the section. This couple will tend to twist the piece about its axis. The forces acting upon any section of a piece in pure flexure is therefore equivalent (may be resolved into), to (1) a shearing force equal and parallel to S, and in the same direction acting in the section and through its center of gravity; (2) a twisting couple composed of forces in the plane of the section, having a moment equal to Sc; and (3) a couple composed of forces at right angles to the plane of the section, acting at points in L, and having a moment equal to Sd. The moment Sc is a *twisting moment;* the moment Sd is a *bending moment.* The effect of a twisting moment has been considered in Chap. VIII.

3. Beams: Kinds of Beams.—A piece subjected to pure flexure is called a *beam.* If it is also subject to a resultant direct stress upon its sections, it is a combination of a beam with a tie or with a strut, according as the direct stress is tension or compression. Sometimes, however, a piece is called a beam if it is exposed to any flexure, whether accompanied by a resultant direct stress or not. It will often be used in this sense in this book. The axis of a beam may be straight or curved. It is evident that a section of a beam is exposed to shearing, tension and compression.

Kinds of Beams.—Beams differ with regard to material and shape, character of loading, and manner of support. The only effect of the material is on the safe stresses, and on the deformations, which will differ as the value of E differs. Beams may be of constant cross-section, or varied. The reason for varying the section is to proportion the section to the stresses, and to approximate to uniform strength at all sections, though this result can never in practice be obtained exactly.

The loading is either by concentrated loads, or by distributed loads. Strictly speaking, a load cannot be concentrated at a point, for that would mean an infinite

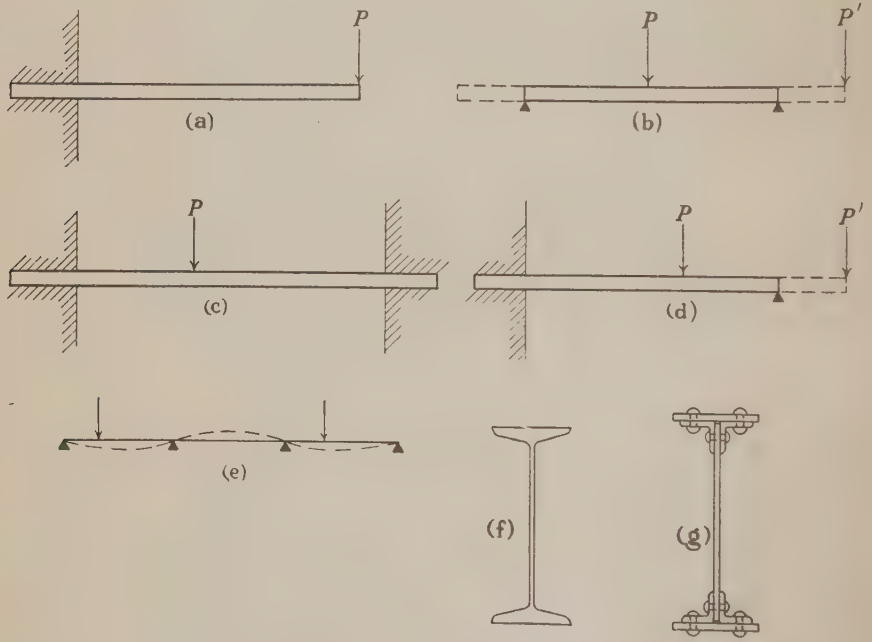


Fig. 152.

pressure per square unit, which would cause indentation, and thus a distribution of the load over some area, even if small. But loads are frequently assumed to be concentrated, as, for instance, the load on the wheel of a car or wagon. A *distributed load* may be *uniformly distributed, i.e.*, with the same load on each unit of length of the beam, or not uniformly distributed; the manner of distribution must of course be known or assumed before the beam can be calculated.

With reference to the manner of support, the ends may be *simply supported, i.e.*, each reaction may be a single force acting normal to the beam, with no bending moment at the end; or the ends may be fixed in direction in some way, which requires the existence of a restraining moment to prevent the tendency to rotation which would occur if the beam were simply supported. A beam may be *fixed at one end* (Fig. 152*a*), in which case it is called a cantilever; it may be simply

supported at each end (Fig. 152*b*); it may be fixed at both ends (Fig. 152*c*); or it may be fixed at one end and supported at the other (Fig. 152*d*).

Strictly speaking, fixed ends mean ends which are held rigidly, by some restraining forces (as by the resistance of a wall in which the ends are imbedded), in the direction of the original axis; *i.e.*, there is *no rotation* at the end. This condition is practically never fulfilled, for the wall or other agency which restrains the ends will be itself deformed, if only slightly, by the tendency of the ends to rotate, so that the ends are really only partially fixed. In fixed-ended beams there is a moment at the end sufficient to preserve the original direction; in partially fixed-ended beams there is a moment at the end, not quite large enough to do so. If the moment at the end is *greater* than necessary to preserve the original direction, the axis is given a rotation in the other direction and the end is more than fixed. A beam supported at each end, or fixed at one end and supported at the other, may also be extended beyond the simply supported points, as shown by the dotted lines in Figs. 152*b* and 152*d*, in which case it is called a cantilever. A beam, of course, need not be horizontal; a telegraph pole is a vertical beam fixed at the bottom. Sometimes the term cantilever is only applied to the projecting portion of a beam, though it is better to term this the *cantilever arm*, and the whole beam a *cantilever beam or truss*.

A *continuous beam* is one which rests on more than two supports (Fig. 152*e*). It is really a series of beams resting on two supports, in which there are bending moments at adjoining ends of sufficient magnitude to bring the ends together and make the axis continuous at those points; *i.e.*, it is a series of beams with ends more or less fixed.

A beam is sometimes called a *girder*, though the latter term is more properly applied to a steel beam made up by riveting several plates and shapes together. Thus the term beam includes *wooden beams* made of a single stick of wood, *steel beams* made of a piece of steel rolled in one piece (Fig. 152*f*) known as an I-beam, and *girders* (Fig. 152*g*) made by riveting several steel pieces together, known as a *plate girder*. Sometimes a steel I-beam is called a girder, and in distinction a plate girder is called a *built-up girder*. But all these structures come under the general title of *beam*. The essential characteristics of a beam are that it is exposed to flexure, and that it is solid throughout. A structure supported at its ends, or at any other points, and exposed to transverse loads, but composed of bars connected together at their ends, would have many of the characteristics of a beam, but is termed a *truss* or a *framed structure*.

4. The main problems to be solved in connection with beams are to determine (1) the fiber stresses; (2) the work done by the forces acting; (3) the deflection or deformation.

In common practice, almost all beams exposed to pure flexure are straight pieces, prismatic in shape (uniform sections) or with sections varying symmetrically on each side of the axis, so that the axis is a straight line. Moreover, the outer forces acting on beams generally all lie in one plane, and this plane contains the axis of the beam. In this case there is no twisting moment.

5. The Normal Stresses in Beams Subject to Pure Flexure.—The formulae given in Chap. IX cover this case completely. The intensity of the normal stress is, in the most general case (since $P = 0$), and for the case where the two axes OX and OY are principal axes of the section ($K = 0$),

$$f = \frac{M_y I_x x + M_x I_y y}{I_x I_y}$$

$$= \frac{M_y x}{I_y} + \frac{M_x y}{I_x}. \qquad (2)$$

If the plane of the forces cuts the section in the principal axis OY, or if the resultant outer force on one side of the section cuts the section in OY, $M_y = 0$, and

$$f = \frac{M_x y}{I_x}. \qquad (2a)$$

For the sake of completeness, the simplest and most usual case will be again treated here, namely, the case in which the section is symmetrical, with one axis of symmetry in the plane of the paper, this plane being also the plane in which all the outer forces lie.

Consider the beam in Fig. 153. The plane of the section is ss, the resultant outer force on the left of the section is S, acting parallel to the section and at a

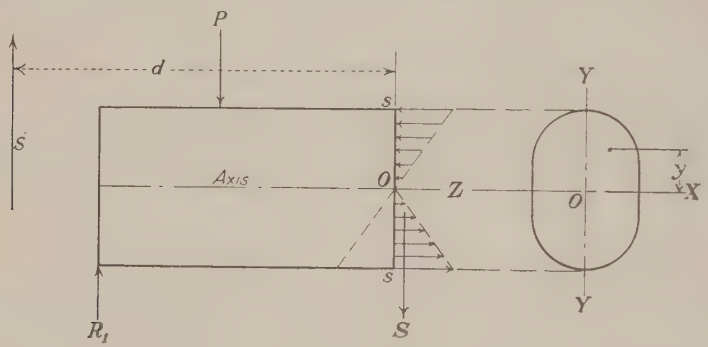


Fig. 153.

distance d from it. On the right is shown the section of the beam, YY being the plane of the paper, and being also an axis of symmetry of the section. O is the center of gravity of the section.

The only statical equations available to determine the inner stresses on the section, express the equilibrium of the portion of the beam shown, under the action of S and those stresses, namely:

$$\Sigma X = 0;\ \Sigma Y = 0;\ \Sigma Z = 0;\ \Sigma M = 0$$

OY and OZ are in the plane of the paper; OX is perpendicular to the paper at O, and in the plane of the section.

On account of the symmetry of the section, the neutral axis will be parallel to OX, or perpendicular to the plane of the forces (plane of the paper).

The equation $\Sigma Y = 0$ indicates only that there must be a resultant vertical shearing stress equal and opposite in direction to S acting on the section, as shown.

The equation $\Sigma X = 0$ indicates only that the resultant shearing stress on the section in the direction of the axis OX is zero. At any given point in the section there may be horizontal shearing stresses, but these must balance each other. Since the section is symmetrical, it is clear that at points in the section symmet-

rically situated with reference to OY the intensity of these horizontal shearing stresses must be equal, and their directions opposed.

The two equations just used tell us simply the *value* of the resultant shearing forces, but nothing as to their *distribution.*

The equation $\Sigma Z = 0$, tells us that the resultant normal stress on the section equals zero, since no other horizontal forces are acting parallel to OZ.

We have seen in Chap. IX that it is reasonable to assume planar distribution of the normal stress, as indicated in the figure. We have, however, not yet found the position of the neutral axis. If f is the intensity of stress at any distance y above the neutral axis, it results from planar distribution that $f = cy$, c being a constant representing the intensity at a unit distance from the neutral axis. If dA is a small (infinitesimal) area, the total normal stress on the section is

$$\int f \cdot dA = c\int y \cdot dA = 0 \quad \therefore \int ydA = 0;$$

which proves that the neutral axis passes through the center of gravity of the section. We see, therefore, that

If there is pure flexure on a section, the neutral axis necessarily passes through the center of gravity of the section. This is true no matter what the shape of the section, since it depends simply upon the fact that $f = cy$, or that the distribution is planar.

The equation $\Sigma M = 0$ gives

$$Sd = M = \int fydA = c\int y^2 dA = cI = \frac{f_1 I}{y_1}$$

$Sd = M$ is the moment, about O, of all the outer forces on the left of the section, or what is termed *the moment at the section;* f_1 is the intensity of stress at the extreme fiber, either top or bottom, and y_1 is the distance of that fiber from the neutral axis. The formula

$$M = \frac{fI}{y} \tag{3}$$

f being the intensity of stress at any distance y from the neutral axis, is the well-known formula for beams subject to pure flexure. The quantity $\frac{I}{y_1}$ is called the *section modulus.* It depends only upon the shape of the section.

It is evident that this formula depends upon a single assumption, namely, that the intensity of the stress is directly proportional to the distance from the neutral axis. This single assumption will be fulfilled if three others are fulfilled, namely:

(1) That sections which are plane before bending remain plane after bending.

(2) That Hooke's Law is true; that is to say, that the intensity of stress is everywhere below the elastic limit.

(3) That the modulus of elasticity for tension is the same as that for compression.

It will be pointed out, however, in paragraph 21, that the single assumption first named does not necessarily require the last three, and some further remarks will there be made with reference to this subject.

Note that the neutral axis is a line in each cross-section of the beam, along which the internal normal stress is zero. The locus of all the neutral axes forms a *neutral surface,* extending from one end of the beam to the other.

The internal normal stresses on any section, in pure flexure, constitute a couple whose moment balances the moment of the outer forces, and is called the *moment of resistance.*

A bending moment at any section of a beam is positive when right-handed on the left of the section, and it causes the beam to bend so as to be concave upward, with compression in the top fibers; and *vice versa* for a negative moment. A shear is positive when the resultant outer force on the left of the section is upward, tending to slide the part on the left up and the part on the right down; and *vice versa* for a negative shear.

It is to be observed that if a given bending moment, in pure flexure, is applied to a beam of given section, but if this moment may act in different planes passing through the center of gravity of the section, the plane of action which will give the maximum fiber stress intensity will be the one giving the greatest value of $\frac{y_1}{I}$ about the neutral axis corresponding to that plane of action; it will not necessarily be when the plane is that of the maximum principal axis, *i.e.*, when the bending takes place about the least principal axis as neutral axis (see Chap. IX, Art. 10). Thus, in Fig. 154, let the bending moment M act in the plane making the angle α with the maximum principal axis OX; then $M_y = M \cos \alpha$, $M_x = M \sin \alpha$. The maximum stress will be at D or D', and will be

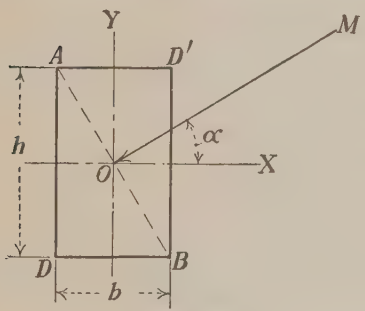


FIG. 154.

$$f = \frac{6M \sin \alpha}{bh^2} + \frac{6M \cos \alpha}{hb^2}.$$

This will be a maximum for $\tan \alpha = \frac{b}{h}$, that is, when the plane of the forces (moment) is perpendicular to the diagonal AB,[1] and

$$\text{max. } f = \frac{6M\sqrt{b^2 + h^2}}{b^2h^2}.$$

6. The Load Line.—The loads acting upon a beam, if concentrated, are represented by arrows, with appropriate letters, generally P, with subscripts. If the load is distributed, however, it may be represented by erecting at each point of the beam a line perpendicular to the axis, proportional to the load per unit of length at that point, and the locus of the points so fixed will be the *load line or curve.* Evidently the load between any two points will be represented by the area included between the load line and the axis between those points, or the area between the beam and the load line represents a sheet of material whose weight is the load.

Concentrated loads, including reactions, may be considered to be represented by a load line consisting of disconnected rectangles of infinitesimal width and infinite height, whose areas represent the loads. With this conception in mind, any loading may be said to be represented by a load line. Loads acting in opposite directions must of course be laid off on opposite sides of the axis.

[1] PROFESSOR C. J. TILDEN in *Trans. Am. Soc. C. E.*, vol. XLI, p. 196.

7. The Moment and Shear Lines. -To determine the maximum fiber stress at any section, for a given loading, the moment at that section for the given loading must first be found. Having the moment at every section we may draw what is called the *moment line*, by erecting at each point of the beam a perpendicular to the axis, having a length representing, to some convenient scale, the moment at that point, for the given loading, and joining the points so found by a line, straight or curved as the case may be. Thus, in Fig. 155, l is the length of the beam, and M_x is the moment at a point at a distance x from the left abutment.

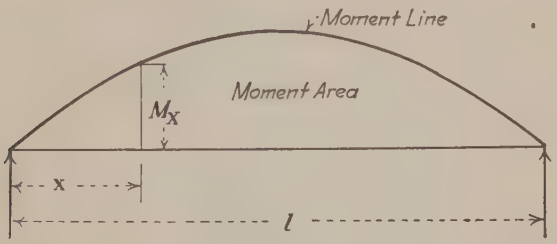


FIG. 155.

The curved line is the moment line. The area between the moment line and the axis is called the *moment area.* Similarly, in Fig. 156 is shown a shear line, in which any ordinate as S represents the shear at the point on the beam at which the ordinate is drawn.

In Chap. XI, the moment and shear lines for different kinds of beams and different loadings will be investigated in detail. It is the purpose here only to explain the general properties of these lines.

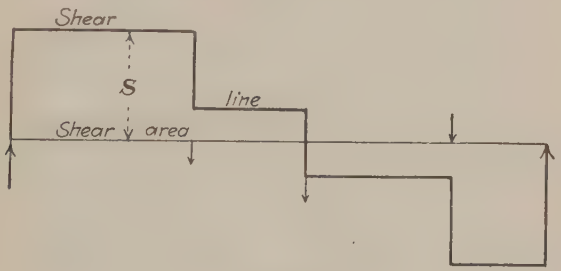


FIG. 156.

8. Relation between Moment and Shear for Beams in Pure Flexure.—In Fig. 157, consider the left-hand portion of a beam cut by a section ss. R_1 is the reaction and P_1 and P_2 are loads.

The resultant outer force on the left of the section is $S = R_1 - P_1 - P_2$.

This force acts to the left of R_1 if $R_1 > (P_1 + P_2)$, otherwise to the right of R_1. The shear at the section is S, the moment is $Sd = R_1e - P_1d_1 - P_2d_2 = M$.

Take now another section s_1s_1 to the right of ss and distant a from it. If there is no load between the two sections, then at s_1s_1 the shear will be S, as on section ss, while the moment will be $M + Sa$.

If between the two sections there are concentrated loads such as P_3 (Fig.158), distant d_3 from s_1s_1, then at s_1s_1

$$\text{the shear} = S_1 = S - \Sigma P_3 \tag{4}$$

$$\text{the moment} = M_1 = M + Sa - \Sigma P_3 d_3. \tag{5}$$

If, instead of a concentrated load, or loads, between the two sections, there is a distributed load whose value per longitudinal unit at any point is the variable p, then at s_1s_1

$$\text{the shear} = S_1 = S - \int_{ss}^{s_1s_1} p\,dx \tag{6}$$

$$\text{the moment} = M_1 = M + Sa - \int_{ss}^{s_1s_1} p\,dx\,(e_1 - x) \tag{7}$$

where x = distance from R_1 to the load $p\,dx$.

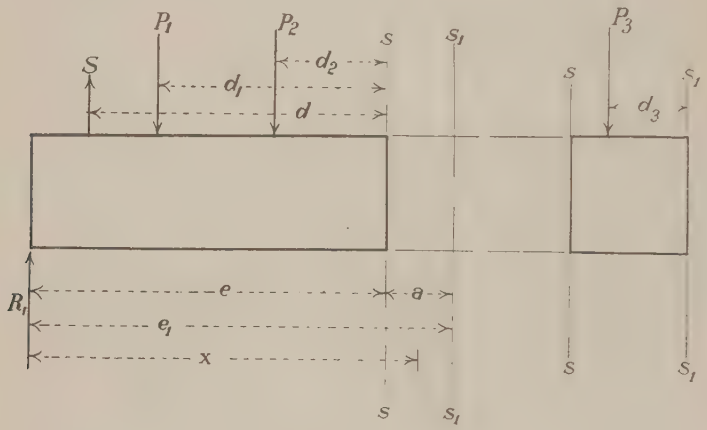


Fig. 157.

Fig. 158.

If $e_1 - e = a = dx$, *i.e.*, if the two sections are an infinitesimal distance apart,

$$S_1 = S - p \cdot dx$$

$$M_1 = \mathrm{M} + S \cdot dx - p \cdot \frac{dx^2}{2} = M + Sdx$$

since the last term is infinitesimal compared with the others. From this it follows that

$$S_1 - S = dS = -pdx$$

$$\frac{dS}{dx} = -p \tag{8}$$

$$M_1 - M = dM = Sdx$$

$$\frac{dM}{dx} = S. \tag{9}$$

The differential coefficient of the moment with respect to x always equals the shear except in the case when a concentrated load is at the section where this coefficient is found, in which case there is a sudden change in $\frac{dM}{dx}$ at that point,

and we can only find the value of this differential coefficient on either side of the point.

Differentiating the last eq. again,

$$\frac{d^2M}{dx^2} = \frac{dS}{dx} = -p. \tag{10}$$

These results should be interpreted.[1] In the first place, the significance and utility of Eqs. (4), (5), (6) and (7) should be fully grasped. In the usual method of finding the shear and moment at a given point we consider all the portion of the beam to one side of that point; in other words, we start with the end of the beam, where the moment and shear are generally zero. These equations show that we may equally well start at any other point, provided the moment and shear at that point are known, and consider the portion of the beam between that point and the point at which the moment and shear are desired. This principle is of frequent use.

It is evident that the shear is constant at all points between two concentrated loads, if there is no distributed load acting, and that at each concentrated load it changes suddenly by the value of such load, giving a shear curve like that in Fig. 156. In this case $\frac{dM}{dx}$ is constant between consecutive loads, *i.e.*, the moment line is a straight line between such loads, and the tangent of its inclination to the axis of x is S. At each concentrated load, and at no other points, there is an angle in the moment line. *For a system of concentrated loads the moment line is therefore a series of straight lines between the points of application of the concentrated loads; and the shear line is a series of steps parallel and perpendicular to the axes.*

If the load is distributed, whether uniformly or not, the shear changes continuously over the portion of the beam covered by such load; hence the shear line is not a broken line consisting of a series of steps as for a series of concentrated loads but a continuous straight line or curve. If the load is uniformly distributed (p = constant) the shear line is an inclined straight line, dropping in each unit of length by the value of the load per unit length. If the load is distributed, but not uniformly, the shear line is a curve. The tangent to the shear line at any point makes an angle with the axis of x equal numerically to p at that point. Similarly, for a distributed load, the moment line is a continuous curve, with no sudden changes in direction. The tangent to the moment line only changes suddenly at points when S changes suddenly, *i.e.*, at concentrated loads. The tangent of the angle between the axis and the line tangent to the moment curve at any point is equal to the shear (S) at that point.[2]

From the equation $dM = Sdx$ we further see that, between any two points of the beam

$$\int dM = \text{change of moment} = \int Sdx.$$

Hence, *the change of moment between any two points equals the area of the shear curve between those points.*

[1] It is most important that the reader should acquire the habit of *interpreting* all algebraical expressions, so far as possible, otherwise they remain mere abstractions or combinations of letters in his mind, and he fails really to grasp them. The fact that students fail to acquire this habit accounts for much of their lack of power and their parrot-like methods of study.

[2] *Question.*—State the relation between shear and moment, and the shape of the shear and moment lines, if the load consists of a combination of concentrated and distributed loads.

Furthermore, it is evident that the area of the shear curve between any two points where the moment is zero, or between any two points at which the moment is the same, is zero.

Another exceedingly important result is the following. *The moment is a maximum numerically where the shear is zero, or at the load where it passes from positive to negative.*[1] In order, then, to find the maximum moment on a beam, for any given loading, it is simply necessary to find the points where the shear is zero, or where it changes from positive to negative values, and to find the moment at those points. There may be several such points, and the real maximum moment will be found after computing the moment at each such point.

The moment at any point may evidently be found from the shear curve, since *the moment at any point equals the area of the shear curve between that point and the free end of the beam;* (i.e., the end of the beam, if there is no moment there; or in general between that point and any point where the moment is zero.

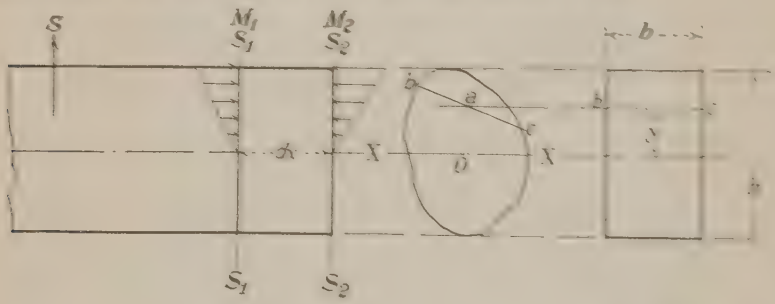


FIG. 159. FIG. 159a.

9. Distribution of Shearing Stress over a Section of a Beam. Longitudinal Shear.—The equations of equilibrium afford no means of determining the distribution of the shearing stresses over the section, but simply give the value of the resultants. To find the distribution, therefore, some indirect method is necessary. Such a method is afforded by the principle that on two planes at right angles, passing through any given point in a body, the intensity of shearing stress, in directions perpendicular to the intersection of the planes, is equal (see Chap. IV). If, then, at any point a of a section Fig. 159 the vertical shear has the intensity s, it follows that on a horizontal plane through a there is a longitudinal shear of the same intensity. If we can determine the latter we shall have determined the former. The latter may be easily found by the following procedure, but in certain cases only.

Take any longitudinal section bac, parallel to the axis of the beam, and consider the portion of the beam above this section and between the cross-sections s_1 and s_2. The only outer horizontal forces acting upon this portion of the beam are the normal forces on the ends, i.e., on those portions of the sections s_1 and s_2 which lie above bac. These total normal forces are not equal, in general, because the moments at s_1 and s_2 are not the same, hence the difference between them must be resisted by horizontal shear on the portion of the longitudinal section bac lying

[1] Since max. M occurs where $\frac{dM}{dx} = S = 0$.

between s_1 and s_2. Let M_1 and M_2 be the bending moments on s_1 and s_2, respectively. Then at any point in s_2 at a distance y from the neutral axis, the intensity of the normal force is $f = \frac{M_2 y}{I}$; hence, the total normal force on that section above *bac* is

$$N_2 = \int \frac{M_2 y}{I} da = \frac{M_2}{I} \int y da = \frac{M_2 Q}{I}$$

in which the integration is extended to cover the area above *bac*, and Q is the statical moment of that area about the neutral axis. Similarly, on section S_1

$$N_1 = \frac{M_1 Q}{I}.$$

The section of the beam is supposed constant, with a moment of inertia I about the axis XX. If the distance between the sections is dx, the longitudinal force per *unit of length* is, if the section of the beam is uniform,

$$\frac{N_2 - N_1}{dx} = \frac{Q}{I} \frac{(M_2 - M_1)}{dx} = \frac{Q}{I} \frac{dM}{dx} = \frac{QS}{I}.$$

Calling s_l the longitudinal shear on the longitudinal section, *per unit of length*, we therefore have, at any cross-section of a beam

$$s_l = \frac{QS}{I}, \tag{11}$$

S being the (transverse) shear at the section; Q the statical moment of the portion of the (transverse) section above the longitudinal section, taken about the neutral axis, and I the moment of inertia of the entire section about the neutral axis.

It was shown in Chap. IX that, with definitions as above, the formula for f is perfectly general for any beam, no matter what the shape or the loading, that is to say, even if it is not subject to pure flexure. Equation (11) is therefore equally general.

This equation, however, gives us no information, in general, regarding the *intensity* of the longitudinal shear, and there is no way of determining the latter, except in one special case. All that can be said is that if the cross-section of the beam is symmetrical about a vertical axis, with the loads all lying in the plane of this axis, and if the line *bac* is also symmetrical about the same axis, then the shear is distributed symmetrically with reference to that axis, half being carried on each side of the longitudinal section; but as to exactly how it is distributed, there is no means of knowing.

The special case referred to is that in which the section is a rectangle, with sides parallel to and perpendicular to the plane of the outer forces, and the line *bac* horizontal (Fig. 159*a*), in which case, since each vertical strip of the section acts like any other of the same width, the total longitudinal shear per unit length given by Eq. (11) (and hence the vertical shear), is distributed uniformly along *bc*; hence, if b is the width of the rectangle,

$$\left.\begin{array}{l}\text{longitudinal shear per unit length} \\ = \text{vertical shear per unit height}\end{array}\right\} = \frac{QS}{I}$$

$$s = \text{intensity of horizontal or vertical shear} = \frac{QS}{Ib}.$$

Let y be the distance of bc from the center of gravity of the section; then

$$I = \frac{1}{12}bh^3;\ Q = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

$$s = \frac{3S}{2bh}\left(1 - \frac{4y^2}{h^2}\right). \tag{12}$$

This is the equation of a parabola, with its vertex at the center of gravity of the section, where, y being equal to zero

$$\text{max. } s = \frac{3S}{2bh} \tag{13}$$

or $\frac{3}{2}$ of what it would be if S were uniformly distributed over the section. From the center of gravity, the intensity diminishes upward or downward by a parabolic curve, becoming zero at the extreme top or bottom of the section.

Since the longitudinal shear on a horizontal section, per linear unit, is equal to $\frac{QS}{I}$, the same quantity equals the vertical shear at the same horizontal, per unit of height of the cross-section. Hence, if for any section we plot this value at right angles to the cross-section, at each point, as in Fig. 160, we shall obtain a

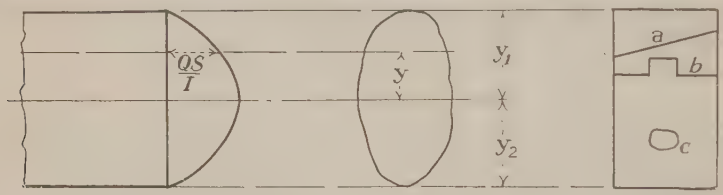


FIG. 160.

FIG. 160a.

curve which will indicate the distribution of the vertical shear S over the cross-section, no matter what the shape of the cross-section may be. The total area enclosed by this curve must equal S, or, if the integration covers the entire section,

$$\int_{-y_2}^{+y_1} \frac{QS}{I}\,dy = \frac{S}{I}\int_{-y_2}^{+y_1} Q\,dy = S.$$

This is clearly true, since $\int_{-y_2}^{y_1} Q\,dy = I$.[1]

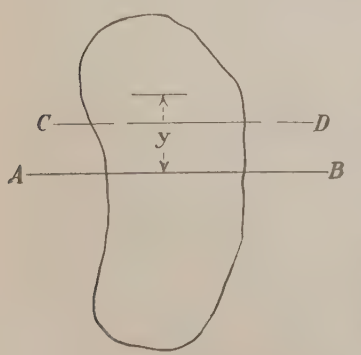


FIG. 161.

In the case of a rectangular beam, this curve is a parabola, with a maximum value of $\frac{QS}{I}$ at the neutral axis of $\frac{3S}{2h}$, corresponding to an intensity of $\frac{3S}{2bh}$. Since the total area of

[1] It is easily shown by the calculus that if Q is the statical moment, about any axis AB (Fig. 161), of the portion of the section above any line parallel to AB, then the integral of $Q \cdot dy$, extended over the entire area on both sides of AB, equals the moment of inertia of that area about AB; or if the integration is extended from the top down to any given line CD, the integral equals the moment of inertia, about AB, of the area above CD.

the curve must equal S, and since the area of a parabolic segment equals two-thirds of the circumscribing parallelogram, all that is necessary to remember is that in a rectangular section the shear is distributed according to a parabolic curve.[1]

Equation (11) enables us to find the shearing stress per longitudinal inch on *any longitudinal section*, no matter what its shape, so long as it is a section generated by a straight line, perpendicular to the cross-section, moved parallel to the axis of the beam. Thus the total longitudinal shear per unit length, on any longitudinal section such as that represented by lines a, b, c, etc. on the cross-section (Fig. 160a) can be found, but the distribution on none of these can be determined (except on the horizontal parts of b if the section of the beam is a rectangle).

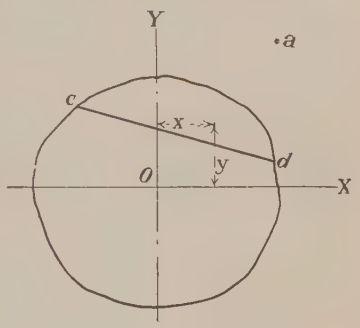


FIG. 162.

In the more general case of flexure, in which the forces do not all lie in one plane, and in which there may be a resultant direct normal force on a section, the determination of longitudinal shear is equally easy. Here the resultant normal force N acts at a point a (Fig. 162) and there is a moment M_x about the principal axis OX, a moment M_y about OY, a shear S_x parallel to OX, and a shear S_y parallel to OY. Then the intensity of the normal force at any point (x, y) is

$$f = \frac{N}{A} + \frac{M_x y}{I_x} + \frac{M_y x}{I_y}.$$

The relation between the moments at two sections distant dx apart, there being no concentrated load between and therefore no change in N, is

$$\frac{dM_x}{dx} = S_y; \frac{dM_y}{dx} = S_x.$$

Hence between these two sections

$$df = \frac{dM_x \cdot y}{I_x} + \frac{dM_y \cdot x}{I_y}.$$

and the longitudinal shear on any longitudinal section cd, per unit of length, is

$$s_l = \int \frac{dM_x}{dx} \cdot \frac{y da}{I_x} + \int \frac{dM_y}{dx} \cdot \frac{x da}{I_y} = \frac{S_y Q_x}{I_x} + \frac{S_x Q_y}{I_y} \qquad (14)$$

Q_x and Q_y being the statical moments, about Ox and Oy respectively, of the area above cd. The integrals are taken to include the area above cd.

It may be supposed that if the section of the beam is not a rectangle, but is composed of or contains rectangles, then the shear will be uniformly distributed along a horizontal line in any such rectangular portion. Such, however, is not necessarily the case. Consider the case of pure flexure, with loads in the plane of symmetry,

[1] *Questions.*—(1) If M is the maximum moment on a rectangular beam (b, d), x the distance from the point of maximum moment to the end of the beam, and s the maximum shear intensity on a longitudinal section, is it correct to write

$$s = \frac{3}{2} \frac{M}{bdx}.$$

If so, why so? If not, why not? If so, is it always so? (See *Eng. News Record*, Aug. 25, 1921, p. 335.)

2. Is the statement in the same article, that the loading to produce maximum longitudinal shear is the loading that will produce the maximum bending moment, correct? If so, why? If not, why not? If so, is it always so; if not always so, when is it so?

applied to an I-beam section, supposed, for simplicity, to have the corners angular instead of rounded (Fig. 163). The longitudinal section through ac sustains a longitudinal shear per unit length, $\frac{QS}{I}$, in which Q is the statical moment of the area $aceg$ about the neutral axis AB. The longitudinal section through bd sustains an equal shear. The longitudinal section ab sustains a shear given by the same formula if Q is the statical moment of the area $abfe$. The part of this last shear due to the area $abdc$ will be uniformly distributed over ab, but the portion due to the areas $aceg$ and $bdfh$, which are applied to the extreme sides ac and bd of the body of which $abdc$ is the section, will not be uniformly distributed on ab, but will cause the greatest intensities at a and b, diminishing from these points toward the center of the web. The same is true, though perhaps in different degree, at any section of the web below ab. On any horizontal section between ef and gh, on the other hand, the shear will be uniformly distributed.

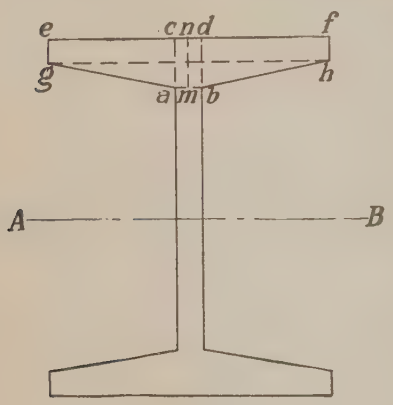


FIG. 163.

This matter will be further investigated in the next paragraph.

10. Further Investigation of Distribution of Shear in a Beam. Shear in Web of I-beam.

Consider the I-beam section shown in Fig. 163.

Let Q = statical moment about AB of area $bdfh$
Let Q_1 = statical moment about AB of area $acfhb$
Let Q_3 = statical moment about AB of area $mnfhb$.

Then, longitudinal shear per unit length on $bd = \frac{QS}{I}$

longitudinal shear per unit length on $ab = \frac{(Q + Q_1)S}{I}$

longitudinal shear per unit length on $nmb = \frac{Q_3 S}{I}$.

But since mn represents a symmetrical longitudinal section, there can, by the law of symmetry, be no shear upon the portion mn of the broken section represented by nmb, hence the longitudinal shear per unit on $mb = \frac{Q_3 S}{I}$. This is also evident from the fact that the longitudinal shear on am must equal that on mb (by symmetry) or $\frac{(Q + Q_1)S}{2I}$, and $\frac{Q + Q_1}{2}$ clearly equals Q_3. The question is, how is the shear on mb distributed?

To investigate this, consider the portion of the beam shown in Fig. 164, representing what is shown in the section $mbdn$. The sum of the moments of all the forces acting on this portion of the beam, about any vertical axis, must equal zero. The forces which will have moments about an axis mn are the following:

(1) The longitudinal shear $\frac{Q_3 S}{I} \cdot dx$, on $mbb'm'$.

(2) The longitudinal shear $\frac{QS}{I} \cdot dx$, on $bb'd'd$.

(3) The horizontal shear on $mbdn$ and on $m'b'd'n'$.

(4) The difference between the normal forces on $mbdn$ and on $m'b'd'n'$, equal to $\frac{(Q_3 - Q)S}{I} \cdot dx$, acting in the line midway between mn and bd.

There may also be normal forces on $mm'n'n$ and on $bb'd'd$, and a horizontal transverse shear on $mbb'm'$, and since the length of the portion considered is dx, these forces would act at the center of this length. We neglect these.

The third of these forces may now be ascertained. At the line $d'b'$ the shearing force on $d'b'm'n'$ must equal in intensity that on the face $bdd'b'$, or it must be $\frac{QS}{I}$ per unit of length along $m'b'$. At the line $m'n'$ it must be zero, because there is no shear on the face $mnn'm'$; hence it must vary between these two values from the line $d'b'$ to the line $m'n'$. There is no means of determining its distribution, but, if it varies uniformly, its average value is one-half the maximum, and the total force is,

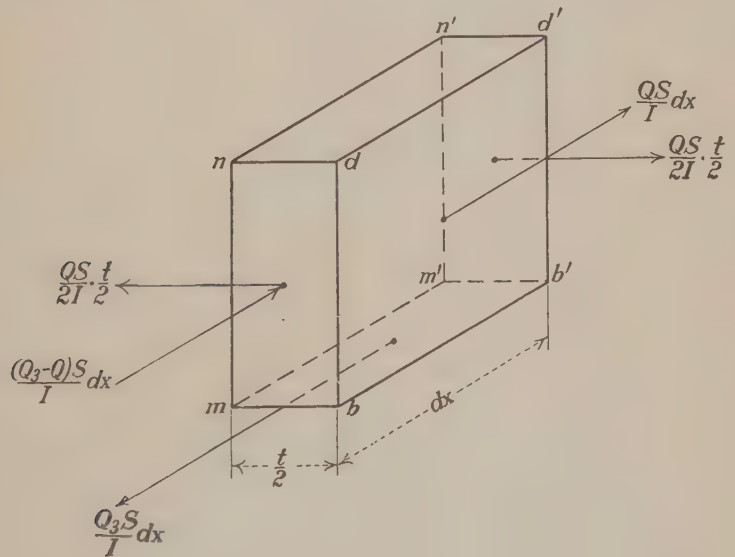


Fig. 164.

as shown in the figure, $\frac{QS}{2I} \cdot \frac{t}{2}$ if t is the thickness of the web. The force on $bdnm$ will be different from this by an infinitesimal amount, and the infinitesimal difference is properly to be neglected.

The equation of moments may now be written as follows, calling d the distance of the resultant shear on $mbb'm'$ from mm':

$$-\frac{QS}{I}dx \cdot \frac{t}{2} + \frac{QS}{2I} \cdot \frac{t}{2} \cdot dx - \frac{(Q_3 - Q)S}{I} dx \cdot \frac{t}{4} + \frac{Q_3 S}{I} \cdot dx \cdot d = 0$$

or

$$d = \tfrac{1}{4}\, t.$$

In this case, then, and with the assumption that the shear on the end sections varies uniformly from zero at $m'n'$ to a maximum at $b'd'$, the resultant shear on the half web acts at the center of mb, or *is probably distributed uniformly.*

We may make two other assumptions regarding the distribution of the horizontal shear on $mndb$ and the opposite face, (1) that in any horizontal line it varies as a parabola with its vertex on mn and axis parallel to bb', in which case the average is one-third the maximum (Fig. 165); (2) that it varies as a parabola whose vertex

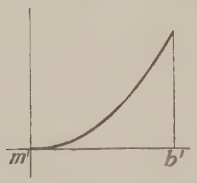


FIG. 165.

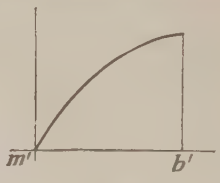


FIG. 166.

is on bd and axis parallel to bb', in which case the average is two-thirds the maximum (Fig. 166). In the first case we shall find

$$\frac{d}{t} = \frac{1}{4} + \frac{Q}{12Q_3}.$$

In the second case we shall find

$$\frac{d}{t} = \frac{1}{4} - \frac{Q}{12Q_3}.$$

The values of Q and Q_3 for several sizes of I-beam, taking the corners as sharp, are shown in the following table:

Size of beam	t	Q	Q_3	$\frac{Q}{Q_3}$
24 inches—79.9 pounds	0.50	31.92	35.18	0.908
12 inches—31.8 pounds	0.35	7.23	7.957	0.908
6 inches—12.5 pounds	0.23	1.565	1.72	0.91

It will be observed that uniformly $\frac{Q}{Q_3}$ equals very nearly 0.91; using this value, we have

$$\text{First case; } \frac{d}{t} = 0.326$$

$$\text{Second case; } \frac{d}{t} = 0.174.$$

These results mean that *in the first case*, if the shear on $mbb'm'$ is supposed to vary uniformly from bb' to mm', the maximum is at bb' and is $\frac{3.82}{t} \cdot \frac{Q_3 S}{I}$; while the minimum (at mm') is nearly zero, or but $\frac{0.18}{t} \cdot \frac{Q_3 S}{I}$, both per unit of length. Since the shear on $m'n'd'b'$ and $mndb$ has been assumed to vary, not uniformly, but according to a parabola with vertex at m, it is reasonable to assume the shear on $mbb'm'$ to vary in the same manner, though the intensity at m will not be zero, under which assumption the maximum (at b) equals $\frac{4.44}{t} \cdot \frac{Q_3 S}{I}$, and the minimum (at m) equals $\frac{0.72}{t} \cdot \frac{Q_3 S}{I}$, per unit of length.

In the second case, if the shear on $mbb'm'$ is supposed to vary uniformly, the maximum is at mm' and is the same as in case one, or $\frac{3.82}{t} \cdot \frac{Q_3 S}{I}$, and the minimum is at bb', and equals $\frac{0.18}{t} \cdot \frac{Q_3 S}{I}$, per unit of length, while if the shear on $mbb'm$ is supposed to vary in the same manner as on $mbdn$, the shear at b would be in the opposite direction,[1] hence this method of distribution is inadmissible.

These results are not very important in themselves, but they illustrate the uncertainties of the subject, the practical impossibility of finding the stresses accurately, and the methods of investigation that may be employed.

11. Normal Forces on Longitudinal Section of Beams.—Reference has just been made to the possibility of the existence of normal forces on the surfaces $mnn'm'$ and

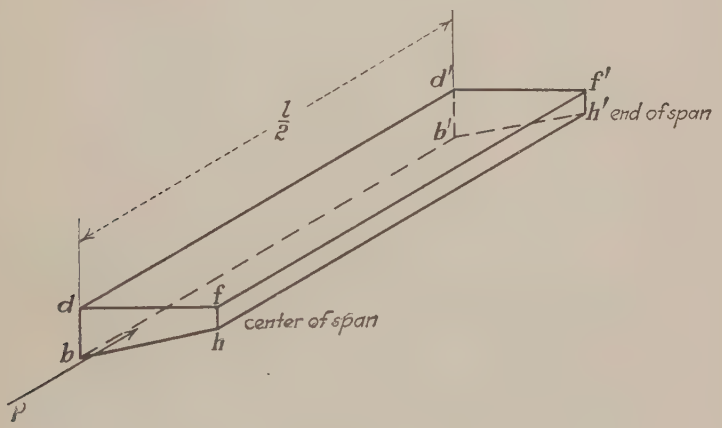


Fig. 167.

$bdd'b'$ (Fig. 164), which are longitudinal sections of an I-beam. As a matter of fact, there will be such normal forces—tension in the central portions of the span, and compression at and near the ends. This will be obvious when it is remembered that the portion $bdfh$ of the flange (Fig. 163) has no force acting upon it at the ends of the span, and a gradually increasing compression toward the point of maximum moment.

Consider the I-beam supported at each end and loaded symmetrically about the center. Then at the center there will be no shear on $bdfh$, either vertically or horizontally, hence the portion of the beam shown in Fig. 167, is a cantilever beam fixed at one end ($bdd'b'$), and loaded with a distributed load on $bdfh$. Hence there is tension at bd and to a point half way to $b'd'$, and compression on the rest of the area $bdd'b'$. The force P equals

$$\int \frac{Mybdy}{I} = \frac{M}{I}\int bydy = \frac{MQ}{I}.$$

If l is the span, d the distance bd, and c the distance from bd to the point of application of P, then obviously the max. tension or compression on the section $bdd'b'$ is given by the formul

$$\frac{MQc}{I} = \frac{1}{6}fd\frac{l^2}{4} = \frac{1}{24}fdl^2$$

$$f = \frac{24MQc}{Idl^2}. \qquad (15)$$

[1] The advanced student should confirm these statements.

Or, if we assume that the flanges are subject to a uniform stress of intensity f_1, and if A is the area of $bdfh$, the formula becomes

$$f = \frac{24 f_1 A c}{dl^2}. \tag{16}$$

Taking the three sizes of I-beams shown in this paragraph, and using the latter formula, we find, for a span as indicated

	A, square inches	l, feet	d	f_1	c	f, pounds per square inch
24 inches—79.9 pounds...	2.83	24	1.142	16,000	1.50	17
12 inches—31.8 pounds...	1.265	12	0.738	16,000	1.03	33
6 inches—12.5 pounds...	0.556	6	0.488	16,000	0.69	59

The insignificance of this stress is evident, and it is therefore never considered, though the writer has never seen its existence pointed out.

It is somewhat interesting to inquire how this stress could be computed if the loading on the beam were unsymmetrical about its center. It might be thought that, after finding the point of maximum moment, the same procedure could be followed, but it will at once be apparent that in this case, since this point would not be in the center of the span, the stress at that point which would be found by considering the flange length on one side would be quite different from that obtained by considering the other side, being less when the longer side is taken. There would thus be two different values obtained for the normal stress at the point of maximum moment, or, in other words, there would be a sudden change in the intensity at the point of maximum moment. As shown in Chap. V, this is impossible. It follows that at the point of maximum moment there would be a transverse shearing stress (parallel to df in Fig. 167) on the section, in such a sense as would decrease the moment on the longer segment of the flange, and increase it on the shorter, and to such an extent as would make the normal stress the same when computed from either segment.

The procedure explained for finding the normal stress on $bdd'b'$ (Fig. 167) cannot be used for finding that on the central section $mnn'm'$ in Fig. 164, because the moment would be affected by the shearing forces acting on the other faces of the solid shown, the distribution of these forces being uncertain.

12. Distribution of Longitudinal Shear in a Non-rectangular Section.—We have seen that in a rectangular section the longitudinal shear on a section parallel to the neutral axis of a transverse section is uniformly distributed (if the neutral axis is parallel to one of the principal axes), but that this is probably untrue on the rectangular portions of other sections. How is it on non-rectangular sections? Suppose a circular section (Fig. 168), and take a longitudinal section having a width b. The total longitudinal shear on this section, per unit of length, is $\frac{QS}{I}$, Q being the statical moment about NN of the area above b.

It is frequently assumed that this too is distributed uniformly, with an intensity of $\frac{QS}{Ib}$. This is easily shown to be incorrect. At the surface of the body, as

shown in the small figure, take an infinitesimal element of the area, having dimensions dx and $dy = dx \cdot \tan \alpha$. Then the longitudinal shear on both these surfaces together per unit of length, is

$$\frac{QS}{I} = \frac{S}{I}\left(\frac{dx \cdot dy}{2}\right)y = \frac{Sy}{2I}dx^2 \cdot \tan \alpha.$$

The average intensity of stress on these surfaces would, therefore, be

$$f_s = \frac{QS}{I(dx + dy)} = \frac{Sy}{2I} \cdot dx \cdot \frac{\tan \alpha}{1 + \tan \alpha}$$

which is an infinitesimal. It is therefore clear that *the longitudinal shear on either of the two planes dx and dy is zero at the surface of the body;* from which it follows that the *vertical* (or transverse) *shear is distributed over the section in such a way that it is zero at the surface of the body, if that surface is not perpendicular to the neutral axis.* Indeed, looking at the question objectively and concretely, and realizing how the longitudinal shear is caused, it is at once evident that it is reasonable to assume that a length dx' at the center of the breadth b, having a considerable area of section above it (comparatively speaking) would have a much greater longitudinal shear upon it than an equal length dx at the edge of b, having practically no area above it.

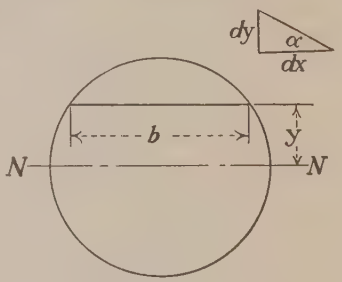


FIG. 168.

It is also obvious from what has preceded that in non-rectangular sections there may be longitudinal shear—not only on planes parallel to the neutral axis—but on planes perpendicular thereto; on a transverse section there may therefore be transverse shearing stresses parallel to the neutral axis, although the resultant of these stresses will be zero.

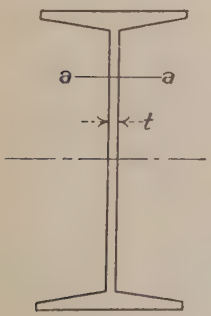


FIG. 169.

13. Numerical Example of Distribution of the Vertical Shear for an I-beam Section.—An I-beam section is shown in Fig. 169. The web is comparatively thin, and the bending moment is resisted mainly by the flanges. It is generally assumed that the shear is carried entirely by the web, and it will be instructive to work out several typical sections to determine the actual distribution.

The vertical shearing stress per unit of vertical height, at any given horizontal section aa, is equal to $\frac{QS}{I}$. If this be assumed uniformly distributed across the thickness t, the intensity of stress will be $\frac{QS}{It}$. Take, now, from the book of sections, a beam 24 inches high, weighing 79.9 pounds per foot, and construct the following table:

y, inches	Moment of area up to y	Q	$\frac{Q}{I}$	$\frac{Q}{It}$	Proportion of total shear on the several areas
0	0	101.32	0.0485	0.0970	0.04845
1	0.25	101.07	0.0484	0.0968	0.0482
2	1	100.32	0.0480	0.0960	0.0477
3	2.25	99.07	0.0474	0.0948	0.0470
4	4	97.32	0.0466	0.0932	0.04605
5	6.25	95.07	0.0455	0.0910	0.04485
6	9	92.32	0.0442	0.0884	0.0434
7	12.25	89.07	0.0426	0.0852	0.0417
8	16	85.32	0.0408	0.0816	0.0398
9	20.25	81.07	0.0388	0.0776	0.03765
10	25	76.32	0.0365	0.0730	0.03049
10.86	29.49	71.83	0.0344	0.0688	0.01661[1]
11.40	52.18	49.14	0.0235	0.0034	0.00705
12	101.32	0	0	0	0.49895

$$
\begin{aligned}
I = {} & \tfrac{1}{12} \times \tfrac{1}{2} \times 24 \times 24 \times 24 && = 576 \\
& + \tfrac{1}{12} \times 1\tfrac{3}{2} \times \tfrac{6}{10} \times \tfrac{6}{10} \times \tfrac{6}{10} \times 2 && = 0.234 \\
& + 1\tfrac{3}{2} \times \tfrac{6}{10} \times 11.7 \times 11.7 \times 2 && = 1{,}070.86 \\
& + \tfrac{1}{36} \times 1\tfrac{3}{2} \times 0.54 \times 0.54 \times 0.54 \times 2 && = 0.056 \\
& + 1\tfrac{3}{2} \times 0.54 \times 11.22 \times 11.22 && = 441.87 \\
& && \overline{2{,}089.02}
\end{aligned}
$$

To find the total shear on the web between $y = 4$ inches and $y = 5$ inches, for instance, it is only necessary to multiply the average shear per vertical unit $\left(\frac{0.0455 + 0.0466}{2}\right)$ by unity, giving the numbers in the last column. These

[1] This figure is obtained by assuming the curve to be a parabola with horizontal axis between points 10.86 and 11.40; thus

$$0.0235 \times 0.54 + \times \tfrac{2}{3} \times 0.54 \times (0.0344 - 0.0235) = 0.01661.$$

The total in the last column is less than 0.5 because the line of distribution is a curve and not a series of straight lines; although the figure 0.01661 is too large, so that the total may in some cases exceed 0.5.

should add up to 0.5, being the total proportion of the shear carried by half the section; actually, the total is slightly different, due to numerical inaccuracies. The results are plotted in Fig. 170. The area between the axis AB and the line to the right represents the total shear, and shows clearly its distribution. It

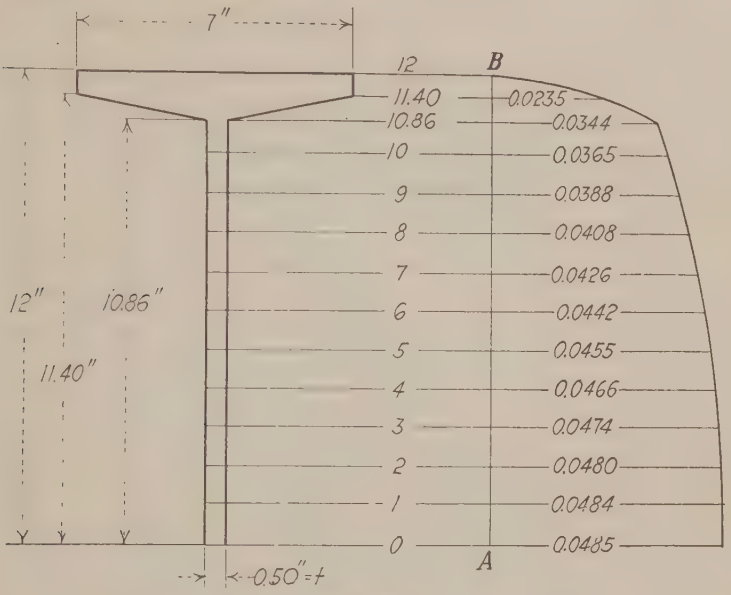


Fig. 170.

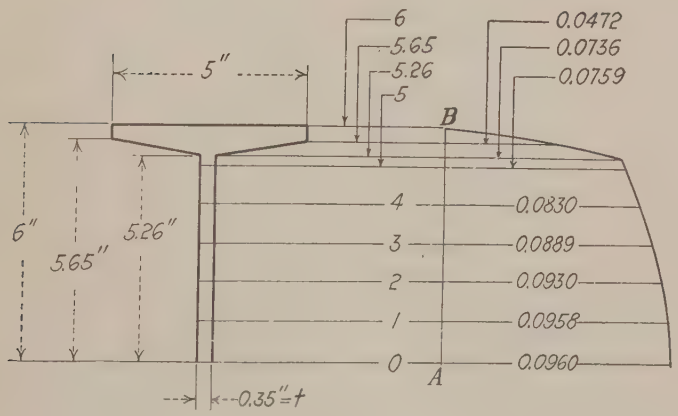


Fig. 171.

appears that 95 per cent of the shear is carried by the web, and a still larger proportion if the web is considered to extend through the flanges ($24 \times \frac{1}{2}$ inches).

Thus

$$\text{max. } f_s = 0.097S.$$

Total shear carried by actual web $= 2(0.49895 - 0.01661 - 0.00705)S$
$= 0.95S$ (approx.).

If it is assumed that the total shear is carried (uniformly distributed) on a web $24 \times \frac{1}{2}$ inches

$$\text{max. } f_s' = \frac{S}{12} = 0.0833S = 0.86f_s$$

or 86 per cent of the actual maximum intensity, which shows the error made in assuming the shear to be uniformly distributed over the web.

As another example, take an I-beam of intermediate size, 12 inches—31.8 pounds. The results are given in the following table and Fig. 171:

y	Moment of part up to y	Q	$\frac{Q}{I}$	$\frac{Q}{It}$	Proportion of total shear on the several areas
0	0	20.78	0.0960	0.274	
					0.0959
1	0.175	20.61	0.0958	0.273	
					0.0944
2	0.700	20.08	0.0930	0.266	
					0.0910
3	1.595	19.20	0.0889	0.254	
					0.0861
4	2.80	17.98	0.0832	0.238	
					0.0796
5	4.38	16.40	0.0759	0.217	
					0.0194
5.26	4.84	15.94	0.0736	0.210	
					0.0256
5.65	10.58	10.20	0.0472	0.0944	
					0.0083
6.00	20.78	0	0	0	
Total.....	.....	.....	.....	.....	0.5003

COMPUTATION OF I

Part	Dimensions	Area	Distance to CG	M	I
1	5.02×0.35	1.75	5.825	10.20	59.4
2	$\frac{1}{2}(4.65 \times 0.388)$	0.902	5.521	5.00	27.6
3	0.388×0.35	0.136	5.454	0.74	4.04
web.....	5.26×0.35	1.84	$\frac{1}{3} \times 0.350 \times 5.26^3$		16.9
		$4.63 \times 2 = 9.26$			$107.9 \times 2 = 215.8$

$$\text{max. } f_s = \frac{0.096}{0.35}S = 0.274S.$$

Total shear carried by actual web $= 0.93S$

$$\text{max. } f_s' = \frac{S}{12 \times 0.35} = \frac{S}{4.2} = 0.238S = 0.87f_s.$$

As a last example, take a small sized beam, 6 inches -12.5 pounds. The results follow:

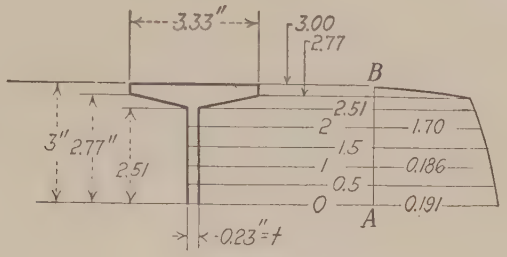


FIG. 172.

y	Moment of part up to y	Q	$\frac{Q}{I}$	$\frac{Q}{It}$	Proportion of total shear on the several areas
0	0	4.166	0.191	0.827	
					0.0953
0.5	0.02875	4.137	0.190	0.826	
					0.0940
1.0	0.115	4.052	0.186	0.809	
					0.0912
1.5	0.259	3.907	0.179	0.779	
					0.0873
2.0	0.460	3.706	0.170	0.739	
					0.0837
2.51	0.726	3.440	0.158	0.686	
					0.0401
2.77	1.956	3.210	0.147	0.440	
					0.0209
3.00	4.166	0	0	0	
		$I = 21.76$			0.5125

$$\text{max. } f_s = 0.827S.$$

Total shear carried by actual web $= 0.903S$.

$$\text{max. } f_s' = \frac{S}{6 \times 0.23} = \frac{S}{1.38} = 0.725S = 0.88f_s.$$

The deeper the beam, the greater the proportion of the shear which is carried by the web, as would be expected.

14. Deformation of a Cross-section.—Assuming that the intensity of the normal stress has a planar distribution, and that Hooke's law holds true, it follows that if one originally plane section remains plane, all others will, and that if one

does not remain plane, none remain plane. If they remain plane, they would be normal to the elastic line.

But shear must distort a section, even if there is planar distribution of the normal stress. If at any point the intensity of vertical and horizontal shear is f_s, a particle dv in height (Fig. 173) would be distorted by $\frac{f_s}{E_s} dv$. The next particle above it would be similarly distorted, and the sum of the distortions would be the total horizontal distortion of a', supposing that a does not change its position and ab remains fixed in direction. If ab is the neutral axis, the shear and distortion are greatest there, and diminish as we go upward. We have seen that the intensity f_s can be found accurately only for a rectangular cross-section, and

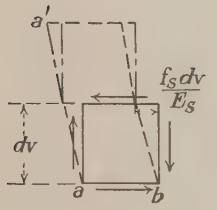


Fig. 173.

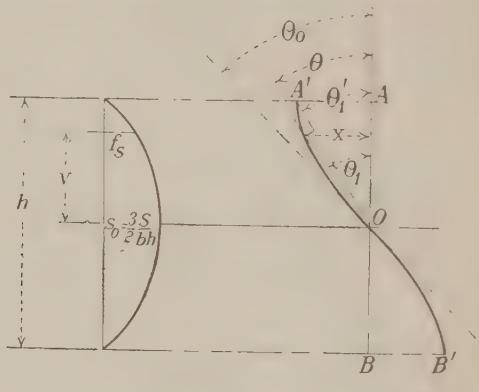


Fig. 174.

Fig. 174a.

that in this case the total shear on a vertical section is distributed according to a parabolic curve (Fig. 174), the shearing intensity at the neutral axis being

$$s_o = \frac{3}{2} \cdot \frac{S}{bh}$$

where S is the total shear. At any distance v from the neutral axis, the shearing intensity is

$$f_s = \frac{3}{2} \cdot \frac{S}{bh} \left[1 - \frac{v^2}{\left(\frac{h}{2}\right)^2} \right] = s_o \left[1 - \left(\frac{2v}{h}\right)^2 \right].$$

Hence the horizontal movement of the point distant v from the neutral axis would be

$$x = \int_o^v \frac{S_o}{E_s} \left[1 - \left(\frac{2v}{h}\right)^2 \right] dv = \frac{s_o}{E_s} \left(v - \frac{4v^3}{3h^2} \right).$$

The original section AB becomes the warped section $A'B'$ (Fig. 174a).

The angular distortion of a particle distant v from the neutral axis, that is, the slope of the curve $A'B'$ at that point, is

$$\tan \theta = \frac{f_s}{E_s}.$$

At the top and bottom of the section, $f_s = 0$ hence $\theta = 0$; that is, the section at those points is in its original direction; the tangent at A' and B' is parallel to

AB, or normal to the axis of the beam. At the neutral axis, f_s has its maximum value, and

$$\tan \theta_o = \frac{3}{2}\frac{S}{bhE_s}.$$

The horizontal movement at the top or bottom of the (rectangular) beam, or

$$AA' = BB', \text{ is } \left(\text{for } v = \frac{h}{2}\right)$$

$$x_1 = \frac{1}{3}\cdot\frac{s_o h}{E_s} = \frac{S}{2bE_s}.$$

If a line be drawn from O to the deformed position of a point distant v from O, it will make an angle θ_1 with AB given by

$$\tan \theta_1 = \frac{x}{v} = \frac{s_o}{E_s}\left(1 - \frac{4v^2}{3h^2}\right).$$

For the point A'

$$\tan \theta_1' = \frac{2}{3}\cdot\frac{s_o}{E_s} = \frac{2}{3}\tan \theta_o = \frac{S}{bhE_s}.$$

WORK AND RESILIENCE IN FLEXURE

15. Work Done in Flexure.—When external loads are applied to any structure, such as a beam, *distortions* or *deflections* are produced, by which the shape of the structure, as well as of its individual particles, is changed. The character and the amount of change of shape of any particle depend upon the character and magnitude of the forces acting upon it. The external forces do work during this change of shape, because their points of application are changed by the deflection, and in general have components of motion in the direction of the forces. The internal forces or stresses do work because they too change their points of application on account of the distortions of the particles. One obvious principle (a consequence of the conservation of energy) is of universal application, namely: *the work done by the external forces equals that done by the internal forces:* or

the external work = the internal work.

The process of applying the loads must be considered to take place in such a manner that there shall be no vibration produced, *i.e., the loads must be assumed to be gradually applied, increasing uniformly from zero to the final value.* The internal forces, or stresses, will therefore increase in the same manner.

If the structure rests upon unyielding supports, or upon rollers, without friction, the work done by the supporting forces will be zero, hence the external work will be that done by the loads alone.

The work done by any external force will be the average force multiplied by the distance through which it moves in the direction of the force, or if P is an external force, and its point of application moves a distance Δ in the direction of P

$$\text{Work done} = \tfrac{1}{2}P\Delta.$$

The same expression will give the work done by any internal force, the motion being, of course, a deformation of the particle on which the internal force acts.

The internal forces are either tension, compression, or shear. We must, therefore, find the work done by each of these kinds of stress.

Work Done by a Tensile Stress.—Let a fiber whose cross-sectional area is A, and whose length is L, be acted upon throughout its length by a force S. Then the work done will be

$$W = \tfrac{1}{2} S \cdot \text{change of length} = \frac{1}{2} S \cdot \frac{SL}{AE} = \frac{1}{2} \frac{S^2 L}{AE}. \tag{17}$$

Work Done by a Compressive Stress.—The same equation will obviously apply to this case, assuming the modulus of elasticity in tension equal to that in compression, and neglecting the bending which may occur in a piece exposed to longitudinal compression, and the consequent shortening of length due to this cause.

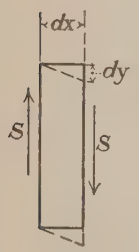


FIG. 175.

Work Done by a Shearing Force.—Let Fig. 175 represent a portion of a beam, dx in length, with a shearing force S on the vertical sections. This force will not be distributed uniformly over the section, but let the intensity at any point be s. Under the shearing forces S, the right section will be moved downward a distance dy with reference to the left section, and the work done by S, gradually applied, will be $\tfrac{1}{2} S dy$. On any infinitesimal area dA of the section the shearing force will be $s \cdot dA$, and the distance it will move will be $\frac{s}{G} dx$, where G is the shearing modulus of elasticity; hence the work done will be $\frac{1}{2} s \cdot dA \cdot \frac{s}{G} \cdot dx = \frac{1}{2} \frac{s^2 \cdot dA \cdot dx}{G}$, and the total internal work done on the section, which equals the work of S, will be

$$W = \frac{1}{2G} dx \int s^2 dA = \frac{1}{2} S dy,$$

whence

$$\frac{dy}{dx} = \frac{\int s^2 dA}{GS} = n \frac{S}{AG} \tag{18}$$

where A is the area of the section and n a constant which takes account of the unequal distribution of S over the area. If that distribution is uniform $s = \frac{S}{A}$ and $n = 1$.

The work done on the infinitesimal particle dA in area and dx in length, is, as above

$$dW = \frac{1}{2} \frac{s^2 dA \cdot dx}{G} = \frac{1}{2} \frac{s^2 V}{G} \tag{18a}$$

in which V is the volume of the particle. This, then is the work done on a rectangular particle by shearing forces of intensity s on four sides.

These principles make it possible to find the deflection of any point of the structure in any assumed direction. We will return to them after considering the deflection from other points of view.

We may now, however, proceed to find the expression for the total work done in flexure.

(*a*) *Work done by the normal stresses only:* Let the moment at any section of the beam be M. Then the work done by the normal stresses on the fibers between this section and another distant dx therefrom, due to a gradually applied load, will clearly be

$$dW_n = \frac{1}{2E} \cdot f^2 dA \cdot dx = \frac{1}{2E} \frac{M^2 y^2 dA \cdot dx}{I^2}$$

$$\therefore W_n = \frac{1}{2E} \int_o^l M^2 dx \int_{-y_2}^{+y_1} \frac{y^2 dA}{I^2} \tag{19}$$

the latter of the two integrals is to be extended over the entire cross-section area, while the former integral is to be extended over the entire length. Hence, if the section is constant

$$W_n = \frac{\int_o^l M^2 dx}{2EI} \tag{19a}$$

$$= \frac{1}{EI} \begin{pmatrix} \text{(statical moment of the moment area} \\ \text{about the axis of } X, \text{ not taken} \\ \text{algebraically but numerically).} \end{pmatrix}$$

(*b*) *Work done by shearing stresses only:* We have seen that it is in general impossible to determine the intensity of the shearing force at any point of the cross-section. Consequently, it is impossible to find an accurate expression for the work done by the shearing stresses. On any horizontal plane at a distance y from the neutral axis, the *longitudinal* shearing force *per unit of length*, as we have seen, is $\frac{QS}{I}$ in which S is the total vertical shear on the cross-section, and Q is the statical moment about the neutral axis of that portion of the cross-sectional area on one side of the horizontal plane. If we divide this by b, the width of the section at the plane referred to, the *average* intensity of horizontal shear, and therefore of vertical shear, along this width b, will be $\frac{QS}{Ib}$. Assuming that the work done by the average stress is the average work, the work done by the vertical shear on the strip whose area is bdy will be (for a load gradually applied),

$$dW = \frac{1}{2} \frac{QS}{Ib} \cdot bdy \cdot \frac{QS}{IbG} dx = \frac{1}{2} \frac{Q^2 S^2}{I^2 bG} \cdot dx\, dy$$

$$\therefore W = \frac{1}{2G} \int_o^l S^2 dx \int_{-y_2}^{+y_1} \frac{Q^2 dy}{I^2 b} \tag{20}$$

the last integral being extended over the cross-section and the first over the length.[1]

This result is cumbersome to use, and its meaning is not clear to perceive. For the case of a rectangular section, however (bh), it reduces to

$$W = \frac{6}{10\, bhG} \int_o^l S^2 dx \tag{21}$$

$$= \frac{1.2}{bhG} \begin{pmatrix} \text{(statical moment of shear curve about} \\ \text{axis of } X, \text{ taken numerically).} \end{pmatrix}$$

[1] *Question.*—Is it also necessary to ascertain the work done by the horizontal shearing forces?

SLOPE AND DEFLECTION OF BEAMS, NEGLECTING THE EFFECT OF SHEAR

16. A beam, when loaded, will be deflected or deformed. If originally straight, the axis will take the form of a curve. Wherever the moment is positive, the fibers above the neutral axis will be compressed, and those below will be extended, consequently the curve of the axis will be *concave upwards*, and when the moment is negative, it will be *concave downwards*. Where there is no moment the axis will be straight. The student should fix in his mind the epigrammatic statement *wherever there is no bending moment there is no bending: wherever there is bending moment there is bending.*

We have now to study the general principles governing the slope and deflection, and to determine what shape the axis of a beam originally straight will assume when subject to pure flexure.

It has been shown in Chap. IX, that if the outer forces are all in the plane of the paper (which is assumed in this discussion) the neutral axis will be perpendicular to that plane only when one principal axis of the section lies in the plane of the paper. Now in any cross-section the intensity of stress, and hence the deformation of a fiber per unit length, is constant along any line in the section parallel to the neutral axis. It follows that the deformation of any slice of the beam lying between two cross-sections at a distance *ds* apart will cause all parts of the beam to deflect in a direction perpendicular to the neutral axis at the slice in question.

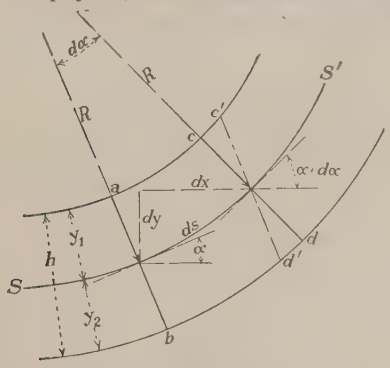


FIG. 176.

(Note that the deformation of any slice causes deflection of *all* parts of the beam.) If, therefore, the beam is of constant section, but with the principal axis of a cross-section not coincident with the plane of the forces, the axis will not deflect in the plane of the forces, but in a plane perpendicular to the neutral axis, which in this case will have the same direction at all cross-sections. The general case, then, of a beam of varying section, acted upon by forces not in a plane, would have to be treated by considering each load by itself, finding the deflection (in magnitude and direction) which it would produce at any given point and combining the effects of the separate loads. But even by this treatment the problem would be complicated, because of the varying position of the neutral axis at the different sections, and the consequent varying directions of the deflections produced by the deformation of the different slices. This problem is not of practical importance, and we shall confine ourselves for the present to the simple case in which the outer forces all lie in the plane of the paper, and one principal axis of each cross-section of the beam is also in the same plane. In this case the deflection of all points will be also in the plane of the paper.

Suppose two sections *ab* and *cd*, *ds* apart, and originally parallel, to be deformed as in Fig. 176. Draw *c'd'* parallel to *ab*. Since we are considering pure

flexure, the length of the axis ds remains unchanged under the load, and $c'd'$ is the original position of cd with reference to ab; the stress, and therefore the strain, varying directly with the distance from the neutral axis of the section, which passes through the axis of the beam. The topmost fibers are subject to a stress intensity $f = \frac{My_1}{I}$, hence the shortening of those fibers, originally ds in length, is $cc' = \frac{My_1}{IE}ds$. Similarly, the extension of the lowest fibers is

$$dd' = \frac{My_2}{IE}ds;$$

$$\therefore cc' + dd' = \frac{Mhds}{IE}.$$

Let R be the radius of curvature of the axis. Then

$$R : ds :: h : cc' + dd'$$

whence

$$\frac{1}{R} = \frac{M}{EI}.$$

Now the calculus tells us that

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{3/2}}.$$

Now as the axis was originally straight, let us take its original position as the axis of X; and further, since the deformations are exceedingly small, assume $ds = dx$, or, in other words, $1 + \frac{dy^2}{dx^2} = 1$, or $\frac{dy^2}{dx^2} = 0$ in comparison with unity, so that we have finally, as the equation of elastic line,

$$\frac{d^2y}{dx^2} = \frac{M}{EI}. \tag{22}$$

This equation is the starting point of all analytical investigations of slope and deflection due to the normal stresses alone. From this explanation, it is clear that it is based upon the following assumptions:

(1) The deflections are so small that $dx = ds$; $\tan \alpha = \alpha$.

(2) The stresses are within the elastic limit, so that Hooke's law applies.

[1] The student should here recall the proof of this equation, in order to have before himself the entire subject, so that he may realize what assumptions, if any, are involved.

$$R \cdot d\alpha = ds;\ R = \frac{ds}{d\alpha} = \frac{ds}{dx} \div \frac{d\alpha}{dx}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} : \tan \alpha = \frac{dy}{dx}, \text{ or } \alpha = \tan^{-1}\frac{dy}{dx}$$

$$\therefore \frac{d\alpha}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \tan^2 \alpha} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}$$

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

This formula, clearly, is strictly correct.

(3) The intensity of normal stress at any point of a cross-section is directly proportional to its distance from the neutral axis.

(4) It is also assumed that sections originally plane before bending, remain plane (see Arts. 28, 29, and 30); that is to say, the deflections thus far studied are those due only to the extension and shortening of the fibers, and not to shear.

The meaning of Eq. (22) should be perceived. It may be written in less mathematical language

$$\frac{\text{Change of slope of axis between two points } ds \text{ apart}}{dx} = \frac{M}{EI}$$

$$\text{or, change of slope of axis between two points } ds \text{ apart} = \frac{M dx}{EI} \qquad (23)$$

in which M is the moment at the point (points), I is the moment of inertia of the cross-section, and E is the modulus of elasticity of the material.

In using Eq. (22), as in using any such equation, it is very important to study the *signs* of the quantities involved, and to adopt suitable conventions so that the equation may be algebraically as well as numerically correct. This point will be

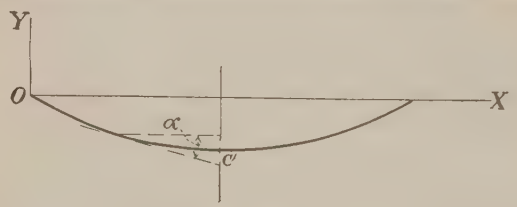


Fig. 177.

explained at some length, because of its importance, and the generality of its application.

We have seen that a positive M causes compression in the top fibers of a horizontal beam, and that if the moment is positive at any point the deflection curve is concave upward, as indicated in Fig. 177. OX and OY are the axes of coordinates. Now if X is positive *toward the right* and y is positive *upward*, y is negative throughout the length of the beam with positive M, and is continuously increasing numerically till it reaches the point c where the tangent to the curve becomes horizontal, that is to say, $\frac{dy}{dx}$ is a negative quantity; beyond c, y decreases numerically, or $\frac{dy}{dx}$ is positive. Also, as we pass to the right from O, the angle α, whose tangent is $\frac{dy}{dx}$, is continuously decreasing numerically, till the point c is reached; but $\frac{dy}{dx}$ being a negative quantity $\frac{d^2y}{dx^2}$ is positive. Beyond c, $\frac{dy}{dx}$ increases numerically, but as it is here a positive quantity, $\frac{d^2y}{dx^2}$ is still positive. With these conventions, therefore, the equation is correctly written $\frac{d^2y}{dx^2} = \frac{M}{EI}$. If, however, x is positive, toward the right and y positive *downward*, the equation must be written

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}.$$

We shall consider M positive, as above, when it causes *compression in the top fibers*, or when the resultant moment of all the outer forces on the left of the section is right-handed, and we shall consider x positive *to the right*, and y positive *upward*.[1]

17. Interpretation of the Moment Line. *Moment Area Method.*—Viewed in the light of Eq. (23), the moment line will be found to have a deep significance, and to be an efficient agent in the solution of problems not only involving moments, but also involving deflection.

We have seen that, under the stated conventions as to sign, the differential equation of the deformed axis, or, as it is called the *elastic line*, is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{d\alpha}{dx}$$

α being the slope of the tangent at the point (x, y) with reference to OX, which is so small that the tangent may be considered equal to the angle $(dx = ds)$. Referring to Fig. 178 let the moment line of the beam for a given loading be *aefhb*, and let the beam be deflected from its original straight position, *ab*, to the curve *ac'd'g'b*, shown much exaggerated in the figure. As just shown, this equation means

$$\text{Increase of slope } (\alpha) \text{ in distance } dx = \frac{M\,dx}{EI}$$

$$= \frac{\text{area enclosed by moment line above } dx}{EI}.$$

From this it follows by integration (summation) that *the increase of slope between any two points c and d equals the area enclosed by the moment line above cd (the area cefd) divided by the constant EI.* This very important result may be restated as a theorem.

THEOREM I.—*The angle which the tangent to the elastic curve at any point (a) makes with the tangent at any other point (b), is represented numerically by the area of the moment curve between a and b; and equal to that area measured to the proper scale, divided by EI. The direction of the angle must be determined by inspection of the moment curve and the character of the moment. If part of the moment area between a and b has one sign and part has the other, the algebraical sum must be taken.*

If, then, we draw ordinates *ce* and *df* at any points of the moment curve, and *gh* at the point where the deformed beam is horizontal, and tangents to the deformed beam at the corresponding points c', d', and g' we have

Area *ace* represents numerically the angle α,
Area *adf* represents numerically the angle α_1,
Area *agh* represents numerically the angle α_2,
Area *cdfe* represents numerically the angle $(\alpha_1 - \alpha) = \alpha_3$.

If desired, the axis of X may be taken at the point where the beam is horizontal (g'); in which case what is usually termed the slope of the beam at $d'(\alpha_4)$ will be represented by the area *ghfd*, the slope at c' (α_5) by the area *ghec*, and that at a (α_2) by *agh*.

[1] *Questions.*—(1) If M is taken as positive when it causes tension in the top fiber, x positive toward the right, y positive downwards, what is the correct differential equation for deflection?

(2) What is it, if M is taken as positive when it causes compression in the top fiber, x positive to the left, y positive downwards?

From this it follows that *the area enclosed by the moment line between the point where the elastic line is horizontal, and any other point b, divided by EI, equals the tangent of the angle (the slope) of the elastic line at b, with reference to the horizontal.*

Remembering, now, that the moment at any point *c* equals the area enclosed by the shear line between the end of the beam (if free) and the point *c*, it is at

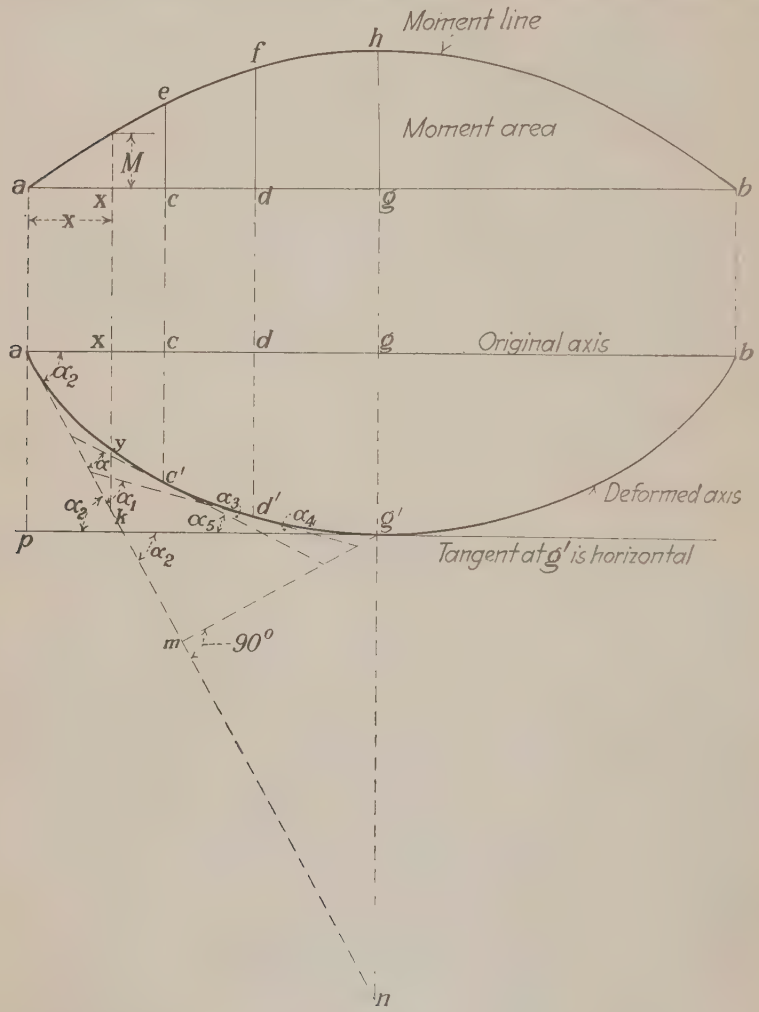


FIG. 178.

once evident that *the angle between the tangent to the elastic line at any point c and the tangent at the free end of the beam, equals the statical moment of the area enclosed by the shear line between the end of the beam and the point c, taken about the point c, divided by EI.*

The slope and deflection are intimately connected, and the moment line gives information, about not only the slope, but also the deflection. Suppose

that the elastic line, at any given point c (Fig. 179) is rotated by an angle $d\alpha$, due to the stresses acting upon a slice dx in length: then any point d at a horizontal distance a from c is deflected vertically a distance $a \cdot d\alpha$ from the tangent at c', due simply to the existence of the deformation $d\alpha$. In other words, starting from the tangent to the elastic curve at c', the deflection of d' from this tangent is the sum of the effects of infinitesimal changes of slope at each point between c and d. If x is the horizontal distance from d to any given point between c and d, then each of these infinitesimal changes of slope produces a vertical deflection at d equal to $x d\alpha$ from the tangent at the point where $d\alpha$ occurs. But the value of $d\alpha$ at any point is the area of a strip dx in width, of the moment area, and x is its distance from d.[1]

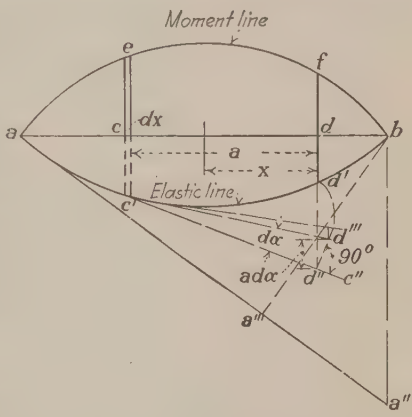


Fig. 179.

Accurately, of course it is $d''d'''$ and not its vertical projection which is equal to $a \cdot d\alpha$, but these two are equal because the theory is based upon the fundamental assumption that $ds = dx$ (Art. 15). Hence, if aa'' is the tangent to the deformed elastic curve at a, and ba'' is perpendicular to ab, while ba''' is perpendicular to aa'', the theory involves the assumption that $ba'' = ba'''$ ($ds = dx$ at a). As a matter of fact, the deformations are so small that this assumption is almost exact, or so near the truth as to be well within the limits of necessary error.

A consideration of the above will show the correctness of the following (second) very important theorem.

Theorem II.—*The deflection of any point d of the original beam (in a direction perpendicular either to the original axis or to the tangent to the deformed axis) from the tangent to the elastic line at any other point c (which moves to c') is equal to the statical moment of the moment area between c and d about d, divided by EI.*

These results may be demonstrated in more strictly mathematical fashion as follows, referring to Fig. 178:

The differential equation of the elastic line is

$$\frac{d^2y}{dx^2} = \frac{M}{EI}.$$

Integrating this twice, we obtain

$$y = \int\int \frac{M}{EI} \cdot dx^2.$$

This, however, is only the general result, and it must be made more definite. Integrated between limits, it means

$$\frac{dy}{dx}\,(\text{at } d) - \frac{dy}{dx}\,(\text{at } c) = \int_c^d \frac{M dx}{EI} = \frac{\text{area } cdfe}{EI}, \text{ or}$$

$$\frac{dy}{dx}\,(\text{at } d) = -\alpha_4 = \frac{\text{area } cdfe}{EI} + \frac{dy}{dx}\,(\text{at } c)$$

$$= \frac{\text{area } cdfe}{EI} - \alpha_5$$

$$\frac{\text{area } cdfe}{EI} = \alpha_5 - \alpha_4 = \alpha_3$$

which is Theorem I.

(Remember that algebraically $\frac{dy}{dx}$ between a and g is negative.)

Integrating between a and d we should have

$$\frac{dy}{dx}\,(\text{at } d) = -\alpha_4 = \frac{\text{area } adf}{EI} + \frac{dy}{dx}\,(\text{at } a)$$

$$= \frac{\text{area } adf}{EI} - \alpha_2$$

$$\frac{\text{area } adf}{EI} = \alpha_2 - \alpha_4 = \alpha_1.$$

Generally, for any point distant x from a

$$\frac{dy}{dx} = \frac{\text{area of moment curve above } ax}{EI} - \alpha_2 = \frac{A_x}{EI} - \alpha_2$$

$$= \int_o^x \frac{M dx}{EI} - \alpha_2.$$

Integrating again, at any point x,

$$y = \int_o^x \left[\int_o^x \frac{M dx}{EI} - \alpha_2 \right] dx$$

$$= \frac{1}{EI} \int_o^x A_x dx - \alpha_2 x$$

$$= \frac{\text{Statical moment about } x \text{ of moment area } A_x}{EI} - \alpha_2 x.$$

The value of $\alpha_2 x$ is, numerically, the distance xk; y is numerically the distance xy, and is negative. Hence

$$\frac{\text{Statical moment about } x \text{ of } A_x}{EI} = xk - xy = yk$$

which is Theorem II.

To obtain what is termed the deflection of g, namely gg', therefore, it would only be necessary to find the statical moment of agh about a, which would give the deflection of a from the tangent at g', or $ap = gg'$.

From Theorem II there follows a simple expression for the deflection of any point a from the tangent at any other point g (Fig. 178); for since this is merely the statical moment of the moment area about a divided by EI, it is clear that if x is the distance from a of any point in ag,

$$\delta = \int_a^g \frac{M x dx}{EI} \tag{23a}$$

so that the deflection is obtained by one integration.

Theorems I and II, together with Eq. (22), are the foundation of all deductions regarding the slope and deflection of beams, and should be thoroughly mastered by the student. The theorems, especially, are very fruitful of results

which can in no other way be reached so easily, so directly, or with so great *perception* of the real relations. Many applications will be given in following pages.

17*a*. Elastic Weights.—Another use of the moment diagram for finding slope and deflection may be mentioned. In Fig. 178*a* let ACB be a curve or line whose ordinate at any point represents, not the moment, but $\frac{M}{EI}$ at that point of the beam AB exposed to any given load. Let the curve below AB represent the deformed elastic line under this given load, and the slope and deflection of this line are desired. When slope and deflection are mentioned, this curve is the one referred to. The upper curve may here be called the moment diagram, though strictly it is the $\frac{M}{EI}$ diagram. Then, as we have seen,

$$\text{change of slope in } dx = \text{area } dA = \frac{M dx}{EI}.$$

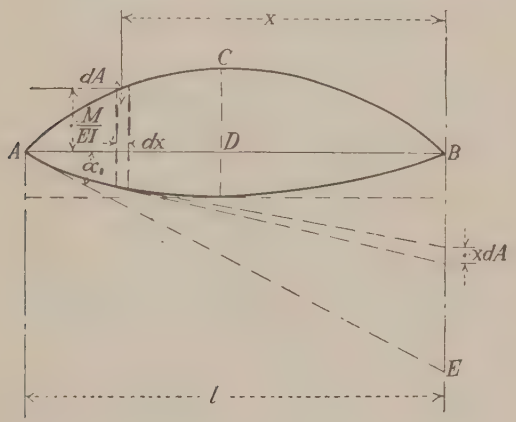


FIG. 178*a*.

Now imagine the same beam AB loaded, not with the given loads, but with an *imaginary distributed load* represented by ACB, the load per running unit at any point being $\frac{M}{EI}$ (where M is the moment due to the given loads). This may be called the *imaginary load;* or the beam AB with this load is sometimes called the *conjugate beam.* Then from Art. 8, change of *shear under imaginary load* in $dx = dA$. From this and the previous statement it follows that the *slope of the original beam at any point equals the shear under the imaginary load, provided that at any one point they are equal.* It is easy to see that they are equal at one of the supports, as A. For drawing tangents to the elastic line at the ends of any distance dx, distant x from B, by the principle of moment areas the angle between these lines is dA, and they intercept on the vertical through B a distance xdA; consequently $\int xdA$ over the whole beam which is the moment about B of the entire imaginary load, is EB, and this divided by l is α_1, and it is also the reaction at A from the imaginary load.

Further, let M' be the moment at any point due to the imaginary load. Then

$$\text{Change of deflection in } dx = \text{slope} = \text{shear on imaginary beam.}$$

But from Art. 8

$$\frac{\text{Change of } M' \text{ in } dx}{dx} = S.$$

From this it follows that *the deflection of the original beam at any point under the given load equals the moment at that point due to the imaginary load*, provided these are equal at any one point. It is easy to see that they are both zero at A.

These theorems comprise the principle of "elastic weights" as applied to beams, stripped of all the unnecessary mathematics with which the subject is usually surrounded, which in one excellent book which is before the writer covers some 7 pages and 20 equations.

The term "elastic weight" is used because the idea is to apply an imaginary load of $\frac{M}{EI}$ on each unit of length, and this expression is related to the elastic deformation of the actual beam, being the change of slope per unit length. The suggestion that the moment diagram could be treated as a load diagram, and from it the slope and deflection under the given loads obtained by finding the shear and moment due to the imaginary load, was made by O. C. Mohr,[1] Professor at the Polytechnikum at Dresden, in 1868.

Illustrations

(1) A beam with a load P at the center, Fig. 178*b*. Here the imaginary load is the triangle shown. The slope at the end of the actual beam is half the area of this

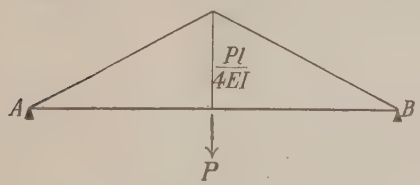


Fig. 178*b*.

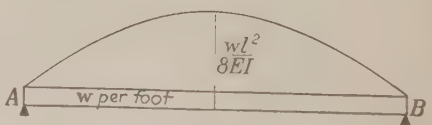


Fig. 178*c*.

triangle, $\frac{Pl^2}{16EI}$ as by the moment-area method; and the slope at any other point is the area between the center and that point. The deflection at the center is

$$\frac{Pl^2}{16EI}\left(\frac{l}{2} - \frac{l}{6}\right) = \frac{Pl^3}{48EI}.$$

(2) A uniformly distributed load, Fig. 178*c*. The same method holds for this case. The end slope is

$$\frac{wl^2}{8EI} \cdot \frac{l}{2} \cdot \frac{2}{3} = \frac{wl^3}{24EI}.$$

The center deflection is

$$\frac{wl^3}{24EI}\left(\frac{l}{2} - \frac{3}{8}\frac{l}{2}\right) = \frac{5}{384}\frac{wl^4}{EI}.$$

[1] "Beitrag zur Theorie der Holz- und Eisenkonstructionen," Zeitsch. d. Arch. u. Ing. Ver. zu Hannover, 1868.

The method is practically identical with the moment-area method, and offers no advantages over it.

18 A Third Method of Finding Slope or Deflection of Beams (or Trusses), Using in This Case the Principle of Work.[1] **Deflection.**—A method of finding the change of position of any point in a structure exposed to stress will now be explained, which is perfectly general in its application (applying to any kind of structure and to any kind of stress), and which leads to many important consequences and many novel points of view. For illustration of this method, a framed structure will be used (Fig. 180).

Suppose it is desired to find the deflection of any given point a of the framed truss shown, in any given direction, ad, under any given system of loads. This deflection is clearly due to the deformation of each individual elementary member or fiber of the structure. It has been shown that the external work equals the internal work. If the intensity of stress is uniform throughout any given length of a fiber or member, and also uniform over its cross-section, all that is necessary to find the internal work done is to divide the structure into fibers or particles possessing the property just stated, to find the deformation of each fiber or particle and the work done upon it, and to add the results. In the case of a truss, the stress is generally assumed to be constant over the entire cross-section of each member and throughout the length of that member. To find the internal work, it is therefore only necessary to find the internal work done in each *member* and to add the results. In the case of a beam, however, the intensity of stress varies continuously along the length of each fiber and over its cross-section. In this case, therefore, it is necessary to divide the structure into particles having infinitesimal cross-section and infinitesimal length, and to integrate the results.

FIG. 180.

The principle of work would give

$$\Sigma(\text{load}) \times (\text{its movement along its line of action}) = \Sigma(\text{work done in each member}).$$

But this clearly does not enable the deflection at a point to be found. It is therefore necessary to resort to some device or stratagem, and this is done as follows: imagine that the *only* load on the structure is *a single load of unity* acting at the given point a in the given direction in which we wish to find the deflection at that point under the given set of loads, and let Δ = the deflection of this point in the given direction under this load unity. The principle of work may now be used, and the internal work may be found and placed equal to the external work. The internal work, if t = the total stress in any member (or fiber) due to the unit load, and l is the length of the member, is

$$W = \Sigma \frac{t}{2} \times (\text{change of length}) \tag{24}$$

and this equals the external work $\frac{\Delta}{2}$, the unit load being gradually applied. The summation, or integration, must extend over every member or fiber in the

[1] This method is more fully treated and exemplified in a later volume, but is introduced here merely for completeness.

structure. Hence, the deflection, stress, and change of length being solely due to the unit load,

$$\Delta = \Sigma t \times (\text{change of length}). \tag{25}$$

But this is not at all what is wanted, for what is desired is the deflection due to given loads, and not to a single unit load.

It must now be perceived that the phenomenon of deflection is a purely geometrical one, and that if any member (or fiber) experiences a given change of length, *from whatever cause*, it will produce the same deflection of the point a in the given direction. Hence Eq. (25) will give the deflection of a in the given direction due —not to the load unity at a—*but to the given set of loads, provided that for (change of length) in the equation we use the change of length due to those given loads.* This must be so, because—since the deflections are very small, and the changes of stress due to the deformations themselves are never considered, the stresses being computed for the form of the structure when undeformed —the deflection must be proportional to the change of length. Now this change of length will be $\frac{sl}{EA}$, if s is the total stress in the member (or fiber) due to the given load, l its length and A its area. Hence the result is reached that the deflection Δ in the given direction, due solely to the given set of loads, is

$$\Delta = \Sigma \frac{stl}{EA} \tag{26}$$

in which

t = total stress in the member (or fiber), due to the load unity at the point a in the given direction,
s = total stress in the member (or fiber) due to the given loading,
A = cross-section of the member (or fiber),
l = length of the member (or fiber).

If the structure is not a truss, in which the stress is uniform over the cross-section and length of any member, but a beam, in which the stress varies in intensity over any cross-section and also along the length of any fiber, the formula still holds, but it only takes account of the effect of normal forces, *neglecting the effect of shear.* In this case, if f = intensity of stress at any point of a cross-section due to the load unity at the point where the deflection is desired and in the given direction, and f' = *intensity* of stress on the same fiber due to the given set of loads, da the area of an infinitesimal part of the section, and dx the infinitesimal distance between two cross-sections, the formula for the deflection (of any given point in any given direction, due to any given set of loads) becomes

$$\Delta = \frac{1}{E}\int\int \frac{(fda)(f'da)dx}{da} = \frac{1}{E}\int dx \int ff'da. \tag{27}$$

The first integral extends over the length of the beam, the latter over the cross-section.

Applying Eq. (27) to a beam, the result may be obtained in terms of the moments; thus let

M_t = moment at any section due solely to a load unity at a (the point where the deflection is desired), in the given desired direction.

M_s = moment at the same section due solely to the given loads, then if the cross-section is uniform, and the case is one of pure flexure

$$f = \frac{M_t y}{I}; f' = \frac{M_s y}{I}$$

$$\therefore \Delta = \frac{1}{E}\int_o^l dx \int_{-y_2}^{+y_1} \frac{M_s M_t y^2 da}{I^2} = \frac{1}{EI}\int_o^l M_s M_t dx.^1 \qquad (28)$$

This is the formula for deflection of a beam, neglecting shearing, and is known as Fränkel's formula, having been first published by Professor W. Fränkel, of the Polytechnic School of Dresden, in *Der Civilingenieur* for 1875.

19. Principle of Reciprocity of Deflections.—Suppose it is desired to find the deflection of a given point a in a given direction B due to a load P acting at a given point c in the direction D. This is given by the formula

$$\Delta = \Sigma \frac{s_1 t_1 l}{EA} \qquad (26)$$

in which

s_1 = stress due load P at c in direction D

t_1 = stress due load unity at a in direction B.

Suppose it is now desired to find the deflection of the point c in the direction D due to a load P at a in direction B. This will clearly be

$$\Delta_1 = \Sigma \frac{P t_1 \cdot \frac{s_1}{P}}{EA} = \Sigma \frac{s_1 t_1 l}{EA} = \Delta.$$

This principle, which of course holds true in flexure also, is expressed by the following:

THEOREM.—*The deflection of any given point a of a body, in any given direction B, due to a single load P at any other point c in the direction D, is the same as the deflection of c in the direction D due to a load P at a in the direction B.*

This is the principle of reciprocity of deflections, sometimes called Betti's principle, but in reality first stated by Clerke Maxwell in an article entitled "On the calculation of the equilibrium and stiffness of frames," published in the *Philosophical Magazine* for April, 1864.

Applications of this principle and of the others developed in this chapter will be given in Chaps. XI and XII which treat of Shears and Moments, and of Deflection and Slope, respectively.

20. Slope.—The change of slope of the axis of a beam, or of the axis of any bar of a frame, at any point, may also be found by the principle of work, in a manner similar to that just explained for finding deflection. This change of slope, like the deflection, is due to the distortion (extension or shortening only, since shear is here neglected) of the individual bars or fibers. Only the case of a beam will be here considered.

[1] This equation may appear to be not homogeneous, the left-hand side being a distance, and the right-hand side a distance times a force. It must be remembered, however, that in the derivation the left-hand side was a force unity times a distance.

Let it be required to find the change of slope of any point a of the axis of a straight beam due to given loads. Imagine a couple whose moment is unity to be applied at that point, concentrated there (Fig. 181). It will cause a change of slope α (expressed in radians), and since the couple may be considered to be

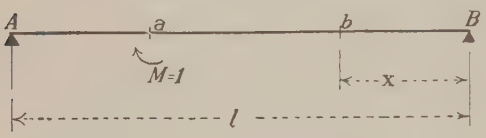


Fig. 181.

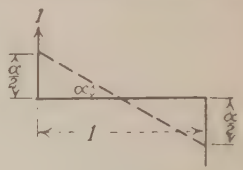


Fig. 182.

composed of unit forces at a unit distance apart (Fig. 182), the work done by the couple, if applied gradually, will be

$$W = \tfrac{1}{2} \cdot \alpha$$

This must equal the work done in the beam, or, if r is the intensity of stress in a fiber, due only to the unit couple, dx the length of the fiber, and da its area,

$$W = \tfrac{1}{2} \cdot \alpha = \tfrac{1}{2} \int_o^l \int_{-y_2}^{+y_1} r da \times \text{(change of length)}.$$

But, as before, the value of α, being due solely to the change of length, and not at all to the cause of that change of length, may be found for the given loads if the change of length is that due to those loads. Hence, if s is the stress in the fiber due to those loads the change of slope which they will cause will be

$$\alpha = \int_o^l \int_{-y_2}^{+y_1} r da \cdot \frac{s}{E} dx.$$

But if M_r is the moment at the section where the fiber considered is situated, due to the unit couple, and M_s the moment there due to the actual loads

$$r = \frac{M_r y}{I}$$

$$s = \frac{M_s y}{I}.$$

Hence

$$\alpha = \int_o^l \frac{M_s M_r dx}{EI} \tag{29}$$

or exactly the same as Eq. (28), substituting for M_t in (28) (which was the moment due a *load* unity) the quantity M_r (which is the moment due to a *couple* unity).

The principle of reciprocity of deflections, explained in Art. 19, will only apply to slope in a limited manner. As applied to slope, or rotation, the principle would be as follows:

The rotation at any point a of the axis of a member, in the sense B, due to a couple M applied at any other point c in the sense D, is the same as the rotation at the point c in the sense D due to a couple M at a in the sense B.

But it would *not* be true of the rotation at a and c due to a load P at the other point, since there is no relation between the rotation due to a load and that due to a couple similar to the relation between the effect of two loads or two couples.

21. Summary of Methods for Finding Slope and Deflection.—From the preceding discussion it will be clear that there are three different methods of procedure for finding the slope and deflection of beams, namely

(1) To start with Eq. (22), substitute for M its value in terms of the loads and dimensions, and proceed by the methods of the calculus.

(2) To use the demonstrated properties of the line of moments, using area to obtain slope and statical moments of area to obtain deflection: or use elastic weights. This is the moment-area method.

(3) To use Eqs. (28) and (29) substituting values of M_s, M_t and M_r and integrating once.

Each of these methods has its uses. In some cases one is much simpler than the others and should be chosen. All, however, should be thoroughly understood by the student and he should be able to use them all, and in any given case to select the one most easily applicable Many illustrations of their use will be given in Chap. XII.

SLOPE AND DEFLECTION OF BEAMS DUE TO SHEARING FORCES

22. The Deformation Due to Shearing.—The previous paragraphs have treated of the deflection and slope due to the extension and shortening of the fibers, assuming that there was no other deformation, and that originally plane sections remain plane during bending. A shearing force, however, distorts an originally plane section, and it is now necessary to consider the effect of shearing upon deflection and slope.

This subject involves many difficulties and contradictions, which may be indicated. It has already been shown that the principle of planar distribution of stress on a section does not in itself necessarily require that originally plane sections should remain plane, but simply that their distance apart, along any fiber, shall after deflection be proportional to the distance of that fiber from the neutral axis. If, therefore, one section, originally plane, becomes distorted (no longer plane), and planar distribution is assumed, the adjacent section must be distorted; or, if one section, originally plane, remains plane, the adjacent section remains plane. The usual beam theory, therefore, both as regards stress and deflection, founded upon planar distribution of stress, requires that if any one section, originally plane, becomes distorted, all must become distorted; or, if any one such section remains plane, all must remain plane. Now the central section of a symmetrical beam, symmetrically loaded, must necessarily, in any case, by the principle of symmetry, remain plane (for any reason that can be adduced to show that it would be distorted in one direction would hold equally in showing that it would be distorted in a symmetrical manner in the other direction). The same is probably true at any section on which there is no external shear. Hence, the usual theory would seem to necessarily require that all originally plane sections must remain plane. But a section originally plane, upon which shear acts, *must* become distorted, since the shear is unequally distributed, and is greatest at the neutral axis, so that a particle originally rectangular, becomes an oblique parallelogram. A slice, therefore (Fig. 183*a*), acted upon by shearing forces, which vary from a maximum at the neutral axis to zero at the top and bottom, must become distorted into the shape shown in Fig. 183*b*, in which a rectangle at the center is most distorted, and one at the top or bottom is not distorted at all. Here then,

Fig. 183*a*.

Fig. 183*b*.

is a contradiction at the start. It probably means, as stated in Chap. IX, that the assumption of planar distribution is not strictly true, although experiments cannot be made with sufficient accuracy to prove or disprove it.

Suppose the deflection of a horizontal beam *due to shear alone* is to be found. Here then, it must be assumed that the *longitudinal* fibers are absolutely rigid, or incapable of extension or compression, while a rectangular particle may be distorted by shear into an oblong parallelogram, which of course requires that one of its diagonals is extended, while the other is equally compressed. We may thus imagine a slice of the beam to be made up of rectangular cells whose sides are absolutely rigid, and which are hinged together, and connected by diagonals which are elastic. Now, if this beam deflects into a curve, due to shearing alone under, say, a load at the center, in which case, as seen, the center section must remain plane and vertical, then any section originally at a (horizontal) distance d from the center, must after deformation remain with every point at this same distance d from the center (since the deflections are so small that $ds = dx$) because the longitudinal fibers are rigid, so that this section, and every section, must remain vertical, that is, plane; hence, every portion is distorted equally, or the shearing force is uniformly distributed. Here is the same contradiction again.

The situation, then, seems to be this: the assumption of planar distribution of normal stress seems to be the only practicable one; this requires that if one plane section remains plane, all must, and also that the shear be not uniformly distributed over a section (see Art. 9); this requires that sections exposed to shear, if originally plane, cannot remain plane; but the central plane section of a beam symmetrical and loaded symmetrically about the center must necessarily remain plane, and probably also any section on which there is no shear; under this condition planar distribution of normal stress requires every other plane section to remain plane.

These discrepancies cannot be easily reconciled, and they serve to show the approximate character of all these theories. The exact theory of beams has been attempted by writers on the Theory of Elasticity, notably by de St. Venant and by Clebsch, but even their results are hardly satisfactory, and cannot well be used in practice owing to their complication.

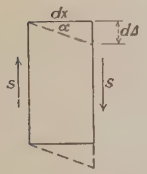


Fig. 184.

How, then, may the deflection due to shear be ascertained, even approximately. There are several methods. First, we may assume the shear to be distributed uniformly over the cross-section, and then find the distortion of a section with reference to another section distant dx from it, and integrate the result. Let Fig. 184 represent such a section, the area being A, the total shear S, and the average intensity $\frac{S}{A}$. Under these conditions the angle α, which measures the average distortion, will be $\frac{S}{AG}$, in which G is the shearing modulus of elasticity. Hence we shall have

$$d\Delta_s = \alpha dx = \frac{S}{AG} \cdot dx.$$

This must now be summed up between one end of the beam and the point where the deflection is desired, or the deflection Δ_s due to shear, is

$$\Delta_s = \frac{1}{AG}\int_o^a S dx \tag{30}$$

$$= \frac{1}{AG}\text{(area of shear curve between either free end of beam and the point whose deflection is desired).}$$

By this method we find the deflection between any two sections, beginning at one end and proceeding to the point where the deflection is desired.

The slope at any point due to shear would simply be the distortion at that point, or

$$\alpha_s = \frac{S}{AG}. \tag{31}$$

23. Deflection Due to Shear Found by Principle of Work.—A better solution may perhaps be reached by making use of the principle of work, avoiding (but not eliminating) the uncertainties just alluded to. The procedure is the same as that followed in the case of normal stresses.

Let s_1 be the intensity of shear at any point in a cross-section, due to a load unity at the given point on the beam in the given direction: then, as before, if Δ_s is the deflection at the point, due to shear, caused by the unit load,

$$\tfrac{1}{2}\cdot\Delta_s = \int \tfrac{1}{2}\cdot s_1 dA \times \text{(shearing deformation)}.$$

Substituting now for the shearing deformation that which is due to the actual system of loads for which the deflection is desired, namely, $\frac{s}{G}dx$, we obtain the result

$$\Delta_s = \int\int \frac{s s_1 dA\cdot dx}{G} = \frac{1}{G}\int_o^l dx \int_{-y_2}^{+y_1} s s_1\, dA. \tag{32}$$

If S_1 = total shear in the section due to load unity at the given point and in the given direction,

S = total shear on the section due to *given loads*, then it is slear that

$$s_1 : s :: S_1 : S$$

$$s_1 = \frac{S_1}{S}s$$

$$\therefore \Delta_s = \frac{1}{G}\int_o^l \frac{S_1}{S}dx \int_{-y_2}^{+y_1} s^2 dA. \tag{33}$$

By this method, which is substantially different from the previous one, we do not sum up successive deflections, but find *the deflection of the given point due to the deformation of every slice of the beam*, and sum up the results.

24. Example of Deflection.—Let it be desired to find the deflection, at the center, of a beam loaded with a concentrated load at the center (Fig. 185). The moment curve is given by Fig. 185*b*; and by Theorem II the deflection due to bending moment may be written at once as

$$\Delta_b = \frac{Pl^3}{48EI}.$$

By Eq. (28) we have

$$M_s = \frac{Px}{2} : M_t = \frac{x}{2}; \therefore \Delta_b = \frac{\int_o^{\frac{l}{2}} Px^2 dx}{2EI} = \frac{Pl^3}{48EI} \tag{34}$$

as before.

To find the deflection due to shear, Eq. (30) gives at once

$$\Delta_s = \frac{Pl}{4AG}. \tag{35}$$

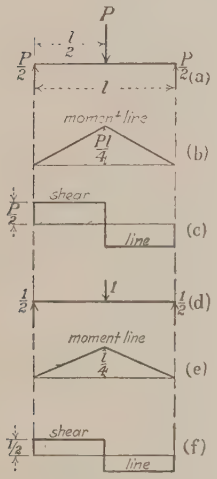


FIG. 185.

To use Eq. (33) we have

$$S_1 = \pm\frac{1}{2}; S = \pm\frac{P}{2};$$

hence, for constant section,

$$\Delta_s = \frac{1}{GP}\int_o^l dx \int_{-y_2}^{+y_1} s^2 dA. \tag{36}$$

The shear per unit height of cross-section is $\frac{SQ}{I}$, and, if we assume this uniformly distributed along the width, the value of s is $\frac{SQ}{Ib}$, and dA equals bdy, hence, for constant section,

$$\Delta_s = \frac{P}{4GI^2}\int_o^{\frac{l}{2}} dx \int_{-y_2}^{+y_1} \frac{Q^2 dy}{b} = \frac{Pl}{4GI^2}\int_{-y_2}^{+y_1} \frac{Q^2 dy}{b}. \tag{37}$$

This cannot be further reduced, except for special shapes of cross-section, because the quantity under the integral will be different for different shapes. For this reason Eq. (30) is more easy of application.

For a rectangular section, Eq. (35) becomes

$$\Delta_s = \frac{Pl}{4bhG}, \tag{38}$$

while Eq. (37) becomes (see Art. 9)

$$\Delta_s = \frac{3Pl}{10bhG} = \frac{Pl}{3.33bhG}. \tag{39}$$

Assuming $G = 0.4E$ these equations become

$$\Delta_s = \frac{5}{8}\cdot\frac{Pl}{bhE} \tag{38a}$$

$$\Delta_s = \frac{3}{4}\cdot\frac{Pl}{bhE}. \tag{39a}$$

The deflection due to bending, from Eq. (34), is

$$\Delta_b = \frac{1}{4}\cdot\frac{P}{bE}\frac{l^3}{h^3}. \tag{34a}$$

The deflection due to shear is almost always neglected. It is, however, not always insignificant. The ratio of Δ_s to Δ_b is given by the following equation, assuming Eq. (39a) to be more nearly accurate than Eq. (38a):

$$\frac{\Delta_s}{\Delta_b} = 3\cdot\frac{h^2}{l^2} \text{ (for a rectangular section)}. \tag{40}$$

This shows that the relative importance of Δ_s decreases as $\frac{h}{l}$ decreases. If we assume $\frac{h}{l}$ as $\frac{1}{5}$, $\frac{1}{10}$, and $\frac{1}{20}$, we find

$$\frac{\Delta_s}{\Delta_b} = \left\{ \begin{array}{l} \frac{3}{25} = 12 \text{ per cent for } \frac{h}{l} = \frac{1}{5} \\ \frac{3}{100} = 3 \text{ per cent for } \frac{h}{l} = \frac{1}{10} \\ \frac{3}{400} = 0.75 \text{ per cent for } \frac{h}{l} = \frac{1}{20} \end{array} \right\} \tag{41}$$

The ratio will be less than 5 per cent if $\frac{h}{l} < \frac{1}{8}$, so that in general the deflection due to shear may properly be neglected.

From the above it follows that owing to their greater simplicity and ease of application, Eqs. (30) and (35) are quite close enough for practical purposes.

25. The Five Related Lines.—From what has preceded it is clear that in the theory of flexure there are five typical lines related to each other in a singular and important manner, *viz.*,

(1) The load line (ordinate at a point equals load per unit length at that point).

(2) The shear line (ordinate at a point equals shear at that point).

(3) The moment line (ordinate at a point equals moment at that point).

(4) The slope line (ordinate at a point equals slope at that point).

(5) The deflection line (ordinate at a point equals deflection at that point).

The relations between these lines may be summarized as follows:

(*a*) The area of the load line between any two points (algebraically) equals the change of shear between those points. The area from the free end of the beam to any point (algebraically) equals the shear at that point. Its area from any point where the shear is zero to any other point (algebraically) equals the shear at that other point. Its area-moment from any point where the moment is zero (end of beam) to any other point, taken about that other point, is (algebraically) the moment at that point.

(*b*) The area of the shear line between any two points (algebraically) equals the change of the moment between those points. The area of the shear line from the free end of the beam to any point (algebraically) equals the moment at that point, and equals the moment of the load area (area-moment) about that point.

(*c*) The area of the moment line between any two points (numerically) divided by EI equals the change of slope between those points. The area between the point where the deformed elastic line is horizontal (*c*) and any other point (*b*), divided by EI, equals the slope at that point (*b*).

(*d*) The area of the slope line between any two points (numerically) equals the change of deflection between those points; the area between the point where elastic line is horizontal and the end of the beam equals the max. deflection, and equals the area-moment of the moment line between the end and the point where the elastic line is horizontal, taken about the end. The point where the elastic line is horizontal may be found from the moment line by the condition that the area-moments of the moment line on each side of this point, taken about the corresponding end, must be equal.

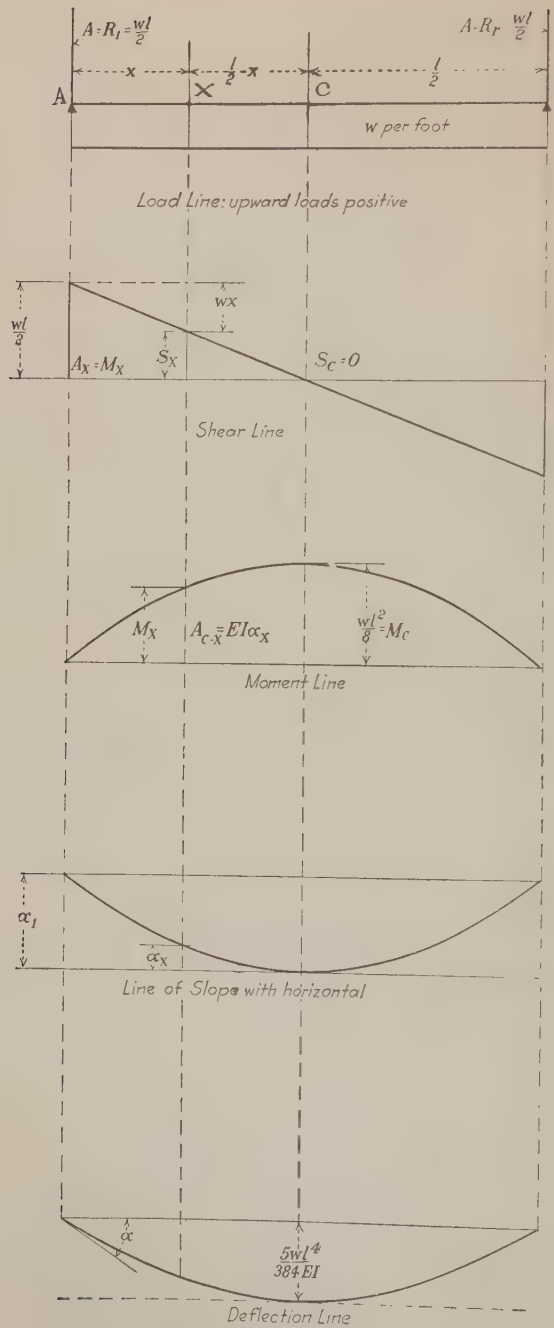

A-R1 = wl/2
A-Rr wl/2
x
l/2 - x
l/2
A
X
C
w per foot
Load Line: upward loads positive
wl/2
wx
Ax = Mx
Sx
Sc = 0
Shear Line
Mx
Ac-x = EIαx
wl²/8 = Mc
Moment Line
α1
αx
Line of Slope with horizontal
α
5wl⁴/384EI
Deflection Line

Fig. 186.

(*c*) The area-moment of the moment line between any two points taken about one of them, equals the deflection of that point from the tangent at the other.

Not all of these principles can be applied to advantage, but the student should understand them, and be prepared to apply, in any given case, such as may be properly used. The slope line, in particular, is seldom or never used. They are illustrated in Fig. 186, for the case of a simple beam supported at the ends and uniformly loaded. The student should construct and test similar figures for other cases. Many applications of these principles will be given in Chap. XII.

Referring to Fig. 186, the following are the relations indicated:

The Load Line.—Here a concentrated area equal to $\frac{wl}{2}$ must be imagined at each end, represented by the heavy lines,

$$\text{Algebraical area from } a \text{ to } x = A_x = \frac{wl}{2} - wx = S_x$$

$$\text{Algebraical area from } a \text{ to } c = 0 = S_c$$

$$\text{Area-moment from } a \text{ to } x = \frac{wx(l-x)}{2} = M_x$$

$$\text{Area-moment from } a \text{ to } c = \frac{wl^2}{8} = M_c.$$

The Shear Line.

$$\text{Area from } a \text{ to } x = \frac{wx(l-x)}{2} = M_x$$

$$\text{Area from } a \text{ to } c = \frac{wl^2}{8} = M_c$$

$$\text{Area-moment from } a \text{ to } c \text{ about } c = \frac{wl^3}{24} = (\text{slope at } a)EI.$$

The Moment Line.

$$\text{Area from } x \text{ to } c = (\text{slope at } x)EI$$

$$\text{Area from } a \text{ to } c = \frac{wl^3}{24} = (\text{slope at } a)EI$$

$$\text{Area-moment from } a \text{ to } c \text{ about } a = \frac{5wl^4}{384} = \partial_c \cdot EI.$$

The Slope Line.

Beam is horizontal at center.

$$\text{Slope at end} = \alpha_1 = \frac{wl^3}{24EI} \text{ (from moment line).}$$

$$\text{Slope at } x = \alpha_x = \text{area of moment line from } c \text{ to } x \div EI$$

$$= \frac{w}{EI}\left(\frac{x^3}{6} + \frac{l^3}{24} - \frac{lx^2}{4}\right), \text{ obtained from moment line.}$$

$$\text{Area from } a \text{ to } c = \int_o^{\frac{l}{2}} \alpha_x dx = \frac{5wl^4}{384EI} = \text{deflection at center} = \partial_c.$$

26. Deflection in Terms of Stress Intensity.—If a beam of uniform section supported at its two ends is loaded uniformly with w pounds per linear foot, and if the allowable intensity of stress is f, per square inch, the deflection Δ at the center will be

$$\Delta = \frac{5}{384} \cdot \frac{wl^4}{EI} = \frac{5}{48} \cdot \frac{l^2 f}{Ey} \tag{42}$$

since

$$f = \frac{My}{I} = \frac{wl^2 y}{8I}.$$

Hence
$$\frac{\Delta}{l} = \frac{5}{48} \cdot \frac{lf}{Ey}. \tag{43}$$

If the beam is loaded with a single concentrated load W at the center, the formulae become

$$\Delta = \frac{Wl^3}{48EI} = \frac{l^2 f}{12Ey}. \tag{44}$$

$$\frac{\Delta}{l} = \frac{lf}{12Ey}. \tag{45}$$

In these formulae f is the fiber stress at the outside fiber, distant y from the center of gravity of the cross-section. If the section is symmetrical, with a height h, $y = \frac{h}{2}$, and the equations become

(*a*) For a uniformly distributed load,

$$\frac{\Delta}{l} = \frac{5lf}{24Eh}. \tag{46}$$

(*b*) For a concentrated load at the center,

$$\frac{\Delta}{l} = \frac{lf}{6Eh}. \tag{47}$$

If the maximum ratio r of $\frac{\Delta}{l}$ is fixed, then the height of the beam must not be less than given by the following equations:

$$(a)\ \frac{h}{l} \geq \frac{5f}{24Er};\ (b)\ \frac{h}{l} \geq \frac{f}{6Er}. \tag{48}$$

For steel, if E is taken as 30,000,000, and f at 16,000 pounds per square inch, these relations become

$$(a)\ \frac{h}{l} \geq \frac{1}{9{,}000r};\ (b)\ \frac{h}{l} \geq \frac{1}{11{,}250r}. \tag{49}$$

For wood, with E at 1,200,000 and f at 1,500 pounds per square inch

$$(a)\ \frac{h}{l} \geq \frac{1}{3{,}840r};\ (b)\ \frac{h}{l} \geq \frac{1}{4{,}800r}. \tag{50}$$

In the case of plastered ceilings, the maximum allowable deflection is usually assumed at 1 inch in a span of 30 feet, or $r = \frac{1}{360}$; hence, approximately, For steel beams:

$$(a)\ \frac{h}{l} \geq \frac{1}{25};\qquad (b)\ \frac{h}{l} \geq \frac{1}{31}. \tag{51}$$

For wood beams:

$$(a)\ \frac{h}{l} \geq \frac{1}{11};\qquad (b)\ \frac{h}{l} \geq \frac{1}{13}. \tag{52}$$

It should be noticed, that the deflection of a given span *varies directly as f, directly as l^2, inversely as E, and inversely as y.* Hence we may say that *if one beam is twice as deep as another, it is twice as stiff, i.e., it will deflect half as much with the same unit stress, independent of the shape. Stiffness depends alone upon depth and upon the material.*

27. Depth in Terms of Loads for Limiting Deflection.—Suppose a beam is designed to carry a given load, and its depth has been determined. If, now, this

depth is less than the values given in the preceding paragraph as necessary to limit the deflection, a greater depth must be chosen; but if a greater depth is taken, the fiber stress under the given load may be reduced, so that it may not be necessary to increase the depth to the full values expressed in the preceding paragraph. How great must the depth be? This may be found from Eqs. (42) and (44) of the preceding paragraph, which may be written (W being the total load):

(*a*) Uniformly distributed load: $$I \geq \frac{5Wl^2}{384Er}. \tag{53}$$

(*b*) Single load at center: $$I \geq \frac{Wl^2}{48Er}. \tag{54}$$

From this it is easy to deduce the following, for *rectangular wooden beams*, b being the breadth,

$$(a)\ \frac{h}{l} \geq \frac{1}{1.86}\sqrt[3]{\frac{W}{Eblr}}; \qquad (b)\ \frac{h}{l} \geq \frac{1}{1.59}\sqrt[3]{\frac{W}{Eblr}}. \tag{55}$$

If for E and r the values 1,200,000 and $\frac{1}{360}$ are used, these formulae become

$$(a)\ \frac{h}{l} \geq \frac{1}{27.8}\sqrt[3]{\frac{W}{bl}}; \qquad (b)\ \frac{h}{l} \geq \frac{1}{23.9}\sqrt[3]{\frac{W}{bl}}. \tag{56}$$

The preceding paragraph showed that for given values of f and r, the value of $\frac{h}{l}$ is greater for a uniformly distributed load than for a concentrated load at the center. The above results show that this relation is reversed when the total load is the same in each case, and only r is constant.

28. Summary of the Most Important Results Thus Far Obtained.—(1) The intensity of normal stress at any point of any cross-section of a beam exposed to pure flexure is given by the formula (Art. 5),

$$f = \frac{M_x y}{I_x} + \frac{M_y x}{I_y}.$$

(2) The fundamental equations governing the relations of shear, moment, slope, and deflection, are the following:

$$\frac{dM}{dx} = S$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{\int S dx}{EI}$$

$$\frac{dy}{dx} = \frac{\int M dx}{EI} + C.$$

(3) The longitudinal shear per unit length on any longitudinal section is

$$S_l = \frac{QS}{I}.$$

(4) The work done by a direct force S on a piece of uniform section A and length L is

$$W = \frac{S^2L}{2AE}.$$

(5) The work done by a shearing force of intensity s on a rectangular particle of volume V is

$$W = \frac{s^2 V}{2G}.$$

(6) The work done in a beam of uniform section subjected to pure flexure is

$$W = \frac{\int_o^l M^2 dx}{2EI}.$$

(7) The maximum or minimum moment in a beam exposed to pure flexure, for any given loading, occurs at the point where the shear is zero or passes through zero.

(8) The difference between the moments at any two points equals the area enclosed by the shear line between those points (algebraically). Where the shear is positive the moment is increasing, and *vice versa.*

(9) The (algebraical) area enclosed by the shear line, between any two points at which the moments are the same, as between free ends of a beam, equals zero.

(10) At any point, the tangent of the angle which the moment line makes with the axis equals the shear at that point, algebraically.

(11) The moment at any point equals the area enclosed by the shear line between that point and a point where the moment is zero, as the free end of the beam.

(12) (*a*) The area enclosed by the moment line (algebraical) between any two points, divided by EI, equals the angle between the tangents to the elastic line at those two points.

(*b*) The area enclosed by the moment line between the point where the elastic line is horizontal and any other point, divided by EI, equals the slope of the elastic line at that other point with reference to the horizontal.

(13) The statical moment, taken about any point of a beam, of the area enclosed by the shear line between the free end of the beam and that point (which equals the area enclosed by the moment line between the same points), divided by EI, equals the angle between the tangents to the elastic line at the given point and at the free end of the beam.

(14) The deflection of any point of the axis, g, from the tangent to the elastic line at any other point, c, measured either at right angles to the latter tangent or to the original (undeformed) axis, equals the statical moment of the area enclosed by the moment line between g and c, taken about g, divided by EI.

(15) The deflection of any point of a beam in pure flexure, and having constant section, in any given direction, due to the normal stresses only, is

$$\Delta_n = \frac{1}{EI}\int_o^l M_s M_t dx. \qquad \text{(See Art. 18)}$$

(16) The deflection of any point of a beam in any given direction, due to the shearing stresses only is

$$\Delta_s = \frac{1}{AG}\int_o^a S dx \text{ (by one method)}$$

or

$$\Delta_s = \frac{1}{G}\int_o^l \frac{S_1}{S} dx \int_{-y_2}^{+y_1} s^2\, da \text{ (by another method).}$$

(17) The change of slope of any point of the axis of a beam, due to the normal stresses only, is

$$d_n = \frac{1}{EI}\int_o^l M_s M_r dx. \qquad \text{(See Art. 20).}$$

29. Beams of Uniform Strength.—If the load on a beam is unvarying, there will be some section (or sections) where the moment is a maximum, while on all other sections it will be less. If the beam is of uniform section throughout, and is strong enough at the point of maximum moment, it will be too strong elsewhere, and there will theoretically be a waste of material. The section may be varied, by varying either breadth or depth or both, so that it will be equally strong at all sections, and such a beam is called one of uniform strength, and will make a saving of material, though perhaps not of cost.

The variation of section must conform to the variation of moment. Since $M = \frac{fI}{v}$, and f is to be constant, the condition to be fulfilled is

$$\frac{Mv}{I} = \text{constant.}$$

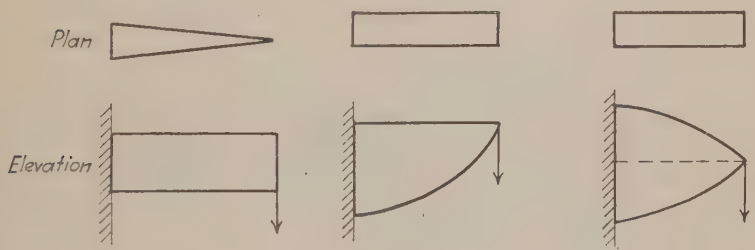


FIG. 187. FIG. 188*a*. FIG. 188*b*.

Beam Fixed at One End and Loaded at the Other.—If the load is W, and if x is the distance from the free end, $M = Wx$, hence the condition is

$$\frac{xv}{I} = \text{constant.} \qquad (57)$$

If the section is rectangular, this becomes

$$\frac{x}{bh^2} = \text{constant.} \qquad (58)$$

Hence; if h is constant, b must vary as x (Fig. 187)
if b is constant, h^2 must vary as x (Figs. 188*a* and 188*b*).

The depth in Figs. 188*a* and 188*b* varies as the ordinates of a parabola with vertex at the free end.

Both breadth and depth may be varied; if they vary so that all cross-sections are similar, or b proportional to h, then any longitudinal section, either vertical or horizontal, would have h^3 or b^3 vary as x.

Beam Fixed at End, with a Uniformly Distributed Load.—Here $M = \frac{wx^2}{2}$, and the condition is for a rectangular section

$$\frac{x^2}{bh^2} = \text{constant.}$$

Hence; if h is constant, b varies as x^2 (Fig. 189)
if b is constant, h varies as x (Fig. 190).

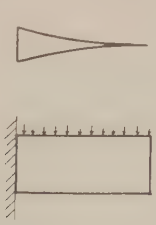


Fig. 189.

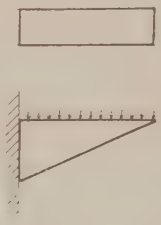


Fig. 190.

Beam Supported at Each End, and Loaded at the Middle.—Each half of such a beam is a beam fixed at one end (the middle) and loaded at the other by the reaction; hence if h is constant, b must vary as the distance from the end (Fig. 191*a*) if b is constant, h^2 must vary as the distance from the end (Fig. 192). In the latter case the curved outline consists of two parabolas meeting at an angle at the center.

Beam Supported at Each End, and Loaded Uniformly.—In this case $M = \frac{wx(l-x)}{2}$, and the condition for equal strength becomes

$$\frac{x(l-x)}{bh^2} = \text{constant}.$$

Hence; if h is constant, b varies as $x(l-x)$, Fig. 193. The curves are parabolas:
if b is constant, h^2 varies as $x(l-x)$ Fig. 194. The curved outline is a semi-ellipse; or it may be a complete ellipse.

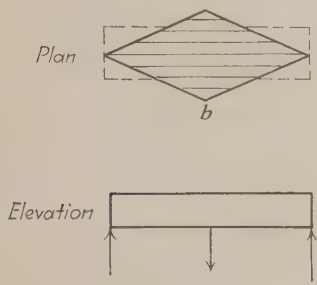


Fig. 191*a*.

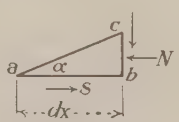


Fig. 191*b*.

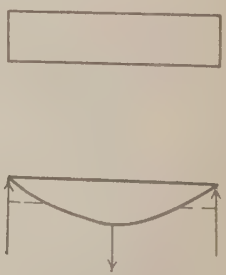


Fig. 192.

These forms, however, are in general impractical, for the following reasons:

(1) Shear must be provided for, and the section cannot be reduced to zero at the ends. This may be allowed for as indicated by the dotted lines in Figs. 191*a*, 192, 193 and 194.

(2) Unless the beam can be cast into the form desired, it would generally cost more to make them than to make them of uniform section. A wooden beam could be cut to the necessary shape, but it would have to be cut from a beam of the maximum section, and would involve no saving in material and an increase in labor. A rolled beam must be made of constant section. Cast-iron brackets

or beams supported at each end, as window lintels, may be cast so as to approximate to beams of equal strength, modified to take account of shear.

(3) The formulae assume that, if the section varies, the distribution of the normal stress will be the same as for constant section. So far as statical equilibrium is concerned, this may be true, but there may be conditions that render it untrue. It has been shown, in Chap. V, that at the surface of a body the stress

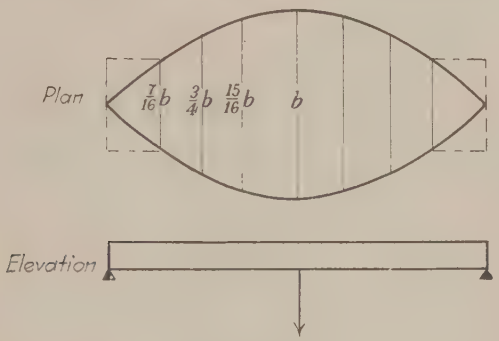


FIG. 193.

on any plane must act parallel to the surface. In Fig. 191*b* if *ac* is the surface and *cb* a small part of a cross-section, the stress N on *cb* may be found by the usual formulae. If $ab = dx$, $cb = dx \cdot \tan\alpha$; and if f is the normal stress intensity on *cb*, $S = fdx \tan\alpha$ = total shear on *ab*; hence shear intensity on *ab* and on *cb* is $f \tan\alpha$; and total shear on *cb* is $fdx \tan^2\alpha$. The shear intensity must be kept within safe limits.

Where there is an angle in the surface there can be no stress, because it cannot be parallel to two different surfaces.

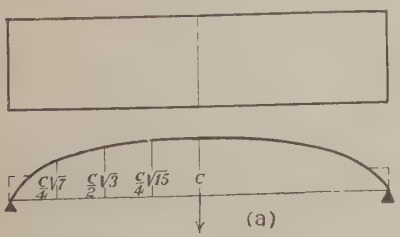


FIG. 194.

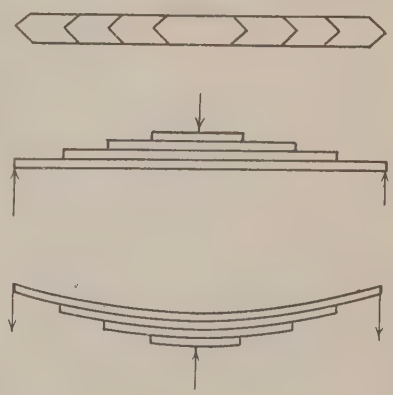


FIG. 195.

Hence the theory by which these beams of equal strength are determined is not always correct.

One case, however, is made practical use of, namely, that shown in Fig. 191*a*. If this beam consists of a thin sheet of constant thickness (h), and is cut into strips as indicated by the longitudinal lines in plan, and these strips are placed one on top of another, the result is the spring shown in Fig. 195, since the supposed strips act independent of each other except for friction. Such a spring is suitable

for carrying a concentrated load at the center, and is essentially a beam of uniform strength. The layers or leaves of the spring are held together by clamps. The lower figure (Fig. 195) shows such a spring inverted, with a load at each end and the reaction in the center.

The deflection of beams of uniform strength is discussed in Chap. XII.

ACTUAL DISTRIBUTION OF NORMAL STRAIN AND STRESS

30. In Chap. IX the distribution of a normal stress over a plane area was discussed, and at the end of the chapter it was stated that the subject would be resumed at the end of the present chapter. In Chap. IX and in the formulae for flexure, the assumption of planar distribution of normal stress was made the basis of the resulting formulae; that is, the assumption that if at each point of the stressed area a normal be erected whose length is proportional to the intensity of stress at that point, the other ends of all these normal lines will lie in a plane. This assumption of planar distribution of *stress* means also planar distribution of *strain within the elastic limit*, that is to say, it means that an originally plane surface remains plane if exposed to normal stress alone.

The problem now before us is to examine this assumption and to see if any conclusions can be reached regarding the actual distribution of stress, and its law of variation, in a homogeneous body.

In all problems of elasticity, the intensity of internal stress on a plane, or on parallel planes, is assumed to vary gradually (never suddenly) from point to adjacent point of a plane, or along a line from one plane to an adjacent parallel plane. Does it vary in this way, and according to what laws?

31. The *outer force* applied to a plane surface may vary in any manner whatever. It may be uniformly distributed (*a*, Fig. 196), or as the ordinates to an inclined straight line (*b*), or gradually but not uniformly, as the ordinates to a curve (*c*), or it may change suddenly in intensity, as where a load may cover the part *ef* and drop to nothing at *f* (line *d*). It cannot, however, be concentrated at a point or line, for that would give an infinite intensity of pressure at that point or line, which is impossible. Yet we continually assume that a load may be concentrated at a point, meaning really that its resultant acts at that point, and that the load is distributed over a small area on each side of that point. If a truly circular wheel rests upon a straight rail, and if neither material yields at all, the load would be concentrated at the point or line of contact; but the materials would actually yield, and the load would be distributed.

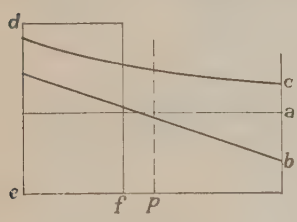


Fig. 196.

But how is it about internal stress? Can this be distributed in any manner? Clearly it cannot be concentrated at a point, for the reason above explained, but can it vary suddenly, or can its rate of variation change suddenly? In thinking of this we must remember that the total internal stress on a surface is the force exerted by the part of the body on either side of that surface upon the part of the body on the other side, and if these two parts are imagined separated, and the actual stress at each point applied to the two parts, the distribution will be identical on each part. This internal stress may be considered as an outer force acting

on the part considered, and balancing the resultant actual outer force acting on it, Thus if ab is a cross-section of a piece (Fig. 196a), and the two parts of the piece are *imagined* separated as shown, there being a force P acting on the surface, the distribution on the upper and lower ab is identical, and at any point c (or c') the stress intensity is $cd = c'd'$. The stress on the upper ab will deform that line, and the same for the lower ab; and the condition to be fulfilled is that the real deformations must be such that the deformed lines will fit together if the parts are supposed joined again. Clearly, if ab remains a straight line in each part of the body, they will fit. Clearly, also, the internal stress P could not be distributed over a part of the surface ab, as at $beff'$, for in that case the upper and lower lines ab would be deformed to some such shape as aeg, $ae'g'$, and when the parts eg, $c'g'$, were put together the parts ac, ae' would not come together. Hence the stress P *must be distributed over the entire area* ab. For similar reasons, there can

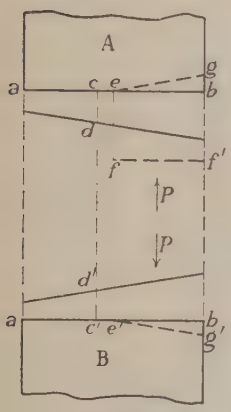


Fig. 196a.

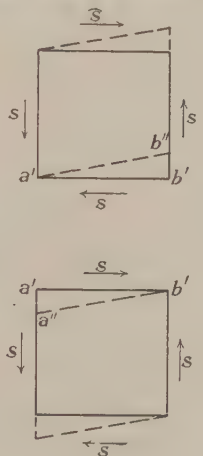


Fig. 196b.

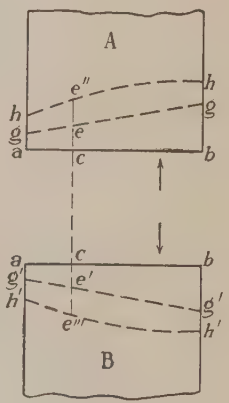


Fig. 196c.

be no sudden change in stress intensity; in other words, the stress intensity must be a continuous function of the coordinates of the point.

It is even difficult to see how there could be a sudden change in the rate of variation of the stress. For such sudden change would imply a sudden change in the rate of variation of strain. Suppose that at e in Fig. 196a there were a planar distribution over ae and also over eb, but with a sudden change in the rate at e. Then ab would be distorted into a broken line with an angle at e; and the lower ab would be similarly distorted. The two would not fit. This may explain why in tests of beams the modulus of elasticity for tension and that for compression are found the same, while direct tests in tension and in compression show them to be different.

32. In studying the distribution of stress on a plane, only the normal stress need be considered, so far as distortion is concerned, and the necessary fitting together of the two parts as in Fig. 196a. For suppose that any infinitesimal element of the surface ab of Fig. 196a takes the position $a'b'$ in Fig. 196b under the normal stress alone. The two lines $a'b'$ (really the same line, the parts being separated in Fig. 196a) fit under the action of normal stresses alone;

but suppose there is a shearing stress on the section having an intensity s on $a'b'$. This shearing stress will distort a rectangular particle above the section, and another such particle below the section as shown by the dotted lines in Fig. 196*b*, and each by the same amount, and the lines $a'b''$ and $a''b'$ will still fit. This will be true no matter what the distribution of the total shear on the section ab over the entire area ab. Hence it is only necessary to study the distribution of the normal stress. The stress at any point c in Fig. 196*c* will distort the part of the body above ab, moving c to e, and it will distort the part of the body below ab, moving c to e'. It is not necessary that $ce = ce'$ although the stress intensity at c is the same in each part, A and B. The distortion of A will depend upon its size, dimensions, etc., and will be the same in A and B, necessarily, only if these parts are in all respects similar; if they are not similar, although the stress intensity at c is the same on part A as on part B, the distortion ce need not be precisely the same as the distortion ce'. But the distortion of c to e will in any case depend upon the stress intensity at c and will increase as this stress increases. The body in question is assumed to be homogenous; then if ab in the upper part is distorted into another straight line gg, and ab in the lower part into the straight line $g'g'$, these two lines would fit together. But it is difficult to conceive that ab in the upper figure could be distorted into *any curve* such as hh; for if e in the upper figure moves to e'' and a to h, ce'' being greater than ah, then if c in the lower figure moves to e''', it follows that a must move to h' and that ah' is less than ce'''. In other words, if the distributed stress on ab in the upper figure distorts ab into a concave curve, the same distributed stress on ab in the lower figure would distort ab into a concave curve also, and the two surfaces hh and $h'h'$ *would not fit.* The distortion must be such that where the upper curve is concave, the lower curve is convex, and *vice versa.*

33. *These considerations appear to show that any plane section in a body exposed to a distributed normal stress must remain a plane surface. This must hold above as well as below the elastic limit,* for the parts must fit for any stress. In other words, *the normal strain has planar distribution.* But *this is not true with reference to the normal stress,* which may be very differently distributed. *Below the elastic limit* the stress will have planar distribution, but not above the elastic limit. The imaginary elastic fibers of a body exposed to a varying normal stress, like the fibers of a beam loaded transversely, are not physically able to deform as they would if separate, and the stress-strain diagram for such a fiber is not at all the usual stress-strain diagram. It shows no yield point and no plastic stage. The fibers are restrained somewhat as those of a notched specimen are restrained, as explained in Art. 4 of Chap. VI. It has been explained in Chap. IV that in such cases the stress-strain diagram, instead of resembling curve A of Fig. 71, more nearly resembled a curve like B. In the present case, that of varying normal stress over a plane section, the same is true. Hence we see the uselessness of the method suggested by Prof. Upton for determining the "true fiber stress" in beams ("Materials of Construction," Art. 139), in which he assumes the usual stress-strain curve to apply for a fiber of a beam.

If, as these considerations appear to prove, plane sections remain plane under normal stress, the effect of a shear is to distort that plane section according to the distribution of the shear. If the shear were zero or uniformly distributed, plane sections would finally remain plane. If the shear is not uniformly distrib-

uted, originally plane sections would not remain finally plane, but would be so distorted that where the part A in Fig. 196c is concave, the part B is convex and *vice versa*. Thus in an ordinary rectangular beam a plane section actually takes the shape shown in Fig. 196d if the shear is positive, as shown in Art. 22.

34. In Chap. V it was assumed that the intensity of normal or shearing stress on two parallel planes, an infinitesimal distance apart, differed only by an infinitesimal, and this must be the case. Consider the prism of which abc, Fig. 197 is a section, and suppose that the shear intensity on ac could change suddenly at d by a finite quantity, from s on ad to s_1 on dc. Then taking two infinitesimal rectangular particles at d, one just to the left and one just to the right of d, with db in common, it would follow that on db there could be simultaneously two different shears, s and s_1. (Remember that $s + ds = s$.) Similarly, the normal stress intensity cannot change suddenly by a finite quantity at d, nor can the direction of the resultant stress on a plane; nor can normal or tangential stress on parallel planes.

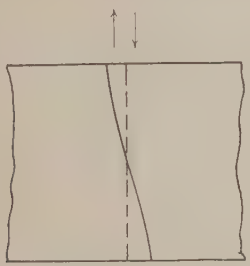


FIG. 196d.

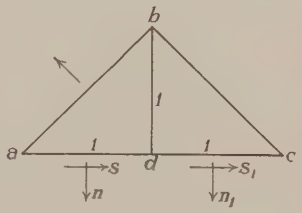


FIG. 197.

35. The results of these articles are of importance in the theory of flexure. They show that the actual stress in a beam (if above the elastic limit), cannot be found by strain measurements, because the form of the stress-strain diagram is unknown. They also show the uselessness of measurements which have often been made by marking two originally plane sections on a rectangular beam and then measuring their distance apart at various points, when the beam is loaded. It has generally been found by such measurements that the distance between the original plane sections varies in a linear manner from a neutral axis. The considerations here shown indicate that this should be recognized as true without testing. The results of such tests, however, are usually stated as proving that plane sections remain plane, which they do not prove at all, and which a moment's consideration of shear should prove cannot be so if there is any shear. We see in such tests, then, an example of taking pains to test a question which a little thought would answer, without testing; and then drawing a conclusion from the test which a little further thought would at once show to be untrue. Also, many books on this subject state that the usual formulae for flexure are based on the assumption that plane sections remain plane, which is not strictly true; they are based on the assumption that the stress is planar distributed stress, which is a different matter.

These considerations also show that strain measurements in beams, or in any case where the stress is varying over a section, are unsatisfactory, and that the

true stress cannot be found from them, because the form of the stress-strain diagram is unknown. The stress may be above the yield point, without any sudden yielding having been observed.

36. Distribution of Torsional Stress.—How is it with regard to torsion? This kind of stress was treated in Chap. VIII, and it was there shown that in a shaft of circular section (the only one here to be considered) the intensity of shear was constant over any circular ring having the axis of the shaft as a center, a radius r and a thickness dr, and that it was proportional to the radius r. What happens when the elastic limit is exceeded at the circumference and for a certain distance within the circumference? In Fig. 126 of Chap. VIII it was assumed that the stress could not exceed the elastic limit, but remained constant and equal to the elastic limit over the area on which, by the torsion formula, that limit was exceeded, because there would be a yielding over that area, similar to that which takes place at the yield point in a simple tension test, where the strain increases with no increase of stress. It will now be shown that this cannot be the case.

At every point of a transverse section of a shaft, normal to the axis, there is pure shear at right angles to the radius. This requires an equal shear on a longitudinal plane containing the axis and radius, and an equal tension on a plane containing the radius and at 45° to the transverse plane. Below the elastic limit the shear and the tension increase in intensity from the center out along a radial line, in proportion to the distance, and the same is true of the shearing and tensile *strains*.

Consider a point at the circumference, visualize a torque on a transverse section through this point and a tension on a plane containing a diameter, and making an angle of 45° with the transverse plane section. This plane at 45° will cut an elliptical section through the shaft, and the point we are considering is at one end of the minor axis of the ellipse. If the reader can visualize this situation, he will see that at the other end of the minor axis there is an equal compression on this plane at 45°, and that at every point along the major axis there is no normal stress, but only a shear at right angles to the major axis, or parallel to the diameter through the point first considered, and in opposite directions at the two ends of the major axis. Along the diameter through the point first considered, the normal stress, below the elastic limit varies uniformly from tension at one end of the diameter to an equal compression at the other end; and at any point of this diameter the normal stress intensity equals the shear intensity on a right section of the shaft. Now if the conclusion is correct that the normal strain on a plane section in any direction must be planar, it follows that even above the elastic limit the normal strain, and therefore the shearing strain, along a radius of the shaft must be proportional to the radius. This means that, even above the elastic limit, a radius of any transverse section before loading remains a radius after loading, or that *each point of that radius undergoes the same angular rotation*, that is, there is no yield point where the elastic limit is exceeded. Yielding is prevented, as in a beam, or as in a notched tension piece.

Professor Upton, in his excellent work on Materials (Art. 96) gives an ingenious mathematical process for obtaining the *true outside fiber stress* from the computed stress, provided the stress-strain diagram is known; based on the assumption that each point of a transverse section sustains the same angular rotation for

any load up to fracture. This assumption is probably true, as has been shown above, but the trouble is that the stress-strain diagram under these conditions is not known. It certainly is not like the usual diagram for a ductile material, with a large yielding at a yield point, but it has no well-defined yield point, and is probably similar to the line B in Fig. 71. Professor Upton's conclusion is that if, at a point in the stress-strain curve corresponding to any computed stress, a tangent to the curve be drawn and extended to meet the axis of stress at a point whose ordinate is S, the real stress on the outside fiber is that in the diagram minus $\frac{S}{4}$, or less than the computed stress by $\frac{S}{4}$. If the stress-strain curve is a straight line, this construction gives the real stress equal to the computed stress, for in that case the assumptions made in the formula are correct; if the curve is convex to the axis of stress, the real stress is less than the computed stress, depending upon the convexity; but in no case can the real stress be found, because the stress-strain curve is unknown.

37. Limits of Application of Formula for Bending Stress.—The formula for the normal stress in flexure, or for any case in which the stress is not uniformly distributed, is based on the assumption of planar distribution of stress, that is, that the stress intensity varies directly as the distance from the neutral axis. We have also seen that there is good reason for believing that in any case there is planar distribution of normal *strain*, not only below the elastic limit, but above it. Therefore, as long as the elastic limit is not exceeded, the assumption underlying the flexure formula is correct; but it will not be true if the formula indicates an extreme fiber stress above the elastic limit. Each fiber *tends* to deform as it would if independent, and to follow a stress-strain diagram such as would be shown by a test in simple direct stress; but in the beam it is constrained by its connection with adjacent fibers and by the necessity of two parts fitting together, as explained in Arts. 30 to 36. It is not able to deform as it tends to deform, and its actual stress-strain is very different from what it would be if alone. Hence the parts of the section which the flexure formula indicates to be stressed beyond the elastic limit do not really have the stresses indicated. If the material is a ductile one, like structural steel, which normally has a large yielding at the yield point, the outside fibers cannot yield in this way, but will throw the stress, by means of longitudinal shear, on to adjoining fibers nearer the neutral axis, which do not tend to deform so much. This will change the entire distribution of stress.

The only formula which applies to all cases is that for direct uniform normal stress in a prismatic piece loaded axially with forces uniformly distributed over the ends, namely $f = \frac{P}{A}$. If a straight vertical rod is held at the top and loaded at the bottom with a uniformly distributed load P, this formula will always hold, subject only to the qualification that as the load increases the cross-sectional area diminishes slightly, so that A does not remain exactly constant. No matter how large P is, the total stress on any section is P and the stress intensity $\frac{P}{A}$ (subject to this qualification). It makes no difference whether the stress intensity is above the yield point or not, because all the fibers are stressed equally, and therefore (with homogeneous material, which is assumed) they will deform equally.

When necking down begins, there is unequal distribution and the situation changes, as shown in Chap. IV.

In a beam the case is very different. Consider stress parallel to the plane of the paper, and let the formula give stresses represented by the line *ab* (Fig. 198). Then as long as the maximum stresses *ca* and *db* are below elastic limit the stresses given by the formula will be correct. But if the bending moment is increased so that the formula gives maximum stresses *ca'* and *db'* which are above the elastic limit—certainly if they are above the yield point—and if the material is ductile, and *ca''* and *db''* represent the stress at the yield point, the fibers which are indicated as stressed above that limit will *tend* to stretch considerably, which they cannot do, and so they will throw stress on the fibers which are below the yield point. The distribution will then be represented by some such line as *a''b''* dotted. The outer moment must still be resisted by the moment of the total stress between *cd* and the dotted line, and the areas on each side of *cd* must still be equal, in pure flexure, because the total tension must equal the total compression. The lever arm of the forces *F* will be somewhat reduced, and hence the total tensile and compressive forces will be increased, so that the areas between the dotted line and *cd* will be greater than those between *a'b'* and *cd*. The stress on the outer fiber will not exceed the yield point if the applied moment can be resisted by a distribution of stress such as that shown by the dotted line *a''b''*.

There is a similar action and constraint against the tendency to deform when flexure is combined with direct stress, or in any case where normal stress is not uniformly distributed, as in the case of a plate with a hole in it and exposed to tension (see Chap. VI). *The point to be kept in mind is that the action of a so-called fiber in a material exposed to a stress not uniformly distributed is very different from what its action would be if independent.*

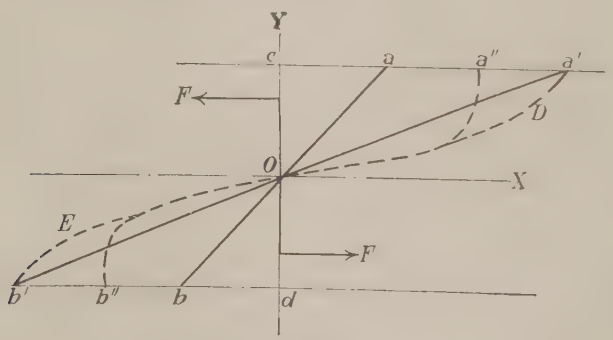


Fig. 198.

If the stress-strain diagram is concave toward the axis of *strain*, as it always is, it is thus easy to see that whenever, in a beam or shaft, the computed maximum stress exceeds the elastic limit, *the actual stress must be less than the computed maximum.* Consider the case of a beam (Fig. 198). Here the stress distribution is represented by a straight line, *ab*, when the maximum, *ca*, is less than the elastic limit. The total compression is represented by the area *Oca*, and the equal tension by *Odb*, and the moment of these two equal forces equals the given outer moment, *M*, on the section. Suppose, now, that the computed maximum, *ca'* or *db'*, exceeds the elastic limit.

The line cd is the axis of strain, and OX the axis of stress, since stress is measured parallel to OX, while the strain is proportional to the distance along cd from the neutral axis. In this case the equal triangles Odb' and Oca' (or the volumes represented by these areas, if the section is not rectangular) have a moment which balances the outer moment M. But if the stress-strain diagram is concave to the axis of strain cd, the maximum stress intensity cannot be equal to ca', or greater than it, such as the curve $a'DOEb'$; for if so, the tension area would be that between Od and the curve OEb', or greater than the triangle Odb' which is all that is necessary to balance M, and the moment of the tension and compression stresses OEb' and ODa' would exceed M, being equal to the moment of the triangles Oca' and ODb' plus the moment of the areas between the curve and $a'b'$. It follows necessarily that the real stresses at c and at d are less than the computed stresses ca' and db'. As a matter of fact, for all engineering materials, the stress-strain diagram, to a point above the elastic limit, is never convex toward the axis of strain; though Morley states that for rolled aluminum the curve is convex to the axis of strain below the elastic limit.

The same argument holds in the case of torsion.

38. Position of Neutral Axis.—If it is assumed that the stress due to a bending moment is distributed over the section in such a manner that the *strain intensity* is proportional to the distance from the neutral axis—which seems correct according to the previous discussion, and which means that plane sections remain plane under normal stress alone—then the *stress intensity* at any point will be represented by the stress-strain diagram if laid off with the strains at right angles to the neutral axis and the stresses at right angles to the section. For, in the stress-strain diagram (Fig. 59), horizontal distances are *proportional* to strain intensity, and either may be laid off to any desired scales; so that, if in flexure the strain intensity is proportional to vertical distances in the plane of the section, stress intensities will be horizontal. Hence, in Fig. 199, if O is the c.g. of a vertical section YY, through which the N.A. passes for pure flexure, and if the stress-strain diagrams for compression and tension be laid off respectively above and below O, then horizontal distances from YY will represent stress intensities to some scale. The question is, Where are the upper and lower limits of the section on this diagram? This will depend upon the magnitude of the bending moment. If the moment is small, the stresses will be correspondingly small. If the moment is large, the stresses may be up to the elastic limit. If the moment is so large that the beam fails by direct tension or compression, the entire stress-strain diagram, up to the ultimate, would be required on the side of the N.A. where failure occurred, always assuming linear distribution of strain.

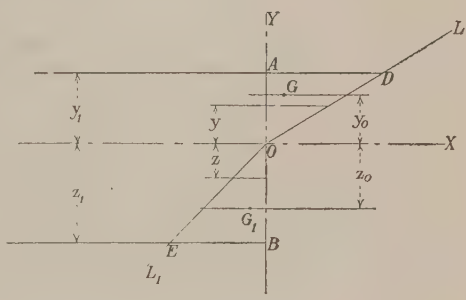


FIG. 199.

(*a*) *Let the elastic limit be not exceeded, either for tension or for compression, and let* E_t differ from E_c.[1] In this case the entire section will be included within the

[1] But see remarks at end of Art. 31. E_t is supposed different from E_c under direct stress, and that this might hold in flexure also.

points L and L_1, which represent the elastic limits for compression and tension respectively. Let A be the top and B the bottom of the section, and AD and BE the max. intensities of compression and tension. The figure is drawn with $E_t < E_c$, since DOE is not straight. The stress intensity in compression at a distance y above the N.A. will be cy, if c is a constant; and the stress intensity in tension at a distance z below the N.A. will be $c_1 z$ if c_1 is another constant. If b is the variable width of the section, perpendicular to the paper, the total compression, C will be represented by the volume OAD, or the area OAD for a rectangular section; or, in general,

$$C = \int_o^{y_1} cyb \cdot dA.$$

Similarly, the total tension will be represented by the volume OEB, or

$$T = \int_o^{z_1} c_1 zb \cdot dA.$$

Since these must be equal (for pure flexure), $C = T$, or

$$c \int_o^{y_1} by \cdot dA = c_1 \int_o^{z_1} bz \cdot dA. \tag{59}$$

If G and G_1 are the centers of gravity of the volumes OAD and OEB, at distances y_o and z_o from O, the condition that the moment M of the outer forces must equal the moment of the stresses requires that the sum of the moments of the volumes OAD and OEB about $O = M$, or

$$M = c \int_o^{y_1} by^2 \cdot dA + c_1 \int_o^{z_1} bz^2 \cdot dA. \tag{60}$$

Equation (59) fixes the position of the N.A. or the point O, and Eq. (60) determines the max. tension and compression.

Considering first Eq. (59), the following results are evident:

1. If $c = c_1$; *i.e.*, if $E_c = E_t$, the N.A. passes through the center of gravity of the section.

2. If $c > c_1$; *i.e.*, if $E_c > E_t$, the N.A. lies above the center of gravity, or toward the compression side.

3. If $c < c_1$; *i.e.*, if $E_c < E_t$, the N.A. lies below the center of gravity, or toward the tension side.

These results are covered by the general statement:

4. *The N.A. lies on that side of the center of gravity where the stress has the larger modulus of elasticity, or on the side which deforms least under a given stress, that is, on the stiffer side.*

Considering now Eq. (60), the following results are evident:

1. If $c = c_1$, or $E_c = E_t$,

$$M = cI = \frac{f_c I}{y_1} = \frac{f_t I}{z_1} \text{ that is, at any point}$$

$$M = \frac{fI}{v}; f = \frac{Mv}{I}, \tag{61}$$

or the usual flexure formula, where v is the distance of any point either above or below the neutral axis, through the center of gravity, and f the stress at that point.

2. If $c > c_1$; $E_c > E_t$,

$$M > c_1 I\left(=\frac{f_t I}{z_1}\right); f_t < \frac{Mz_1}{I} \tag{62}$$

$$M < cI\left(=\frac{f_c I}{y_1}\right); f_c > \frac{My_1}{I}. \tag{63}$$

3. If $c < c_1$; $E_c < E_t$;

$$M < c_1 I\left(=\frac{f_t I}{z_1}\right); f_t > \frac{Mz_1}{I}. \tag{64}$$

$$M > cI\left(=\frac{f_c I}{y_1}\right); f_c < \frac{My_1}{I}. \tag{65}$$

These results are covered by the general statement that

4. *The formula* $f = \frac{Mv}{I}$ *is only true when* $E_c = E_t$; *if that condition is not fulfilled, the fibers on the side of the larger E will have larger stresses than are given by this formula, and the fibers on the side of the smaller E will have smaller stresses than given by the formula.*

(*b*) If the external moment M is so large that it cannot be counterbalanced by an internal moment represented by stresses lying between the points L and L_1 of Fig. 199, there is a *tendency* for the elastic limit to be exceeded, either in tension or in compression (if E_c is not equal to E_t), or in both (if $E_c = E_t$). But the situation is immediately changed, particularly if the max. stress intensity tends to exceed the yield point, and if the actual stress-strain diagram is much curved above the elastic limit.

We do not know the shape of the stress-strain diagram, and there is no way of determining it, because we do not know the actual stress at any given distance from the neutral axis, for that is the very thing we want to find. We cannot measure it. All we can measure is the strain, and even that cannot be measured to near the point of fracture, for the action of the beam would prevent. We do know that the diagram is *not* like the usual one, for steel, shown in Fig. 59 of Chap. IV, but more like curve B in the same figure.

There is no formula possible for such a case, but it is certain that the usual formula will be quite incorrect. By this usual formula, therefore, it is impossible to prove the existence of a stress above the elastic limit, since such a stress is contrary to the assumptions on which the formula is based. Some further conclusions, however, may be arrived at.

We have seen that as soon as the fibers at one side of a beam yield more than those at the other (*i.e.*, E_c and E_t are unequal), the neutral axis will be shifted *toward the stiffer side*, and that the usual formula will give too small stresses on the stiffer side, and too large stresses on the side which yields most. The effect of yielding, from any cause—whether difference in the modulus of elasticity, buckling due to compression, or tendency to exceed the elastic limit—is analogous. It shifts the neutral axis away from the side which yields, and tends to reduce the stress there.

Wrought iron and steel are ductile materials, and E_c is very closely equal to E_t (see Part II), so that these results hold. When an I-beam is tested to failure, if the compression flange begins to bend; the neutral axis is shifted toward the tension flange and the stress in the latter, which can stand it, is increased above

what it would be if the usual formula held, while in the compression flange it is diminished.

In the case of a brittle material which has no elastic limit or yield point, or in any case, if we knew the actual stress-strain diagram in Fig. 199 the actual stress could be found from the principles that the volumes of the diagram above and below the neutral axis (or the areas, for a rectangular beam) must be equal, and the moment of the two equal volumes must equal the applied moment. If the *stress* is not planar, the neutral axis is not at the center of gravity, and the flexure formula is incorrect.

Wood, for example, is very much stronger in tension than in compression, and, when tested to fracture, the upper fibers may fail, or the cell walls collapse to such extent that, even if not noticeable, the fiber can carry but a small load. This throws the neutral axis toward the tension side.[1] The modulus of elasticity of wood is very variable, and while probably somewhat different in tension from that in compression, the facts are not well determined. Probably $E_c < E_t$, and the actual tensile stress at rupture is greater than that given by the usual formula, while the actual compressive stress is less. The compression failure accentuates these results. The actual fracture may be by tension, with a much greater stress intensity than the formula would indicate. The stress-strain diagram for wood is shown in a later volume; but in a beam it would be modified.

In cast iron, $E_c > E_t$ apparently, hence the neutral axis lies toward the compression side, and the compressive stress at rupture is greater, and the tensile stress less, than indicated by the usual formula.

These facts prove the following statements:

(1) The ultimate strength of a material in tension or compression cannot be found correctly from flexural tests and the use of the formula $f = \frac{My}{I}$.

(2) This formula is not correct, even within the elastic limit, unless $E_c = E_t$.

(3) If $E_c = E_t$, the value of E can only be found, by the usual deflection formulae, if the stresses are within the elastic limit.

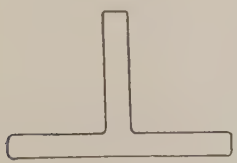


Fig. 200.

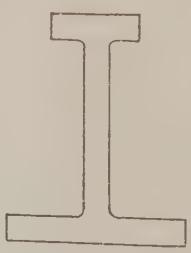


Fig. 200*a*.

39. Unsymmetrical Beams.—In order to take account of different ultimate strengths in tension and compression, beams are sometimes purposely made unsymmetrical about a horizontal axis through the center of gravity. Cast iron, for instance, is much stronger in compression than in tension (see Vol. 2), and as cast-iron beams can be cast into any form, they are often given

[1] Yet even so eminent an authority as Professor Tetmajer thought that the neutral axis moved toward the compression side. See *Proc. Sixth Congress, Internat. Assoc. for Testing Materials*, Second Section, XXIII$_2$, p. 11.

a T-shape, as in Fig. 200, the horizontal flange being in tension. Here the extreme compression fiber is much farther from the center of gravity than the extreme tension fiber, and consequently the max. tensile stress, by the formula, will be less than the max. compressive stress. Inexperienced builders have been known to set these beams wrong side up. Sometimes there is a tension flange smaller than the compression flange (Fig. 200*a*).

40. Beams Composed of Different Materials.—Beams are frequently composed of different materials so connected that they are supposed or intended to act as one. Thus, in reinforced concrete beams, steel rods or shapes are imbedded longitudinally in the solid concrete. Frequently a steel plate is placed between two wooden sticks, and the three firmly bolted together (called a flitched beam).

The fundamental point in such beams is this: that if the materials act together as a solid beam, there is no joint of separation between them on which sliding can take place. Hence, in any cross-section, if there are two small areas, dA_1 and dA_2 at equal distances from the neutral axis and on the same side of it, one in one material and one in the other, the stress on each area must be such that the stretch of each in a small distance, ds, between two adjacent cross-sections, must be the same. If f_1 and f_2 are the intensities of stress on the two areas, the stretch of each will be $\frac{f_1}{E_1}ds$ and $\frac{f_2}{E_2}ds$, hence

$$\frac{f_1}{f_2} = \frac{E_1}{E_2} \tag{66}$$

or the intensities are proportional to the respective moduli of elasticity. The total stress on the first area will be $f_1 \cdot dA_1$, and that on the second area will be

$$f_2 dA_2 = f_1 \frac{E_2}{E_1} \cdot dA_2.$$

The beam may obviously be treated as entirely composed of the first material if instead of the area dA_2 of the second material it is conceived that there exists at the same point the area $\frac{E_2}{E_1}dA_2$. Thus if f_1 refers to concrete and f_2 to steel, and a steel rod with area A_2 be imbedded in concrete (Fig. 201), we must conceive that on the area A_2 there is really concentrated an area of concrete equal to $\frac{E_2}{E_1} \cdot A_2$. Taking E_2 as 30,000,000 and E_1 as 2,000,000, this means that the area of the steel must be multiplied by 15 to reduce it to an equivalent area of concrete, which may be conceived by supposing that for the case of flexure *in a vertical plane* the reinforced beam is the same as a solid concrete beam with wings or projections on the sides, as shown in the figure, the area of both wings plus that of the rod itself being equal to 15 times the area of the rod. This imaginary beam is then to be treated as a pure concrete beam of the section shown. If the flexure is not vertical, this would not be correct; here we must simply suppose that on the actual area of the rod, and in its actual position, there is concentrated—not an area A_2—but an area of $15A_2$ *of concrete*. This is easily done, and by this method

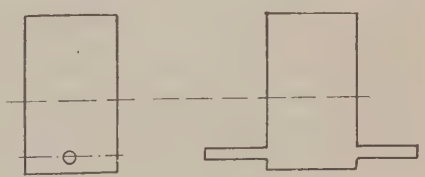


FIG. 201.

any reinforced concrete beam may be treated by the formulae already given for solid beams, no matter where the resultant outer force on the section acts. When, by these formulae, the stress intensity on the rod has been found, the actual total stress on the rod will be this intensity multiplied by $15A_2$, or the real intensity on the rod will be 15 times the intensity as computed on the concrete.

In the case of a flitched beam, similarly (Fig. 202), the steel must be considered replaced by an equivalent wooden beam of 25 times the area.

All such cases of beams of different materials may therefore be solved by methods already explained for homogeneous beams, provided each material is able to take the stress which, according to the formulae used, would be brought upon it. In the case of reinforced concrete, on account of the small tensile strength of concrete and the possibility of shrinkage cracks, it is generally assumed that the concrete can take no tension. Under these circumstances the usual formulae will not apply, and must be modified. The first problem is to find the position of the neutral axis, assuming that the concrete on one side of that axis (the side which according to the usual theory would be in tension) carries no stress. In this

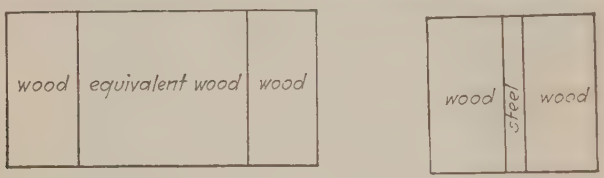


FIG. 202.

case the neutral axis is not where it would be if the concrete could take tension, and the new position must be found. At the tensile side the only areas to be considered as part of the section would be those which can actually take tension, and which are so constructed, or connected with the rest of the beam, that tension will really be brought upon them; and similarly for the compression side. The neutral axis, for pure flexure, will pass through the center of gravity of those areas; and similarly for the case in which there is not pure flexure. But as those areas cannot be known until the neutral axis has been found, the latter must be determined by a process of trial, although in some cases general formulae may be employed.

As an illustration take the case of a reinforced concrete beam 6×12 inches, with two steel rods 1 inch in diameter, as shown in Fig. 203. Then if the neutral axis is 3 inches down from the top, since the only area which can take tension is that of the steel rods, and since the neutral axis must pass through the center of gravity of the area, we have

Area of rods = 1.5708 square inches

Equivalent area = $1.5708 \times 15 = 23.562$ square inches

Moment of compression area about N.A. = $6 \times 3 \times 1\frac{1}{2} = 27$

Moment of tension area about N.A. = $23.562 \times 7 > 27$.

Hence the assumed N.A. is too high.

If it is assumed as 6 inches from the top:

$6 \times 6 \times 3 > 23.562 \times 4$; hence N.A. is too low.

If it is assumed as 5 inches from the top:

$$6 \times 5 \times 2\tfrac{1}{2} < 23.562 \times 5$$

At $5\frac{1}{2}$ inches: $6 \times 5\frac{1}{2} \times 2\frac{3}{4} < 23.562 \times 4\frac{1}{2}$

At 5.75 inches: $6 \times 5.75 \times 2.875 = 99.19$

$$23.562 \times 4.25 = 100.14.$$

Hence the neutral axis is almost exactly at 5.75 inches down from the top.

This may also be done as follows: let x = distance down from the top; then the equation of condition is

$$6x \cdot \frac{x}{2} = 23.562(12 - x - 2); \; 3x^2 = -23.562x - 235.62;$$

from which is found, $x = 5.76$ inches.

The algebraical method, however, is not applicable in some cases, while the method of trial may always be used.

This problem cannot well be treated generally, for cases in which there is a normal resultant force. By Eq. (11) or (12) of Chap. IX if the value of f is negative at any point, it follows that some portion of the section would be in tension

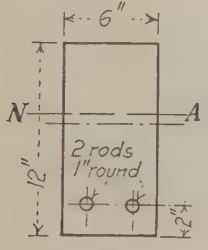


Fig. 203.

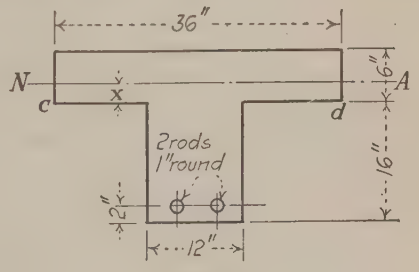


Fig. 204.

if the entire section were in action. Hence some part of the concrete section must be omitted, being assumed unable to take tension. When this is omitted, however, the values of A, I_x and I_y are all changed, as well as the location and inclination of the neutral axis, and it is not possible to arrive at the new position except by trial. The methods of solution indicated above are sufficient for all practical cases.

As another illustration, consider the T-beam in Fig. 204. Here it is not known, at the start, whether the neutral axis is in the slab or in the stem. This can easily be determined by taking moments about cd; thus

$$\text{Moment of upper part} = 36 \times 6 \times 3 = 648$$

$$\text{Moment of rods} = 23.562 \times 14 = 329.9.$$

Hence the neutral axis lies in the slab, or above cd. Its position is determined by the following equation, if x is its distance above cd:

$$23.562(14 + x) = \frac{36(6 - x)^2}{2}$$

from which

$$x = 1.5 \text{ inches.}$$

The moment of inertia may now be found about this neutral axis, and the stress determined for any bending moment.

41. Discontinuous Axis.—The theory of flexure which has been explained has various contradictions and uncertainties, some of which have been referred to, but it is the best that can be arrived at in the present state of our knowledge. Another discrepancy occurs if the axis is discontinuous, or if the beam has a bend in the plane of the axis.

Suppose, for example, the beam to have the shape in Fig. 205. If *aoc* is the axis, what is the section at *o*; *eoe* perpendicular to *ao*, or *dod'* perpendicular to

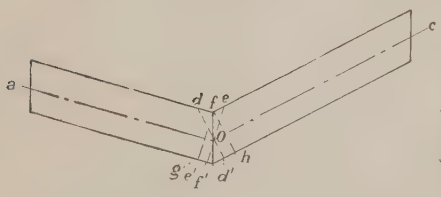


Fig. 205.

Fig. 205a.

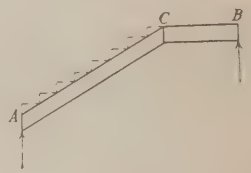


Fig. 206.

oc? If it is taken as *ff'*, then with a positive moment, the theory would give a tension at *f* perpendicular to *ff'* although there are no fibers at *f* in this direction. Probably the best plan would be to compute the stresses on *fh* and on *fg* by the usual methods, and then consider separately the stresses existing in the portion *fgf'h* (Fig. 205*a*). The true condition is unknown, and a larger factor of safety should therefore be used than in usual cases. The best plan, of course, is to avoid such shapes, but this is not always possible. Such forms not infrequently occur in construction. The author has designed built-up beams for buildings having the shape in Fig. 206, for use in a stairway, the part *AC* being the stairway and *CB* being a landing. The supports at *A* and *B* were walls or girders.

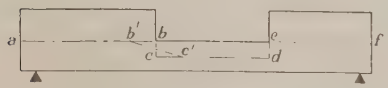


Fig. 207.

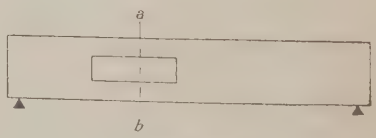


Fig. 208.

Again, in the beam shown in Fig. 207, with an axis *abcdef*, the theory would indicate, for a section just to the left of *bc*, compression above *b* and tension below, for a positive *M*; while for a section just to the right of *bc* it would indicate compression at *c*, which is a contradiction, since the stress cannot change suddenly from one section to another (Art. 31). Probably in this case there is a gradual change of the conditions on each side of *bc*, the axis gradually changing from the line *ab* to the line *cd* as indicated by the line *b'c'*. Nobody knows—at least the author does not—what the real condition is. Here again the only thing to do is to use judgment, and a larger factor of safety (or ignorance) than usual.

A similar situation arises when a beam has holes or openings in it (Fig. 208). Here a section *ab* might be computed by the usual formula; but if *ab* is close to the edge of the opening there is a discrepancy according as it is just on one side or the other of the edge. A redistribution of stress takes place near the edges, in regard to which we are quite ignorant, though with judgment a safe design may be made. The reader must above all realize that this subject is not a mathematical one, in the sense that absolute accuracy is attainable; but that *judgment* must be exercised and trained, as well as theoretical principles understood. (See also Chap. VI for a discussion of the effect of holes in tension pieces.)

42. The reader is urged to give careful consideration to the reasoning in Arts. 30 to 38. Arts. 37 and 38 are given because it is desirable to understand the results which follow if E_c is not equal to E_t, and the effect of a yielding of the material on one side of the neutral axis; and also because many engineers believe that E_c and E_t are actually different in beams because they are different in direct tests of tension and compression. Arts. 31 to 35, however, if the reasoning is correct, indicate that there must always be planar distribution of *strain*, from which it follows that if the moduli of elasticity are different there is planar distribution of stress on each side of the neutral axis, but not on the entire section, and therefore that the neutral axis does not pass through the center of gravity of the section even in pure flexure.

CHAPTER XI

SHEARS AND MOMENTS ON BEAMS: INFLUENCE LINES

1. It has been shown that in order to find the inner forces in a solid piece, the outer forces being given, the *universal method* is to take a section through the body or structure, and then to find the inner forces which, when applied to the section cut, will balance the outer forces acting on that portion of the body to one side of the section. The resultant outer force on this portion of the body, and also the point where this resultant cuts the section may be found by the principles of composition of forces; and the principles explained in Chap. IX will then render easy the determination of the inner forces. In doing this, it is necessary to make use of two very important functions of the outer forces, the *shear* and the *moment*. These have already been discussed to some extent, and they will now be considered in detail. No student can have an adequate knowledge of the theory of structures (either of beams or of framed structures) unless he is thoroughly familiar with shears and moments.

A plane section being taken, the shear on that section is obtained by finding the resultant of all the outer forces on one side of the section, and resolving it into two components, one normal to the plane of the section, and the other in that plane. The latter is the shear on the section. If the outer forces all act in the plane of the paper, the normal force and the shear will act in that plane. The section is generally taken perpendicular to that plane and is represented by a line in the figure. The *shear*, then, in this case, *is the algebraic sum of the components, parallel to the section, of all the outer forces on either side of the section*, and is considered positive when it is upward on the part to the left of the section and downward on the part to the right. The resultant on one side is equal and opposite to that on the other, and either side may be considered.

The moment at a section, or at a point of a beam, *is the moment, taken about the center of gravity of the section, of all the outer forces on either side of the section*, and is considered positive when it is right-handed on the part to the left of the section and left-handed on the part to the right. The resultant moment on one side is equal and opposite to that on the other, and either side may be considered. A positive moment causes compression in the upper fibers, and tends to make the beam concave upward. A moment tends to bend the beam, and a shear tends to slide the two parts of the beam by each other along the section.

In what follows, the outer forces and the horizontal axis of the beam will be supposed to be in the plane of the paper. The shear and moment at a point of the beam will be the shear and moment on a vertical section through a point of the axis. The student, in finding shears and moments, must always choose the part of the beam which gives the result in the simplest way; generally the side which has the smaller number of forces.

The shear at any point being plotted from the axis of the beam, the ends of these ordinates will be on the *shear line*. At the left end of the beam the shear

will equal the left reaction. For a series of concentrated loads the shear line will consist of a stepped line, horizontal between loads and dropping suddenly at a load by the amount of that load. In other words, *the shear always changes suddenly at a concentrated load.* If the load is distributed, the shear line will be a curve, its shape dependent upon the distribution of the load; if the load is uniform per foot, the drop in the shear line will be the same in each foot, or the shear line will be a straight line whose tangent with the axis of the beam will equal the transverse load per foot along the beam at right angles to its length. For a load which is distributed, but not uniformly, it is a curve.

If the moment at each point of a beam is plotted at right angles to the axis, the moment line is obtained. It is clear that for a series of concentrated loads the moment line is a series of straight lines, the angle being at the loads, *i.e.*, it is a broken line, changing its direction at each load. There is never an angle in the moment line except at a concentrated load, and there always is an angle at such a load. If the load is distributed, the moment line is a curve, its shape dependent upon the distribution.

2. The Moment Cannot Change Suddenly, nor Can a Load be Concentrated at a Point.—If the moment line is plotted, it must be a curve or a broken line, but it cannot change suddenly from any point to the next. At a concentrated load it changes its direction, which simply means that its *rate of increase or decrease changes suddenly* because the shear changes suddenly. But if the moment itself should change suddenly, this would mean, whatever the law of distribution of stress on the cross-section, that the normal stress on a fiber would change suddenly, having one value at a certain cross-section, and a different value (differing by more than an infinitesimal—an infinitesimal difference being of course the usual difference) at the section immediately adjoining, or distant dx from the first; and this has been shown to be impossible, in the chapter on flexure. Any change in moment, however great, must take place in a finite distance, even though small. Suppose a beam resting on two supports, A and B (Fig. 209) to have attached to it at c a cross-arm, to which two forces P are applied. The cross-arm may be riveted or welded to the main bar so that it forms an integral part of it. It would seem that there would be a moment $M = Pb$ applied suddenly to the beam at c, and that the moment line would be as shown in the second figure. But this would be impossible, because it would mean a sudden change in the moment at c. But as a matter of fact, the connection at c or the cross-arm if a part of the beam, must have finite dimensions, and it is within those dimensions that the change in moment must take place; and the line of moments, instead of Acc_1B, must be a line like $Ac'c_1'B$, c' and c_1' being some finite distance apart horizontally, within which the change of moment must take place. Nevertheless, for a theoretical investigation, and subject to the qualification explained, the moment may be conceived to be applied at a point.

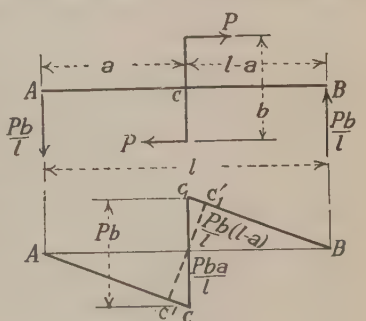


FIG. 209.

This is really not different from the case of a concentrated load, for in fact *no load can be concentrated at a point*, for this would mean an infinite pressure per unit on the material. It must always be distributed over some distance, or rather area, however small, yet we properly consider loads to be concentrated.

3. The Principle of Summation of Effects.—*If several loads act simultaneously upon a structure, every effect produced is the algebraical sum of the effects of the loads taken separately.* (This principle is not true in every case, as will be explained below.)

Thus if it is desired to find the shear, moment, fiber stress, deflection, or any other effect of a system of loads, the effect of each single load may be found, and the results added. The same may be done for a distributed load, which is merely an infinite number of infinitesimal loads. The theory of structures thus reduces, in its essence, to finding the effect of a single load.

3*a*. Limitations on the Principle of Summation of Effects.—The principle is not always true. It is clearly true only *if the effect which is under consideration* (*shear, moment, etc.*) *produced by one load, does not influence the effect produced by any other load.* This is often the case, though not always. One of the loads, for instance, produces shear, moment, and deflection; another load is now put on, and it finds an already stressed beam; but if its own effect is independent of that condition, and its effect is the same as if it were put on the unstressed beam, the principle is true in that case; and it is, in fact, only a truism, namely, that if a load produces the same effect whether or not other loads are on the structure, then its effect is to be added to that of the other loads.

The only limitation to the principle arises from the fact that each load produces deformation. Now in the most exact theory the effect of a load on a distorted structure is not exactly the same as its effect on the undistorted structure. It may therefore be said correctly that the theorem is *never exactly true*. But in most cases the deformation produced by a load, or by all the loads, is entirely neglected and is not considered to have any influence on the effect produced by other loads. For all such cases the principle may be said to hold true.

There are some obvious cases, however, in which it does not hold, namely in compression accompanied by flexure, that is, the case of a long column which bends; or in the case of compression or tension accompanied by transverse flexure. Thus, if a straight piece supported at each end is exposed to a transverse load, it will be bent. If the original piece is exposed to a pure tensile force it will have a uniform tensile stress. But if the piece is first exposed to the transverse load, and then the tensile force is applied, the latter will not produce a uniform tensile stress, because it will have a lever arm equal to the deflection, and it will produce bending also. The same is true in the case of compressive forces which bend a column. If one of them bends the column, the next produces different stresses from what it would if the first load were not acting.

It may be said then, that *the principle does not apply to cases in which the effect considered depends upon deformation.*

4. Influence Lines.—In studying the effect of a single load upon any function, such as shear, moment, stress, deflection, etc., it is convenient to consider *a load whose magnitude is unity* to pass across the span, and at each point of the span to erect an ordinate, perpendicular to the axis of the structure, whose magnitude represents the effect of the unit load upon the function considered. The locus

of the extremities of these ordinates is what is called an *influence line*, since each ordinate of this line shows the influence, upon the function considered, of a unit load at the point where the ordinate is drawn. Thus the influence line for the left reaction of a beam supported at its ends will be as shown in Fig. 210.

Influence lines are much used, and afford a most convenient method of studying many problems. It will be at once evident that an influence line is a means of obtaining the following results:

(1) It shows the effect of a single load of any magnitude at any given point; for, if P is the load, its effect will be Py, if y is the ordinate to the influence line at a given point.

(2) It shows the point at which a single load must lie to produce the maximum or minimum effect.

(3) It shows where a load will produce no effect at all, where it will produce a positive and where a negative effect; therefore,

(4) It shows what portions of the span must be loaded by a distributed load in order to produce maximum positive or negative effects.

(5) It enables us to determine the effect of a system of concentrated loads: for such effect will clearly be ΣPy (taken algebraically).

(6) It thus enables us to determine, by trial, the position of a system of concentrated loads which will give maximum positive and negative effects.

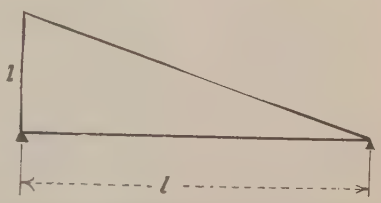


Fig. 210. Influence line for left reaction.

(7) It enables us to find at once the effect of a uniformly distributed load. For, if w is the distributed load per unit of length, then on any length dx the load will be wdx, and if y is the ordinate to the influence line at this point the effect of the load wdx will be $wy \cdot dx$; hence the effect of the entire uniformly distributed load will be $w\int ydx = wA$. Therefore, *the effect of a uniformly distributed load covering any portion of this span, is simply the uniform load per foot of length multiplied by the area enclosed by the influence line and the axis between the ordinates at the ends of the loaded portion.*

Thus, referring to Fig. 210, the left reaction due to a load w per foot over the entire span is $\frac{wl}{2}$, and for a load covering the right half of the span it is $\frac{wl}{8}$.

These principles regarding influence lines should be thoroughly mastered by the student before proceding farther.

The various kinds of beams will now be considered, and for each one the laws governing shear and moment will be studied, together with the load, shear, and moment curves, influence lines, and the formulae for slope and deflection. The beam will be considered straight and horizontal, the loads and reactions all vertical, the outer forces statically determined. In following pages, "influence line" will frequently be denoted by the letters I.L.

5. Stringers and Floor Beams.—The load which comes upon a beam may rest directly upon it, as when a wall is supported by a beam directly beneath; but may be brought to the beam through other beams, and so may be applied to the beam as concentrated loads, even though it is in itself a distributed load. Thus

in Fig. 211 the main girders of a highway bridge spanning an opening are represented in plan by AB, $A'B'$ and in side view by $A''B''$. This bridge carries a floor upon which may be a crowd of people constituting a uniformly distributed load; yet, apart from its own weight, the girder AB is subjected only to three concentrated loads. The weight on the floor is carried directly by so-called *stringers* S, to the so-called *floor beams* F, which in turn carry it to the main girders at a, b, c, a', b', c'. A *stringer* is a longitudinal beam to which the load is directly applied, and which is supported by (carries the load to) the floor beams. A *floor beam* is a transverse beam, loaded only by the loads brought to it by the stringers (aside from its own weight), and supported by (carrying the load to) the main girders or trusses, which are supported by the abutments.

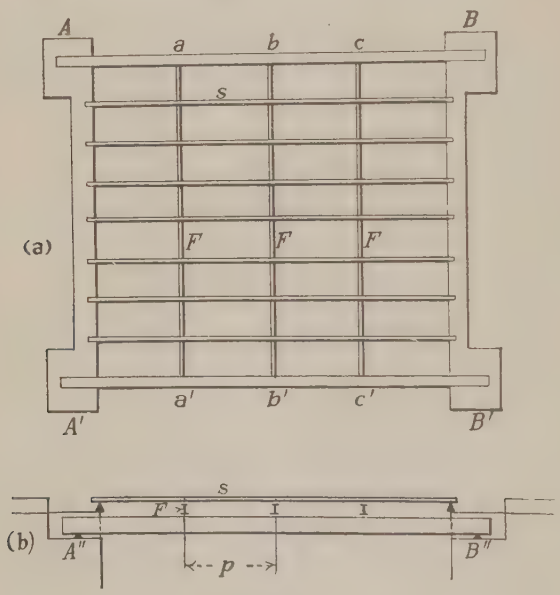


FIG. 211.

Stringers and floor beams are therefore supported at each end, either without moment there, or more or less fixed if riveted to connecting stringers or to main girders. They are usually computed as simply supported.

Strictly speaking, stringers, if continuous over several floor beams, or if rigidly fastened to them, should be considered as continuous girders or as beams fixed at the ends, but owing to many uncertainties (such as varying elevations of the points of support, owing to the deflection of the floor beams) they are generally considered to be simply supported, *i.e.*, with no moments at the ends. The same is true of floor beams, which instead of simply resting upon the main girders, as indicated in the figure, are generally rigidly connected to them.

The space between two adjacent floor beams ($bb'c'c$) is called a *panel*, so that the span of a stringer is the length of a panel. The span of a floor beam is the distance between main girders. One end of end stringers rests on abutment.

6. Effect of Floor Beams on Shear and Moment.—If stringers and floor beams are used, the only load directly applied to the main girder is its own dead weight; all other loads which act upon it are applied at the points where the floor beams are supported by it. In other words, the outer forces consist of a distributed load equal to its dead weight, and a series of concentrated loads at the floor-beam connections.

Similarly, the load acting on a floor beam consists of its own dead weight, and a series of concentrated loads at the points where the stringers are connected to it.

The load on a stringer is a distributed or concentrated load according to circumstances, since the stringers support the floor directly.

Effect of Floor Beams upon the Shear on Main Girder.—Leaving out of consideration the dead weight of the girder, and considering only the other loads, which come upon it as concentrated loads through the floor-beam connections, it is clear that for any given loading on the structure *the shear is constant between*

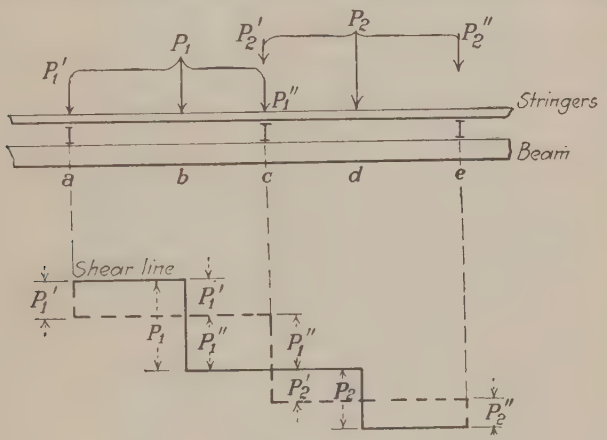


FIG. 212.

adjoining floor beams, or constant in any one panel. Suppose, in Fig. 212 that there are two loads, P_1 and P_2 applied to the main girder through the stringers and floor beams. If there were no floor beams, the shear line for the main girder would be a stepped line dropping at b by an amount P_1 and at d by an amount P_2. But P_1 is resolved into two components P_1' and P_1'' acting at a and c respectively, while P_2 is resolved into two components P_2' and P_2'' acting at c and e respectively. The shear line with floor beams, therefore, will drop at a by an amount P_1', at c by an amount $P_1'' + P_2'$, and at e by an amount P_2'', as shown by the dotted line.

With a distributed load, the shear line without floor beams is a curve or straight line falling toward the right; with floor beams it is transformed into a stepped line. It is evident that the effect of floor beams is to make the shear greater at some points, and less at others, than it would be if there were no floor beams.

Effect of Floor Beams upon the Moment on Main Girder.—Since there is a concentrated load on the girder at each floor-beam connection, and since we

neglect in this consideration the dead load of the girder itself, it follows that *the moment line with floor beams is a series of straight lines between floor beams;* or, in other words, *the moment varies uniformly between floor beams.* Clearly, *the tangent of the angle which the moment line in any panel makes with the axis equals the shear in that panel.*

Moreover, *the moment at a floor beam is the same as if there were no floor beams; while the moment at a point between two floor beams is the same as it would be if there were no floor beams only in the case that there are no loads in the panel containing the point.*

The last proposition is evident when it is remembered that at any floor beam, as c (Fig. 212) the moment is the algebraical sum of the moments about c of all outer forces to the left of c, including, therefore (if there were no floor beams), the moment of all forces to the left of a and also the moment of P_1; and including, if there are floor beams, the moment of all forces to the left of a and also the moment of P_1' (but not of P_1); but the moment of P_1' equals the moment of P_1. At any point between two floor beams, as b', however, the moment about b' without floor beams would not include the moment of P_1', while with floor beams it would include that of P_1'. If the point b' were between b and c, the moment without floor beams would include the moment of P_1, and with floor beams it would include that of P_1', which is not the same about b' as the moment of P_1.

It follows from the above that the moment line may always be drawn as if there were no floor beams. If there are floor beams, the line will be made up of a broken line obtained by connecting the points above the floor beams in the curve without floor beams. It also follows that the moment at a point between floor beams may be greater or less than it would be at the same point if there were no floor beams.

CASE I. BEAM FIXED AT ONE END, WITHOUT FLOOR BEAMS

7. Influence Lines.—The influence lines for shear and moment at any point a are shown in Fig. 213.

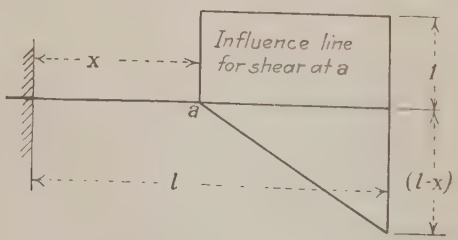


FIG. 213.—Influence lines for shear and moment at a.

8. Shear and Moment.—The lines of shear and moment *for a concentrated load* P at the end are shown in Fig. 214, for a load not at the end in Fig. 215, and for a uniformly distributed load of w pounds per foot in Fig. 216.

From these figures the following principles are evident: For downward loads the shear is always positive, the moment always negative (tension in upper fibers).

(*a*) *For a concentrated load* the shear is zero to the right of the load, and constant (equal to the load) between the load and the support. The moment is zero

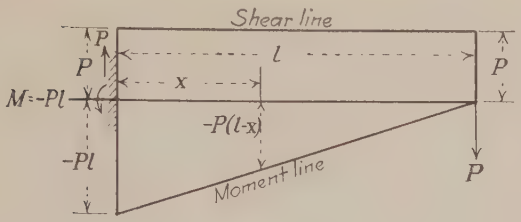


FIG. 214.

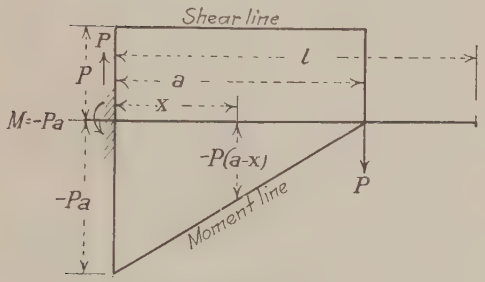


FIG. 215.

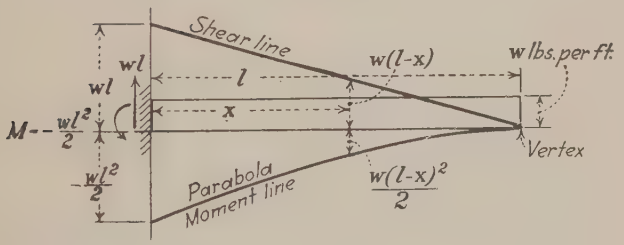


FIG. 216.

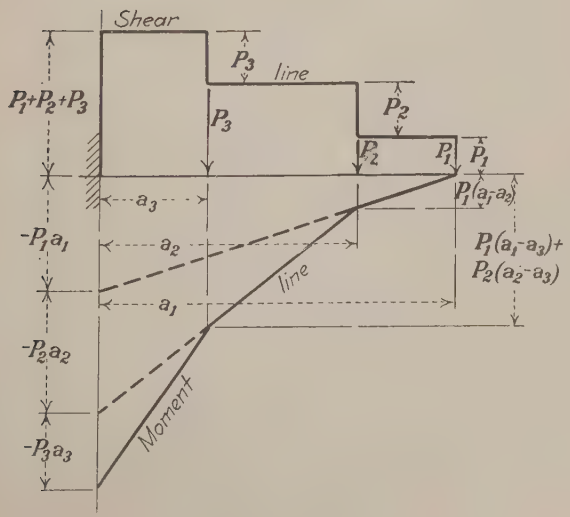


FIG. 217.

to the right of the load, and varies uniformly from zero at the load to $-Pa$ at the support.

At any given point, the shear is greatest when the load is anywhere to the right of the point, and zero when it is to the left; the moment is greatest when the load is at the free end, and zero when it is at the point or to the left.

(*b*) *For a series of concentrated loads*, the effect of each load is to be found and the sum taken. The shear and moment lines are shown in Fig. 217.

The shear is everywhere positive, and at any point equals the sum of the loads to the right; hence with a system of movable loads the shear at any point will be greatest when the loads are so placed that that sum is greatest, otherwise independent of the position of the loads. This max. S will always occur when a load is at the free end of the beam.

The moment is everywhere negative and at any point is equal to the sum of the moments of the loads to the right; and therefore greatest when the heaviest loads are as near the free end as possible. Max. M will always occur with a load at the end of the beam, for a system of movable loads.

(*c*) *For a Uniformly Distributed Load of w Pounds per Foot.*—The shear is greatest at a given point and equals $w(l - x)$ when the load extends from the point to the free end. It makes no difference whether it extends to the left of the point, since loads to the left cause no shear. The moment is greatest for the same loading and equals $\frac{w(l-x)^2}{2}$.

(*d*) *With a Couple at the Free End Having Moment M.*—In this case there is no shear on the beam, and there is a uniform bending moment M throughout.

(*e*) *Effect of a series of concentrated loads. Determination of position of such loads to produce max. shear and moment at any point.*

1. *Shear.*—Since the shear at any point is always positive and equal to the sum of the loads to the right of the point, it is clear that the max. shear will occur when such sum is greatest. All that is necessary, then, is to shift the fixed system of loads until this sum is a max. If the loads are plotted to scale on a thin strip of cardboard, or tracing cloth, this strip can be shifted above the plotted span and the desired position easily found.

2. *Moment.*—In studying this case, it is necessary to adopt a *general method* applicable to all such problems, namely, to start with the system of loads in any assumed position and to investigate the effect of moving this system, as a whole, in one direction or the other. Since the moment at any point is always negative, and greater the greater the loads and the farther they are to the right of the point, let the loads, starting with any assumed position, be moved to the right. Clearly the moment at the given point will increase until the right-hand load arrives at the end of the beam. If, in moving, a new load passes the point from left to right, this simply causes an increase in the total load to the right, and so an increase of moment. Clearly, *the max. M will occur when some load is at the end, and when the heaviest loads are to the right and as near as possible to the end.* This is the only result at which it is possible to arrive. The rest must be left to the common sense and vision of the student. No formula is possible which will replace these qualities.

CASE II. BEAM FIXED AT ONE END, WITH FLOOR BEAMS

9. Influence Lines.—These are shown in Fig. 218.

10. Shear and Moment.—(1) For a concentrated load at the end, or at any floor beam, the shear and moment lines would be the same as if there were no floor beams. For a load on the stringers between two floor beams, the lines are shown in Fig. 219.

To the right of the load, in the panel containing it, the shear is greater, and to the left of the load it is less, than without floor beams.

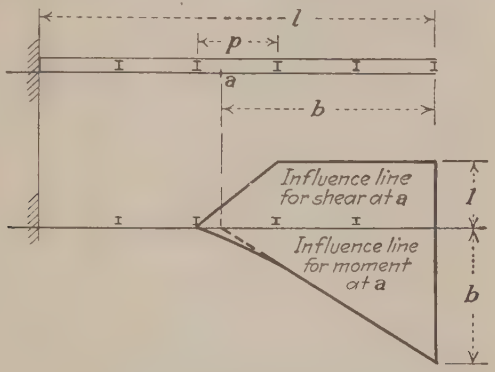


FIG. 218.

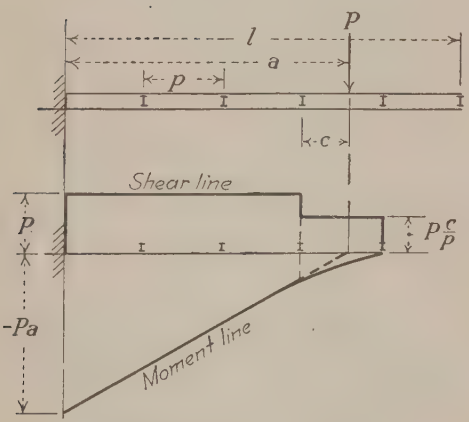


FIG. 219.

At all points in the panel containing the load the moment is numerically greater than without floor beams.

A load, therefore, which produces no shear or moment at a given point if there are no floor beams, may produce both shear and moment if there are floor beams. All loads outside the panel containing the point produce the same shear and moment as if there were no floor beams.

(2) For a system of concentrated loads, there may therefore, be a shear and moment due to loads to the left of the point.

(3) For a uniform load the curves of shear and moment are shown in Fig. 220. Here the shear line drops wp at each floor beam except that at the end. The moment is everywhere greater than if there were no floor beams. The dotted line shows the shear line without floor beams. Max. S will occur, for any point as a, for full loading or when the uniform load extends from C to the free end, and equals $w(b + \frac{1}{2}p)$. Max. M occurs for the same loading.

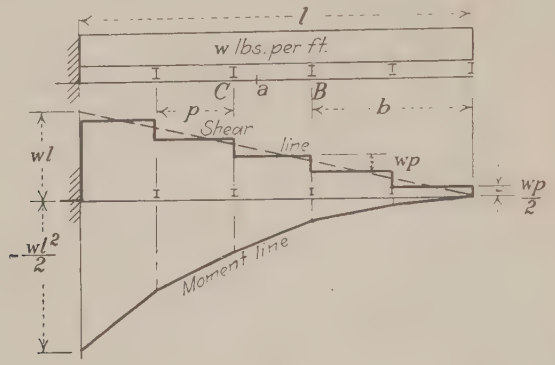


Fig. 220.

CASE III. BEAM SIMPLY SUPPORTED AT EACH END, WITHOUT FLOOR BEAMS

11. Influence Lines.—These are shown in Figs. 221 and 222.

12. Shear and Moment. (1) *For a Single Concentrated Load Anywhere.*—The reactions are found by taking moments about the abutments. The shear to the

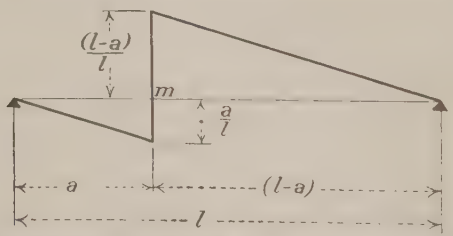


Fig. 221.—Influence line for shear at m. The inclined lines are parallel.

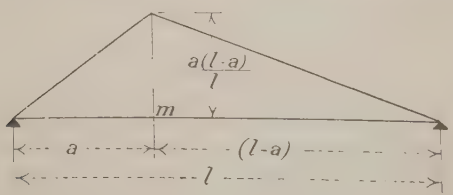


Fig. 222.—Influence line for moment at m.

left of the load is $+R_l$, and to the right it is $-R_r$. *A load anywhere produces a positive shear to the left of its point of application, and a negative shear to the right;* or, in other words, *at any given section the shear is positive if the load is to the right,*

and negative if it is to the left, and numerically greater the nearer the load is to the section. At a section distant a from the left abutment,

$$\text{max. } +S = \frac{P(l-a)}{l} \text{ when the load is just to the right of the section}$$

$$\text{max. } -S = \frac{Pa}{l} \text{ when the load is just to the left of the section.}$$

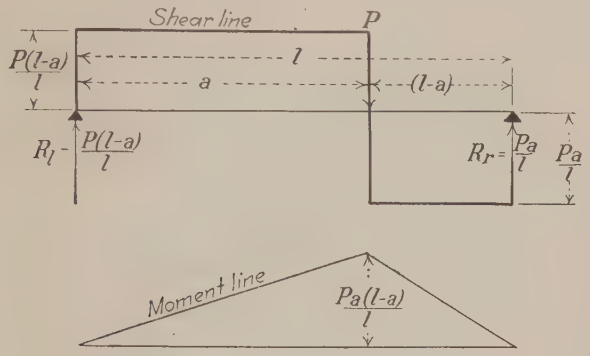


FIG. 223.

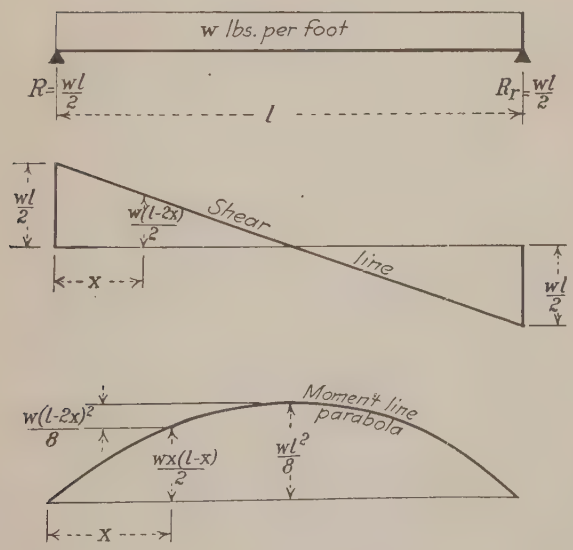


FIG. 224.

The moment at any section is always positive if the load acts downward, and is greater the nearer the load is to the section. At a section distant a from left abutment,

$$\text{max. } M = \frac{Pa(l-a)}{l} \text{ when the load is at the section.}$$

(2) *For a Uniform Load w over the Entire Span* (Fig. 224).—The moment curve is a parabola with vertex at the center. The moment at a section distance x from the left support is

$$M = \frac{wx(l - x)}{2} \tag{1}$$

at the center it is

$$M_c = \frac{wl^2}{8}. \tag{2}$$

At a distance b from the center the moment is

$$M = M_c - \left(M_c \frac{4b^2}{l^2}\right). \tag{3}$$

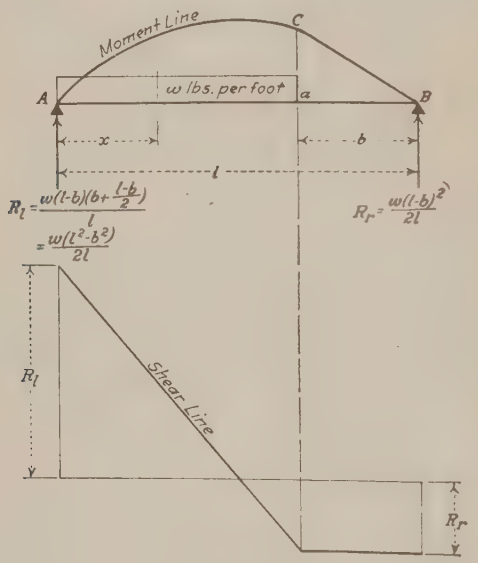


FIG. 225.

The shear at a section x from left support is

$$S = \frac{w(l - 2x)}{2}. \tag{4}$$

At the center it is zero, and at the end it is $\frac{wl}{2}$. The shear line is a straight inclined line dropping from the left to the right at the rate of w per foot.

(3) *If the uniform load covers any part of the span* (Fig. 225), like a train on a railway bridge or a crowd of people on a highway bridge, the following principles are evident:

(*a*) For max. $+S$ at any point x from the left support the load must extend from that point to the right support, and

$$\text{max. } +S = \frac{w(l - x)^2}{2l}. \tag{5}$$

(*b*) For max. $-S$ the load must extend from the point to the left support, and

$$\text{max. } -S = \frac{wx^2}{2l}. \tag{6}$$

The sum of these two is the shear for full loading or $\frac{w(l-2x)}{2}$.

(*c*) The moment at any point x from left support is a max. when the load covers the entire span, and equals $\frac{wx(l-x)}{2}$, as above.

(*d*) Let the uniform load cover a portion of the beam as indicated in Fig. 225. The moment line will be straight from c to the right abutment and a parabola from c to the left abutment. The ordinate ac will be $\frac{wb(l-b)^2}{2l}$.

The point of max. M will be at a distance from the left abutment $x_o = \frac{R_l}{w}$, since the shear will be zero there, and the max. M will be $R_l x_o - \frac{wx_o^2}{2} = \frac{wx_o^2}{2}$. At any other point between A and a, M will be $R_l x - \frac{wx^2}{2}$.

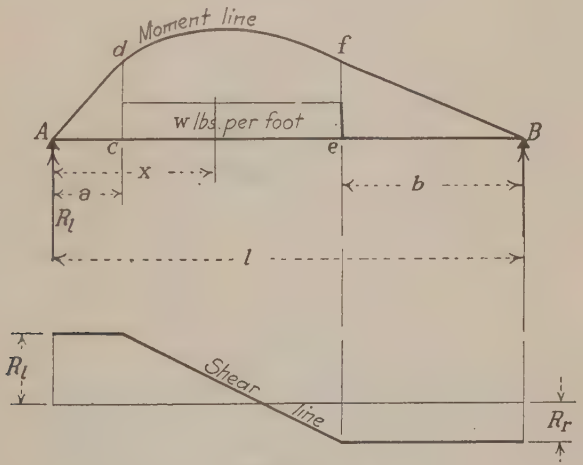


FIG. 226.

(*e*) Let the uniform load be as in Fig. 226.

$$R_l = \frac{w(l-a-b)[b+\tfrac{1}{2}(l-a-b)]}{l}$$

$$= \frac{w(l-a-b)(l-a+b)}{2l}$$

$$R_r = \frac{w(l-a-b)(l-b+a)}{2l}.$$

The moment line consists of two straight lines from the points A and B and a parabola tangent to both. The ordinate $cd = R_l a$; and $ef = R_r b$. The max. M will be at a distance to the right of c equal to $\frac{R_l}{w}$. Between c and e, the moment equals $R_l x - \frac{w(x-a)^2}{2}$, and the shear equals $R_l - w(x-a)$.

(4) *Combined Live and Dead Uniform Loads.*—All structures are exposed to the action of their own weight, known as the "dead" load, and of the applied or "live" loads. The dead load of a beam, if of uniform cross-section, is a uniform load covering the entire length. For practical purposes the dead load of a beam or truss is generally assumed uniformly distributed, even though this may not be exactly true. The live load is often uniformly distributed, as a crowd of people upon a floor or a highway bridge; and even in the case of a railroad bridge, carrying a train of cars, the live load is frequently assumed to be uniformly distributed. But this uniform live load, unlike the dead load, need not cover the entire span, but may be placed wherever it will produce the greatest stress in any part under consideration. Hence it is important to investigate the effect of a combination of live and dead uniform loads. Let us designate the dead and live loads per foot respectively as w_d and w_l. Then in Fig. 227 aa' will be the shear

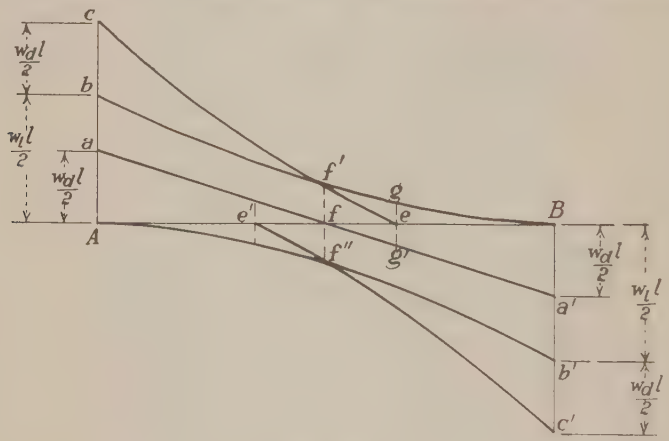


FIG. 227.

line for the dead load; bB will be the line whose ordinates give max. $+S$ for the live load, and Ab' will give max. $-S$ for the live load[1] (Eqs. (5) and (6)).

To find the max. $+S$ due to a combination of live and dead loads the lines aa' and bB must be combined, giving the line $cf'e$. The ordinate from AB to this line represents the max. possible positive shear due to live and dead loads. At the center f the dead load causes no shear, hence the max. possible $+S$ is that due to the live load, when covering the distance fB. At the point e the dead load causes a fixed negative shear eg', while the live load, if covering the distance eB will cause its greatest possible positive shear eg, and $eg = eg'$. Similarly $e'c'$ is the line of max. possible negative shear. The lines $cf'e$ and $c'f''e'$ are parabolas with vertical axes.

The equation of $cf'e$ is (from Eqs. (4) and (5)):

$$y = \text{max. } +S = \frac{w_d(l - 2x)}{2} + \frac{w_l(l - x)^2}{2l} \tag{7}$$

and that of $e'f''c'$ is

$$y' = \text{max. } -S = \frac{w_d(l - 2x)}{2} - \frac{w_l(x^2)}{2l}. \tag{8}$$

[1] These lines will be parabolas with vertical axis and vertex at B and A respectively.

To find the point e', make $y' = 0$.

$$\therefore x = Ae' = \frac{w_d l}{w_l}\left[\sqrt{1 + \frac{w_l}{w_d}} - 1\right]. \tag{9}$$

It is clear that there are three parts of the span, differing in regard to the possible sign of the shear, namely Ae', $e'e$, and eB. *In Ae' the shear is always positive;* no possible loading with the live load can produce a negative shear great enough to counteract the fixed positive shear due to the dead load. Similarly, *in eB the shear is always negative. In ee', however, the shear may be positive or negative, depending upon the position of the live load.* In $e'f$ the max. $+S$ is greater than the max $-S$, while in fe the reverse is true. The distance ee' is greater the greater the ratio of live to dead load per foot; if there were no dead load there could be a $+$ and a $-S$ at each point of the span, while if there were no live load the points e and e' would coincide with f.

(5) *A Series of Concentrated Loads.*—The shears and moments are found by adding the effects of the separate loads. Thus, if a is the distance of any load from the right support, the shear and moment at a point distant x from the left support are

$$S = \sum_x^l P\frac{a}{l} - \sum_o^x P\frac{l-a}{l} \tag{10}$$

$$M = \sum_x^l \frac{Pax}{l} + \sum_o^x \frac{P(l-a)(l-x)}{l}. \tag{11}$$

These expressions are in somewhat different form from those which the student might be inclined to write, and he should compare the different modes of

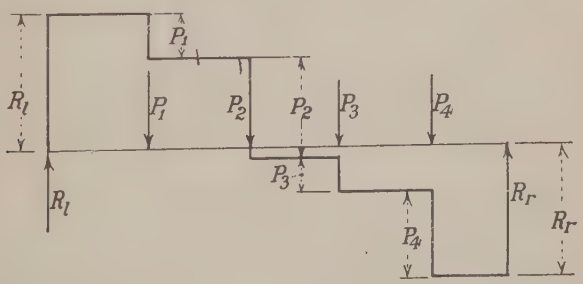


FIG. 228.—Shear line for a series of concentrated loads.

expression. The shear line (Fig. 228) is a stepped line starting at A with a value equal to R_l and dropping at each load by the amount of that load. With loads acting downward the shear is always algebraically greater just to the left of a load than just to the right. The moment line (Fig. 229) is a broken line. From A to P_1 the moment increases as the distance, being simply the moment of $R_l l$. If the line is continued to meet the vertical through B at 1, $B1$ will represent $R_l l$. From P_1 to P_2 the moment of R_l will continue to increase in the same way with the distance, and at any point a will be represented by ab; but from this must be subtracted the moment of P_1, which will be proportional to the distance from P_1.

The distance 12 will represent the moment of P_1 about B, and similarly for the other loads. The fact that $B1$ equals the sum of the intercepts 12 etc. means that the moment of R_l about B equals the sum of the moments of the loads about the same point.

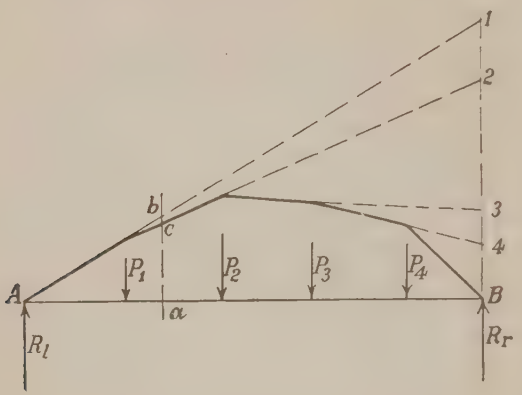


FIG. 229.—Moment line for a series of concentrated loads.

The moment will clearly be a max. at some load, or it may be at two consecutive loads, if the moment line between those loads is parallel to AB. The max. will occur at the load where the shear passes through zero, or P_2 in the figure.

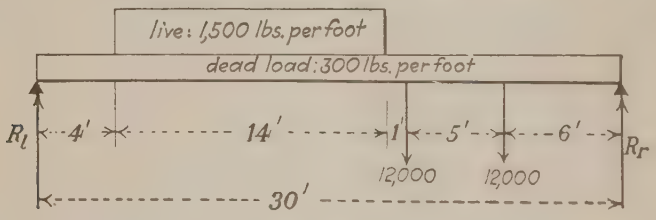


FIG. 230.

Example.—As an example of these principles, assume a beam loaded as in Fig. 230.

To find right reaction:

$$300 \times 15 = 4{,}500$$

$$24{,}000\,\frac{21.5}{30} = 17{,}200$$

$$\frac{14 \times 1{,}500 \times 11}{30} = 7{,}700$$

$$R_r = 29{,}400 \text{ pounds}$$

Shear just to left of left-hand load $= 29{,}400 - 24{,}000 - 11 \times 300$
$= 2{,}100$ pounds negative shear.

Shear at right end of live load $= 2{,}100 - 300 = 1{,}800$ pounds negative.

$\therefore$ Max. M is $\frac{1{,}800}{1{,}800}$ feet $= 1$ foot inside live load, or 13 feet from right abutment.

To find max. M:

$29{,}400 \times 13$	=	382,200
$24{,}000 \times 4.5$	= 108,000	
$300 \times 13 \times 6.5$	= 25,350	
$1{,}500 \times 1 \times 0.5$	= 750	134,100
Max. M	=	248,100 foot-pounds.

13. Moment Diagram.—In the case of a railroad bridge in which the loading consists of a series of a considerable number of concentrated loads, being the loads on the axles of engine and tender, followed perhaps by a train load assumed as uniformly distributed, the computation of moments and shears for different points on a span, and on various spans, is much facilitated by the use of a so-called moment diagram, which is merely a device for finding the moment of a

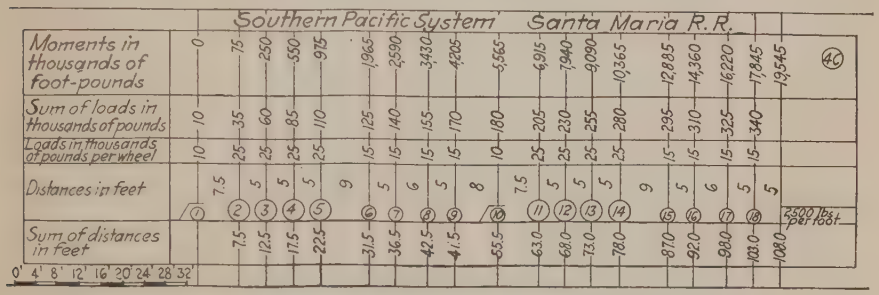


FIG. 231.—Moment diagram.

series of loads, or of any portion of it, about any point. Figure 231 shows such a diagram, for a load consisting of two locomotives with tenders, followed by a uniform load. This diagram, for convenient use, should be plotted to scale, so that it can be slipped along over the span under consideration, which is drawn to the same scale.

The circles indicate the loads; above them are the distances between adjacent loads; just below are the distances of each load from the first load; above the spacing distances are the separate loads; then the sum of all the loads from the first to the load in question; then at the top the moment of all the loads in front of any load about that load. Thus, load 11 is 63 feet from load 1, it is itself 25,000 pounds, the loads 1–11 aggregate 205,000 pounds and the moment about 11 of loads 1–10 is 6,915,000 foot-pounds. The moment, about a point 3 feet to the right of load 14, of loads 1–14 is $10{,}365{,}000 + 280{,}000 \times 3 = 11{,}205{,}000$ foot-pounds. The moment, about a point 6 feet to the right of the head of the uniform load, of all the loads in front of this point would be $(19{,}545{,}000 + 340{,}000 \times 6 + 2{,}500 \times 6 \times 3)$ foot-pounds. The moment about the same point of all the loads, except 1–10 inclusive, would be then above minus $[5{,}565{,}000 + 180{,}000 \times (108 + 6 - 55.5)]$ foot-pounds. More complicated diagrams than this are in use, but the writer prefers the form shown, and it meets all requirements.

Example.—Let is be required to find the moment at a point 30 feet from the left end of a span of 120 feet, with load 4 at the point.

Note first what loads are on the span. The loads to the left of 4 are all on the span, while the right end will be 90 feet beyond load 4, or at 107.5 feet on the diagram, or 4.5 feet to the right of load 18. Taking moments about this end of the span, the moment of all the loads on the span will be

$$17{,}845{,}000 + 340{,}000 \times 4.5 = 19{,}375{,}000 \text{ foot-pounds.}$$

The left reaction will be this $\div 120 = 161{,}458.3$ pounds.

The moment at 4 will be

$$161{,}458.3 \times 30 - 550{,}000 = 4{,}293{,}750 \text{ foot-pounds.}$$

For this position of the loads the shear just to the left of load 4 is $161{,}458 - 60{,}000 = 101{,}458$ pounds.

14. To Find the Position of a Movable System of Concentrated Loads to Give Max. Positive Shear at Any Given Point of a Beam Supported at Each End, without Floor Beams.—This is obviously a very important problem, for upon it depends the proper computation of the max. $+S$.

It is clear that for max. $+S$ as many loads as possible should be to the right of the point in question, and as many heavy loads as possible as near to the point (on the right of it) as possible. Starting with any position of the loads, the shear will be $R_l - \Sigma P_l$ the summation being of the loads to the left. If the loads are moved to the left, R_l will be increased, while ΣP_l will not be changed unless

(1) A load passes the given point.

(2) A load passes off the span.

The shear will therefore be increased by moving to the left until a load reaches the given point, and lies an infinitesimal distance to the right of it. The effect of (2) does not alter this conclusion, since just before leaving the span on the left a load causes no shear at the point. Similarly, the effect of a load coming on the span at the right does not alter the conclusion, since when it comes upon the span it adds to the positive shear at the point. Hence,

For the max. $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right\}$ shear at any section, without floor beams, the loads must be as completely as possible to the $\left\{ \begin{matrix} \text{right} \\ \text{left} \end{matrix} \right\}$ of the section, one load must lie *at* (or just to the $\left\{ \begin{matrix} \text{right} \\ \text{left} \end{matrix} \right\}$ of) the section, and the heaviest loads must lie as near the section as practicable. The question is, what load must lie at the point? It should if possible be a heavy load, with other heavy loads close behind it. To determine this, a method of trial is used. Since $S = R_l - \Sigma P_l$, the loads are moved up step by step and the *changes* of R_l and of ΣP_l are found. If R_l is increased more than ΣP_l is increased, the shear is increased by the movement. No rigid formula which will replace the exercise of intelligence and common sense applied to this case.

The loading giving max. snear is found by letting the train come on from the right, and moving it up till the first load is at the section considered. Then the question arises, shall the load be moved still farther to the left? By moving up the left hand reaction is increased, but at the same time the first load must be subtracted from the said reaction. If moved up at all, the loads must be moved

up till P_2 comes just to the right of the section. The question is, therefore, whether in this operation the reaction is increased by more than the first load, which must be subtracted from it. If any load P be moved a distance d toward the left abutment, the left reaction is increased by the amount

$$P\frac{d}{l}.$$

Hence, in Fig. 232, starting with P_1 at the section, C, let R_1 be the total live load on the beam, or in this case $P_1 + P_2 + P_3 + P_4 + P_5$; then if P_2 is moved up to the

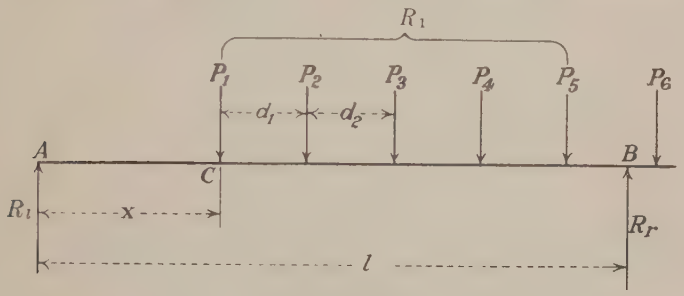


FIG. 232.

section, all the other loads also moving up the same distance d_1, the left-hand reaction is increased by

$$\left(R_1\frac{d_1}{l} + \delta\right),$$

δ being the reaction due to any additional loads which may come on the span at the right. Thus P_6 may come on the span if P_2 is moved up to C and δ is, in this case, the left-hand reaction due to P_6 when P_2 is at C. Clearly the shear will be increased by moving up the loads so that P_2 lies at C provided

$$R_1\frac{d_1}{l} + \delta > P_1.$$

supposing that $x > d_1$, *i.e.*, P_1 does not leave the span.

In the same way, R_2 being the total load on the span when P_2 is at the section, the shear will be increased by moving up the loads till P_3 lies at the section, provided,

$$R_2\frac{d_2}{L} + \delta' > P_2,$$

supposing that P_1 still does not go off the span.

It is to be noticed that the shear might be decreased by moving up P_2 and then increased by moving up P_3 and that it may be a max. with P_3 at the section, because the increase in the second movement may be greater than the decrease in the first. Hence it is sometimes necessary to consider the effect of moving up by two load intervals instead of one. This can only be determined by the use of common sense and vision, and cannot be covered by any general formula.

If a load leaves the span during the operation of moving up, the expressions given will not hold. Suppose $x < d_1$, so that P_1 will leave the span when P_2 is moved up; then the shear will be increased by moving up P_2, provided

$$(R_1 - P_1)\frac{d_1}{l} + \partial > P_1\frac{l - x}{l}.$$

The first term of this inequality is the increase of left reaction, and therefore of the shear, supposing P_1 to stand still at C while the other loads are moved up. But P_1 must be removed from the span. At the start, when it lay just to the right of C, it caused a positive shear of $P_1\frac{l - x}{l}$, and this is now taken away.

Suppose $x > d_1$; $x > d_2$; but $x < d_1 + d_2$; and suppose P_2 at the section (just to the right). Then P_1 will be on the span at the start, but when P_3 is moved up P_1 will leave the span while P_2 will remain upon it. The shear will be increased by moving up P_3 provided

$$(R_2 - P_1)\frac{d_2}{l} + \partial' - P_2 + P_1\frac{x - d_1}{l} > 0.$$

The first term is the increase of left reaction due to moving up all the loads except P_1 a distance d_2, supposing P_1 to stand still. The second term ∂' is the left reaction due to any new loads which come on the span at the right end. At the start P_1 was subtracted from the reaction to obtain the shear, while now P_2 must also be subtracted. P_1 must now be removed. At the beginning it caused a *negative* shear of $P_1\frac{x - d_1}{l}$, so that its removal *increases* the shear; and the first member represents the total *increase* of shear due to moving up P_3.

In these expressions, ∂ and ∂' are generally small and may be neglected, except where the inequality is otherwise small. Further, it is not generally necessary to work out these values completely, or to express ∂ or ∂' other than by the letter. When the general expression is written it is in most cases easy to see at once whether the inequality holds. This method does not reduce the process to a rigid mathematical problem, in which the intellect may go to sleep over mathematical symbols, but the computer must at every step *see* what he is doing, and not merely substitute values in a formula. The process is really very simple if the student understands the principles.

When the position of the loads giving max. $+S$ has been found, the shear itself is found by Eq. (10).

15. Examples of Finding Max. +S.—(1) Let it be required to find the max. $+S$ at a point 20 feet from the left end of a beam of 50-foot span, for the loads shown on the moment diagram Fig. 231.

With P_1 at the section, the right end of the span will be at 30 feet distant, so that loads 1–5 will be on the span. P_2 is moved up if

$$110\frac{7.5}{50} + \partial > 10\text{: clearly it is greater.}$$

P_2 is now at the section, and the right end is at 37.5 feet, so that load 7 is on the span. P_3 is moved up if $140 \cdot \frac{5}{50} + \partial' > 25$: clearly it is less, and the max. $+S$ is with P_2 at the section.

$$2{,}590{,}000 + 140{,}000 \times 1 = 2{,}730{,}000 \text{ foot-pounds}$$

$$\text{Max. } +S = \frac{2{,}730{,}000}{50} - 10{,}000 = 44{,}600 \text{ pounds.}$$

(2) Assume a point 10 feet from the left end of a beam of 30-foot span. With P_1 at the section, load 4 is on the span. P_2 is moved up if

$$85\,\frac{7.5}{30} + \partial > 10\text{: clearly it is greater.}$$

P_2 is now at the section. If P_3 is moved up, P_1 will leave the span. Move up P_3 if

$$100\,\frac{5}{30} + \partial' - 25 + 10\,\frac{2.5}{30} > 0\text{: clearly it is less,}$$

and P_3 is not moved up. It is obvious that $\partial' = \frac{1}{30} \cdot 15 = 0.5$, and will not change the result. No computations are necessary in these two problems other than what are here printed. The process is simple and rapid.

In designing a structure it may be necessary to find, at any section, both max. $+S$ and max. $-S$. Instead of finding negative shears, it is in practice better to find only the max. positive shears, but to find them to the right as well as to the left of the center. Thus, if on a span of 50 feet it is desired to find max. shears at points 5 feet apart, and if the applied loads can move both ways across the structure, it would be necessary to find max. positive and negative shears from the left support to the center, and no farther; *i.e.*, at points 5, 10, 15, 20, and 25 feet from the support. But instead of this it is better to find positive shears every 5 feet from left support to right support. The max. negative shear at a distance a from the left support will be the same as the max. positive shear at the same distance from the right support. By this means the load is always on the same side of the section, and the same kind of shear is found, so that the mind does not have to change its point of view.

16. To Find the Position of a System of Concentrated Loads to Give Maximum Moment at Any Given Point of a Beam Supported at Both Ends, without

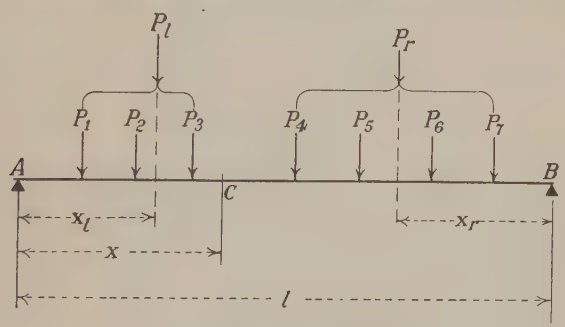


FIG. 233.

Floor Beams.—In Fig. 233 let C be the point at which the max. M is desired. Let a series of concentrated loads be in any position on the span, but with no load at C.

Let P_l = resultant of all loads to the left of C.
x_l = its distance from left support.
P_r = resultant of all loads to the right of C.
x_r = its distance from right support.

Then the moment at $C = M = P_r \frac{x_r}{l} x + P_l \frac{x_l}{l}(l - x)$.

Let the system of loads be moved a small distance Δ to the *left;* then the new moment is $M' = P_r \cdot \frac{x_r + \Delta}{l} x + P_l \frac{x_l - \Delta}{l}(l - x)$. The moment has been increased by the operation provided

$$\frac{P_r}{l - x} > \frac{P_l}{x}.$$

It follows from this that the moment will be increased by moving one way or the other until some load reaches the section, unless

$$\frac{P_r}{l - x} = \frac{P_l}{x}, \tag{12}$$

in which case no change will be made by moving either way.

It follows that the max. M will always occur when some load is at the section. It is obvious that it should be a heavy load, with other heavy loads as near the section as possible. The above expression means that if the average load per foot of length on the right of the given point is greater than the average load per foot on the left, the moment at the point will be increased by moving to the left, and *vice versa.* It follows that the load which must lie at the section for max. M must be such that when it is just to the right the average load on the right is greater than that on the left, and when it is just to the left the average load on the left is greater than that on the right.

To find max. M at any point, therefore, place the system of loads on the span with one of the heavy loads just to the right of the point. Find the average load on each side; if that on the right is greater than that on the left, place the same load *just* to the left of the point and find the new average loads on each side. If now, that on the left is greater, then the load considered must lie at the point. If, on the contrary, when the load is on the left of the section, the load per foot on the right is still greater than that on the left, move the system of loads toward the left till the next load is at the *left* of the section, and try that load. It is clearly not necessary to try this load just to the right of the section, and it is equally clear that it makes no difference what loads go off the span on the left, or come upon it from the right, during the operation.

An actual train may give several loadings corresponding to max. moment at any given point. Generally, it is easy to *see* which will give the *maximum maximorum,* but sometimes it is necessary to compute several moments.

17. Example of Max. M.—Span 50 feet; to find max. M at a point 20 feet from the left support. Use moment diagram (Fig. 231).

The load to be placed at the point will no doubt be one of the driving wheels, probably No. 3 or No. 4. Try No. 3.

With No. 3 just to the right, the right end of the span will be at 42.5 feet, thus load 8 will be just to the right of the support. Hence,

	Left	Right	
No. 3 to right	$35\frac{5}{20}$	$105\frac{5}{30}$	$\therefore \leftarrow$
No. 3 to left	$60\frac{5}{20}$	$95\frac{5}{30}$	$\therefore \leftarrow$
No. 4 to left	$85\frac{5}{20}$	$85\frac{5}{20}$	$\therefore \rightarrow \therefore$ No. 4 lies at point.

It might be thought that since if No. 3 is at the point No. 8 would be over the right support, and hence would cause no moment, that load might be neglected, and hence that it might be correct to write

No. 3 to left	$60\frac{5}{20}$	$80\frac{5}{30}$	$\therefore \rightarrow \therefore$ No. 3 lies at point,

but this is not correct. It would be if there were no loads to the right of No. 7, but not in the actual case, since when No. 4 is moved up, No. 8 moves up too, and then causes a moment. This may easily be tested by working out the moments. No. 4 at point:

$$50)4{,}205{,}000$$
$$84{,}100 \times 20 = 1{,}682{,}000$$
$$550{,}000$$
$$1{,}132{,}000 = \text{max. } M \text{ (foot-pounds)}.$$

No. 3 at point:

$$50)3{,}430{,}000$$
$$68{,}600 \times 20 = 1{,}372{,}000$$
$$250{,}000$$
$$1{,}122{,}000 = M \text{ (foot-pounds)}.$$

No. 4 at point, no loads to right of No. 7:

$$140{,}000 \times 11 = 2{,}590{,}000$$
$$1{,}540{,}000$$
$$50)4{,}130{,}000$$
$$82{,}600 \times 20 = 1{,}652{,}000$$
$$550{,}000$$
$$1{,}102{,}000 = M \text{ (foot-pounds)}.$$

Hence it is necessary to carefully consider all loads that would be on the span, no matter how close to the support.

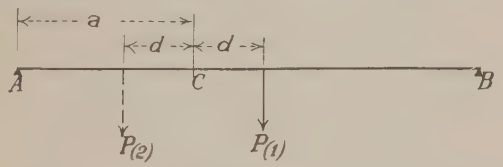


FIG. 234.

17*a*. Maximum Moment Due to Two Equal Concentrated Loads.—The problem sometimes arises to find the maximum moment *at a given point* due to two equal concentrated loads at a fixed distance apart. Let P be each load, and $2d$ their distance apart. Since the max. moment at any given point occurs when a load lies at that point, one of the two loads must lie at the point, and the only question is, on which side of it the other load must lie. In Fig. 234, let the given point be at the left of the center of the span, then if a load P is to the right, the moment it causes at C is $M_1 = Pa \cdot \frac{l - a - d}{l}$; while if it is to the left, the moment at C is $M_2 = P(l - a)\left(\frac{a - d}{l}\right)$. It will be found that $M_1 > M_2$; hence, *the second load must always be in the longer of the two segments into which the span is divided by the point.*

The following principle is therefore useful to bear in mind: If a given *concentrated load may be placed a given distance* (d) *either to the right or to the left of a point C, the moment which it will produce at C will be the greater when it lies in the longer segment into which C divides the span.*

CASE IV. BEAMS SIMPLY SUPPORTED AT EACH END, WITH FLOOR BEAMS

18. Influence Lines.—In this case, as already explained, since the entire load above the beam itself (including weight of floor beams, stringers, and floor, as well as the live load) is applied through the floor beams, the main beam itself is always subjected to a system of concentrated loads, in addition to its own weight. The shear, except for the weight of the beam, in any given panel itself, will not vary from point to point, while the moment will vary uniformly.

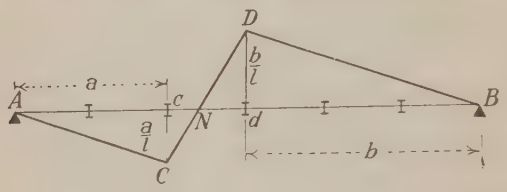


FIG. 235.—Influence line for shear in panel *cd*. Lines *AC* and *DB* are parallel.

19. From the influence lines (Figs. 235 and 236), the following principles follow:

(1) A load lying to the { right / left } of any panel produces { positive / negative } shear in that panel. A load in the panel may produce either positive or negative shear depending upon its position.

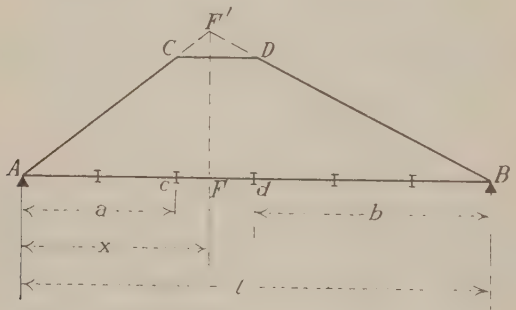


FIG. 236.—Influence line for moment at *F*.

$$cC = \frac{a(l-x)}{l}$$

$$dD = \frac{bx}{l}$$

$$FF' = \frac{x(l-x)}{l}$$

(2) There is a certain point in each panel such that a load at that point produces no shear in that panel. This point may be called the *neutral point*. (*N* in Fig. 235.)

(3) A load to the { right / left } of the neutral point causes { positive / negative } shear in the panel.

(4) If the load is uniformly distributed, the max. $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ shear will occur when the load covers that portion of the span to the $\left\{\begin{matrix}\text{right}\\\text{left}\end{matrix}\right\}$ of the neutral point.

(5) A single load produces max. $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ shear in a panel when it lies at the floor beam just to the $\left\{\begin{matrix}\text{right}\\\text{left}\end{matrix}\right\}$ of the panel.

(6) A single load produces at every point a positive moment. The maximum moment at any point F will occur when the load lies at d, if F is situated between the neutral point and d, or at c if F is situated between the neutral point and c. The moment at the neutral point is the same for any position of the load in the panel.

(7) A uniform load must cover the entire beam to produce max. moment.

(8) The moment at any point is less with floor beams than without; except at the panel points, where it is the same in either case; and except in panels in which there is no load, the moment at any point in such a panel being the same as if there were no floor beams.

It is also clear that the total reactions for any loading are the same as if there were no floor beams. This total reaction, however, includes the load on the end floor beam, or, if there are no end floor beams, it includes the reaction on the end stringer. This end stringer may rest directly upon the abutment, or it may rest upon an end floor beam, which is itself just over the abutment. Since the load on this floor beam is carried directly through the main beam to the abutment, without causing any shear or moment on this main beam, but only a direct vertical compression on the end of the beam, it is best to neglect this load. If, therefore, the span is l and the panel length p, the *effective reaction* at each end, *i.e.*, the reaction causing shears and moments, will, for a uniform load w per foot, be $R = \frac{1}{2}w(l - p)$.

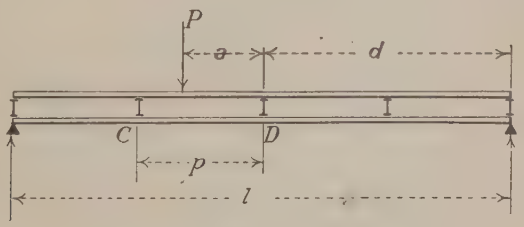


FIG. 237.

20. Effect of a Single Load. Neutral Point in a Panel (Fig. 237).

$$\text{Shear to the left of } C = R_l = P\frac{d + a}{l}.$$

$$\text{Shear to the right of } D = -R_r = -P\frac{l - d - a}{l}.$$

$$\text{Shear in panel } CD = P\left(\frac{d + a}{l} - \frac{a}{p}\right).$$

If the position of the load is such that

$$\frac{d+a}{l} = \frac{a}{p},$$

or

$$a = \frac{pd}{l-p} \tag{13}$$

the shear in the panel CD is zero, or the load is at the *neutral point* in the panel. There is a neutral point in each panel. To find its position by construction

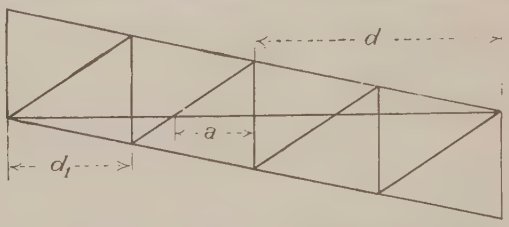


FIG. 238.—Construction of the neutral point.

(Fig. 238), draw any two parallel inclined lines from A and B respectively, erect verticals at the panel points, and draw the short diagonals of the parallelograms so formed. Clearly

$$\frac{a}{p-a} = \frac{d}{l-d-p}$$

and from this equation follows (Eq. 13)

21. Shear and Moment Lines.—These lines, for a single load, are shown in Figs. 239 and 240. The load at the panel point C is $P\frac{a}{p}$ and at D, $P\frac{p-a}{p}$.

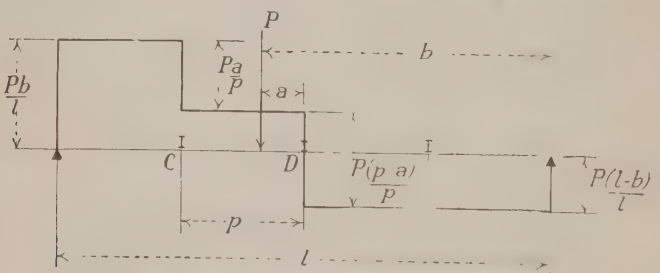


FIG. 239.—Shear line for a single concentrated load.

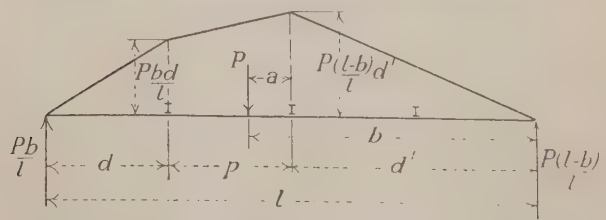


FIG. 240.—Moment line for a single concentrated load.

The shear line for a series of concentrated loads is merely a stepped line, dropping at each floor beam by an amount equal to the concentrated load applied at that floor beam.

The moment line for a series of concentrated loads is found by drawing the moment line as if there were no floor beams, and joining the points over the floor beams by straight lines (Fig. 241).

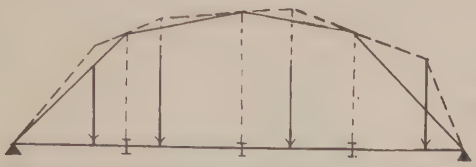


FIG. 241.—Moment line for a series of concentrated loads. Dotted line: no floor beams. Full line: with floor beams.

22. Uniformly Distributed Load.—If the load covers the entire span, the load at each panel point will be wp, and the curve of shears will be as in Fig. 242, dropping by wp at each panel point. The moment curve, if there were no floor

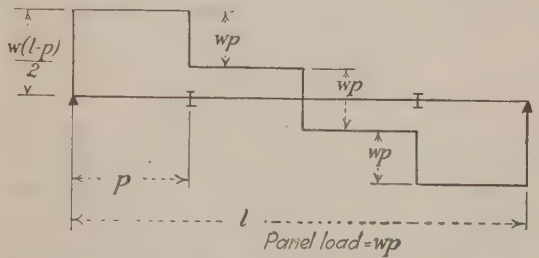


FIG. 242.—Shear line for uniform load over entire length. Beam with floor beams supported at both ends.

beams would be the dotted parabola in Fig. 243, with a center ordinate $\frac{1}{8}\ wl^2$. Since the moment at a panel point will be the same as if there were no floor beams, the moment curve if there are floor beams may be found from this dotted

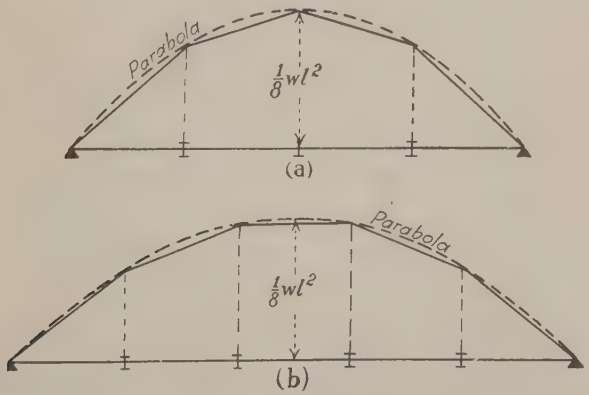


FIG. 243.—Moment line for a uniformly distributed load. Dotted line: no floor beams Full line: with floor beams.

curve as shown. This broken curve will give the maximum moment at each point.

The *max. positive shear* in any panel will occur when the load covers the portion of the span to the right of the neutral point of that panel. Its value will be (Fig. 238 and Eq. (13))

$$\text{Max. } +S = \frac{w\left(d + \frac{pd}{l-p}\right)^2}{2l} - \frac{w\left(\frac{pd}{l-p}\right)^2}{2p}$$

$$= \frac{wd^2}{2(l-p)}. \tag{14}$$

Similarly

$$\text{Max. } -S = \frac{wd_1{}^2}{2(l-p)}. \tag{15}$$

In the $\left\{\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}$ panel there is no $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}$ shear.

In the left panel the max. $+S = \frac{w(l-p)}{2}$, or half the total load, neglecting the loads carried directly to the supports. Call this S_o. Then

$$\frac{\text{max. } +S}{S_o} = \frac{d^2}{(l-p)^2}. \tag{16}$$

This enables a very rapid computation to be made of the shear in any panel. If there are six panels, for example, having found the max. $+S$ in the left end panel, the max. $+S$ in the next one will $\frac{16}{25}$, in the next $\frac{9}{25}$, in the next $\frac{4}{25}$, in the next $\frac{1}{25}$ of the first, and in the right end panel zero.

The curve whose ordinates give max. $+S$, which without floor beams is a parabola with vertex at B (Fig. 227) becomes in this case a stepped line (Fig.

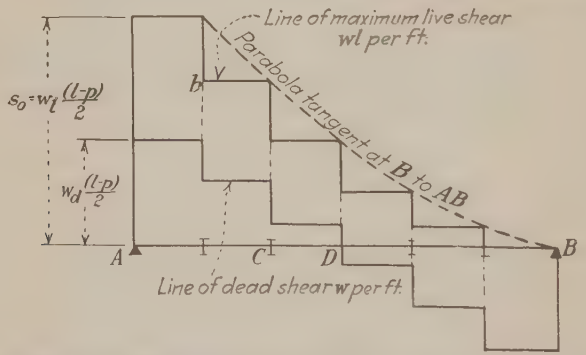


FIG. 244.—Shear lines for uniform dead load and for max. $+S$ for uniform live load up to neutral point in each panel.

244), but the angles of the steps lie on a parabola with vertical axis and vertex at B, Fig. 244, since the max. $+S$ in a panel such as CD is $S_o \frac{d^2}{(l-p)^2}$ by Eq. (16). The effect of floor beams upon shear is to make it greater at some points and less at others than if there were no floor beams.

The assumption is frequently made in practice that for max. $(+S)$ in any panel, all the panel points on the right, up to and including the one just to the

right of the panel, *are fully loaded*, while all the panel points to the left are entirely unloaded. This assumption is impossible to realize; for no panel point can be *fully* loaded unless the panels on either side of it are fully loaded. If, therefore, the panel point D (Fig. 244) is fully loaded, there must be a load of $\frac{1}{2}pw$ on C, and this load causes a negative shear in the panel CD, which is neglected under the above assumption, which therefore gives a larger positive shear than is possible, and is therefore on the safe side.

As an example, let $w = 2{,}000$ pounds per foot, and let the span consist of six panels each 20 feet in length. Then the panel load is 40,000 pounds, and the max. shear in the end panel is 100,000 pounds. In the other panels the max. $(+S)$ will be, up to the center,

$$2^\circ \text{ panel: } \tfrac{16}{25} \times 100{,}000 = 64{,}000 \text{ pounds.}$$
$$3^\circ \text{ panel: } \tfrac{9}{25} \times 100{,}000 = 36{,}000 \text{ pounds.}$$

Under the assumption just referred to, this max. shear would be

$$\text{end panel: } 100{,}000 \text{ pounds}$$
$$2^\circ \text{ panel: } \frac{1+2+3+4}{6} \cdot 40{,}000 = 66{,}667 \text{ pounds.}$$
$$3^\circ \text{ panel: } \frac{1+2+3}{6} \times 40{,}000 = 40{,}000 \text{ pounds.}$$

The assumption may therefore be considerably (11.1 per cent in this case) in error, and should not ordinarily be made, as there is no good reason for it. However, it will frequently be made in following pages, in order to show methods of computation much in use.

In Fig. 244 are shown the shear line for a uniform dead load of w_d per foot, and the accurate curve of max. $(+S)$ for a live load of w_l per foot, the live load extending up to the neutral point in each panel.

23. Load Distributed, but Not Uniformly.—The loads on beams are not infrequently distributed, but not uniformly. For instance, if C (Fig. 245)

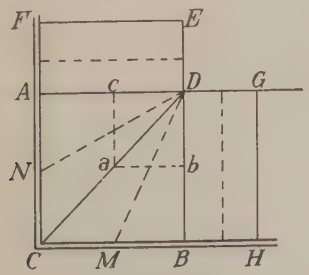


FIG. 245.

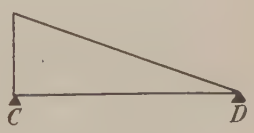


FIG. 246.

is the corner of a building, and CD a beam resting on the outer wall at C and on an inner wall or girder or other support at D, and if DB, GH, DA and EF are other beams, and if the load on the areas ACD and CDB are carried to AD, CD and DB by beams in the directions ac and ab, the beam CD will carry a distributed load increasing uniformly from nothing at D to a max. at C, the whole representing the load on the area NDM. The load will thus be represented by a triangular

area (Fig. 246). The load per foot of length at C, if w_1 is the load per square foot upon the floor, will be $\frac{1}{2}w_1 (CB + AC)$.

If the load on CDB were carried by beams in the direction of DB to the beam CD and the outer wall, the load on CD would increase from nothing at C to a max. at D.

The load on DB, with distribution parallel to ab, would increase from a min. at D of $\frac{w_1}{2}(DG)$ to a max. at B of $\frac{w_1}{2}(CB + BH)$. Similar conditions occur in many structures, especially buildings.

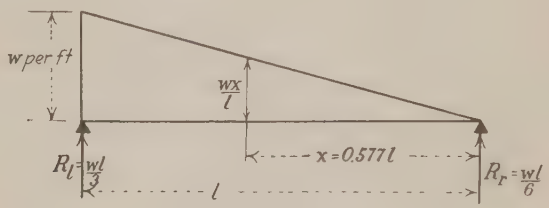


Fig. 247.

(1) *Triangular Distribution* (Fig. 247). The average load per longitudinal foot is $\frac{w}{2}$, the total load $\frac{wl}{2}$, the reactions $\frac{wl}{3}$ and $\frac{wl}{6}$. The max. M will occur at a distance x from the right support such that load over said distance $= \frac{wl}{6}$, or

$$x = \frac{l}{\sqrt{3}} = 0.577l.$$

The max. $M = \frac{0.577l}{3} \cdot \frac{wl}{3} = 0.064\, wl^2$. (17)

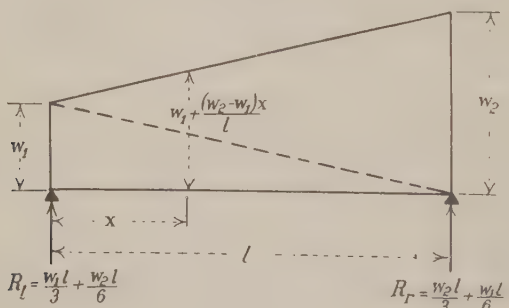


Fig. 248.

(2) *Trapezoidal Distribution* (Fig.248). $R_l = \frac{w_1 l}{2} + \frac{(w_2 - w_1)l}{2} \cdot \frac{1}{3} = \frac{w_1 l}{3} + \frac{w_2 l}{6}$, which may be seen directly by drawing the diagonal of the trapezoid.

$$R_r = \frac{w_1 l}{6} + \frac{w_2 l}{3}.$$

The max. M will occur when

$$\frac{w_1 l}{3} + \frac{w_2 l}{6} = \left[2w_1 + \frac{(w_2 - w_1)x}{l}\right]\frac{x}{2},$$

or

$$x = \frac{l}{w_2 - w_1}\left[-w_1 + \sqrt{\frac{w_1^2 + w_1 w_2 + w_2^2}{3}}\right]. \tag{18}$$

If $w_1 = 0$, this value is the same as for Case (1).

24. A System of Concentrated Loads.—Let the loads be so placed that no single load lies at the panel point at either end of a given panel, and let the shear in this panel be under consideration. Let

ΣP = the total load on the span,
ΣP_l = the total load to the left of the panel,
ΣP_p = the total load in the panel.

Let the system of loads be moved a distance dx to the left, without any load coming on or going off the span, or going into or out of the panel. Then the shear in the panel will be *increased* by

$$\Sigma P \frac{dx}{l} - \Sigma P_p \frac{dx}{p}.$$

The shear will therefore be increased if

$$\frac{\Sigma P}{l} > \frac{\Sigma P_p}{p}. \tag{19}$$

If increased, the loads should be moved to the left until either:

(1) A load passes off the span at the left.
(2) A load passes on the span at the right.
(3) A load passes off the panel at the left.
(4) A load passes on the panel at the right.

(1) Does not alter the fact that the shear is increased by moving still further to the left, but simply changes the rate of increase, because just before passing off the span it causes no shear.

(2) Similarly, does not affect the fact that the shear is increased by moving still farther to the left, because it simply increases ΣP.

(3) Does not alter the fact, because just before passing out of the panel, the entire load so passing is subtracted from the reaction, and by a still greater movement the same load is still subtracted.

(4) May alter the fact, because by passing into the panel, ΣP_p is suddenly increased while ΣP is unchanged.

It follows that *for max. $+S$ in a panel some load must lie at the panel point at the right end of the panel.* Clearly *this load must be such that when just to the right of the panel* $\Sigma P_p < \frac{p}{l}\Sigma P$, *and when just in the panel* $\Sigma P_p > \frac{p}{l}\Sigma P$.

This criterion makes easy the determination of position of the loads to produce max. $+S$ in the panel.

25. Example of Finding Max. +S in Each Panel (Fig. 249).—Let the span be 90 feet with 6 panels of 15 feet each. Then $\frac{p}{l} = \frac{1}{6}$.

For Panel 1.—Using the moment diagram, when P_1 is just inside the panel the right end of the span is at 75 feet, $\Sigma P = 255$, and $\Sigma P_p = 10 < \frac{1}{6} \cdot 255$. $\therefore$

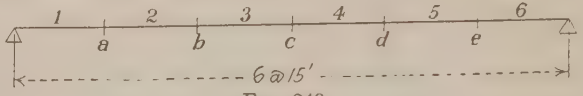


FIG. 249.

move P_2 up. When P_2 is just inside the panel $\Sigma P = 280$, $\Sigma P_p = 35 < \frac{1}{6} \cdot 280$. $\therefore$ move P_3 up. When P_3 is just inside the panel, $\Sigma P = 295$, $\Sigma P_p = 60 > \frac{1}{6} \cdot 295$. $\therefore$ for max. $+S$, P_3 should lie at a.

With P_3 at a; $R_l = \frac{12{,}885 + 295 \cdot \frac{1}{2}}{90} = 144{,}800$ pounds.

(Remember that, in the moment diagram, loads are in thousands of pounds.)

Max. $+S = 144{,}800 - \frac{5}{15} \cdot 25{,}000 - \frac{12.5}{15} 10{,}000 = 128{,}133$ pounds,

$\left(\text{or, from the moment diagram } 144{,}800 - \frac{250{,}000}{15}\right)$.

Panel 2.—P_1 just to left of b; $10 < \frac{1}{6} \cdot 180$.

P_2 just to left of b; $35 > \frac{1}{6} \cdot 205$.

$\therefore$ P_2 should lie at b.

$$R_l = \frac{6{,}915 + 205 \times 4.5}{90} = 87{,}083 \text{ pounds.}$$

Max. $+S = 87{,}083 - \frac{7.5}{15} \cdot 10{,}000 = 82{,}083$ pounds.

Panel 3.—Try P_1; $10 < \frac{1}{6} \cdot 155$.

$\therefore$ P_2 should lie at c. If P_3 was not moved up for panel 2, it will not be moved up for panel 3.

$$R_l = \frac{4{,}205 + 170 \times 5}{90} = 56{,}170 \text{ pounds.}$$

Max. $+S = 56{,}170 - 5{,}000 = 51{,}170$ pounds.

Panel 4.—Try P_1; $10 < \frac{1}{6} \cdot 110$. $\therefore$ P_2 should lie at d.

$$R_l = \frac{2{,}590 + 140}{90} = 30{,}330 \text{ pounds.}$$

Max. $+S = 30{,}330 - 5{,}000 = 25{,}330$ pounds.

Panel 5.—Try P_1; $10 = \frac{1}{6} \cdot 60$. $\therefore$ the shear is not changed by moving to the left, so long as no load comes on the span. But if P_2 is moved up to e, P_4 will come on the span and will cause a positive shear. $\therefore$ P_2 should lie at e.

$$R_l = \frac{975}{90} = 10{,}830 \text{ pounds.}$$

Max. $+S = 10{,}830 - 5{,}000 = 5{,}830$ pounds.

The problem of finding max. + and − shear in each panel is now completely solved.

In panel 1: max. $+S$ = 128,133; max. $-S$ = 0.
In panel 2: max. $+S$ = 82,083; max. $-S$ = 5,830.
In panel 3: max. $+S$ = 51,170; max. $-S$ = 25,330.
In panel 4: max. $+S$ = 25,330; max. $-S$ = 51,170.
In panel 5: max. $+S$ = 5,830; max. $-S$ = 82,083.
In panel 6: max. $+S$ = 0; max. $-S$ = 128,133.

26. Moment.—The moment at a floor beam, as already shown, is the same as if there were no floor beams. The max. M at a floor beam may therefore be found as in §16. It is rarely, if ever, necessary to find the max. M at a point between floor beams. If it should be necessary, it may be approximated to by the following considerations (Fig. 250). Let aa' represent the max. M at a, and bb'' the moment at b for the same loading which causes max. M at a. Let bb' be the max. M at b and aa'' the moment at a for the same loading. Then when the max. M at a occurs, the moment at any point between a and b will be the ordinate to the line $a'b''$; and when the max. M at b occurs the moment at any point between a and b will be the ordinate to the line $b'a''$. The max. M at any point between a and b will therefore be, approximately, the ordinate to the line $b'ca'$. It might be possible for a loading to occur in which the line giving moments between a and b would pass above c, so that near the center of the panel a slightly greater moment might be obtained than that given by $b'ca'$. If absolute certainty of obtaining the maximum is desired, the ordinate to $a'b'$ may be taken.

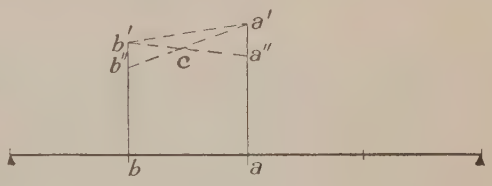


FIG. 250.

27. Absolute Maximum Moment.—The discussions which have been given explain how to find the max. M at any point of a beam. It is clear, however, that these methods will never enable the absolute max. moment on the beam, which is necessary in proportioning the beam if it is of uniform section, to be found exactly; for no matter how close together the points may be at each of which the max. M is found, there will always be the possibility that at some intermediate point the max. M may be greater than at any of the points chosen. To find the absolute max. moment it is therefore necessary to resort to some different method, instead of finding max. M at points chosen, and fortunately a very simple method or device is available. Wherever the point may be at which the max. M is greater than at any other point, we know that for that max. M *some load* must lie at that point, and we know that it will be one of the heavy loads. If, therefore,

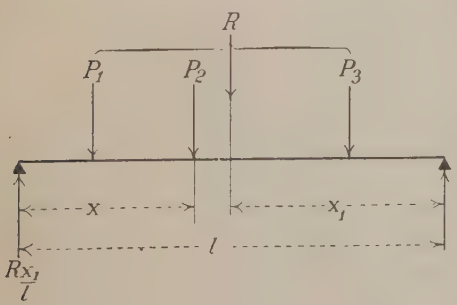


FIG. 251.

instead of finding max. M *at a given point*, we find the max. M *at a given load*, it will only be necessary to find this for a few of the loads in order to determine the absolute max. M on the beam. Let a series of loads (Fig. 251) lie in any position on a beam, and let R = resultant of all these loads. Let us find the moment M_2 at P_2.

$$M_2 = R\frac{x_1 x}{l} - \Sigma P_1 d.$$

P_1 being any load to the left of P_2, and d its distance from P_2.

If now the system of loads is moved to the left a distance Δ, without changing the loads on the span, the new moment at P_2 will be

$$M_2' = \frac{R}{l}(x_1 + \Delta)(x - \Delta) - \Sigma P_1 d$$

$$M_2' - M_2 = \frac{R}{l}(x\Delta - x_1\Delta - \Delta^2).$$

If Δ is an infinitesimal dx, Δ^2 is zero in comparison with the other terms, and

$$M_2' - M_2 = \frac{R\Delta}{l}(x - x_1). \tag{20}$$

If $x \gtrless x_1$ M_2 will be $\left\{\begin{matrix}\text{increased}\\ \text{decreased}\end{matrix}\right\}$ by moving to the left, and the max. M_2 will occur if $x = x_1$. Hence,

The max. moment at any load occurs when that load and the resultant of all the loads on the span are equidistant from the center of the span.

Except for a symmetrical system of loads with one load at the center, therefore, the max. moment on a beam will not be at the center of the span.

In case the beam has a uniform load over its whole length (such as dead load), and a system of concentrated loads is moved over it, the maximum moment at any

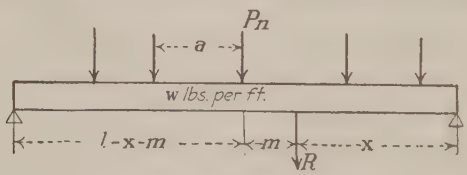


FIG. 252.

concentrated load such as P_n (Fig. 252) may be found in a similar way. Let R be the resultant of all the concentrated loads on the span, x its (variable) distance from the right end, and m its (fixed) distance to the right of P_n; w is the uniform load per foot. Then

$$M(\text{at } P_n) = \left(\frac{Rx}{l} + \frac{wl}{2}\right)(l - x - m) - \sum_a^n Pa - \frac{w(l - x - m)^2}{2}$$

$$\frac{dM}{dx} = R - \frac{2Rx}{l} - \frac{Rm}{l} + \frac{wl}{2} - wm - wx = 0$$

$$x = \frac{l - m}{2} - \frac{wml}{2(2R + wl)}. \tag{20a}$$

If $w = 0$; $x = \frac{l - m}{2}$; or, the center of the span lies midway between R and P_n.

28. Example of Finding Absolute Maximum Moment.—Let it be desired to find the absolute max. M on a span of 30 feet for an electric railway car weighing 50 tons on 4 axles spaced as shown. Use the loads on one rail in proportioning the stringer supporting that rail (Fig. 253).

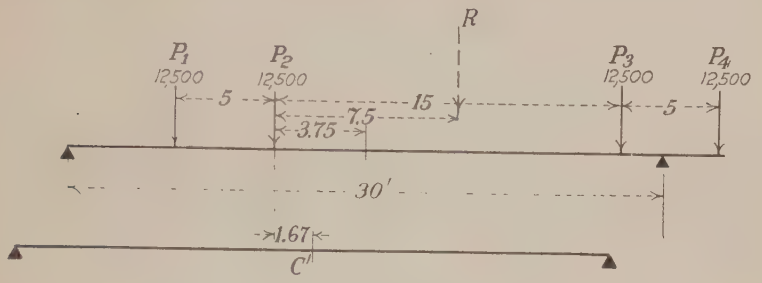


FIG. 253.

Inasmuch as the final position of the loads is not known at the start, it is necessary to assume what loads will be on the span. In doing this, remember that the load considered will lie not far from the center. In the present case it is clear that the max. M will be at P_2 or P_3. Let us find it at P_2, and assume that all the loads will be on the span. The resultant of all the loads is 50,000 pounds, midway between P_2 and P_3. Hence, place the center of the span midway between this load and P_2. Then the right end of the span will be between P_3 and P_4, *i.e.*, *P_4 will not be on the span.* Hence we must find the resultant of P_1, P_2, and P_3: its distance to the right of P_2 will be $\frac{12{,}500 \cdot 15 - 12{,}500 \cdot 5}{37{,}500} = 3\frac{1}{3}$ feet, or more simply, since the loads are equal in this case $\frac{15 - 5}{3} = 3\frac{1}{3}$ feet. Hence the center of the span must be 1.67 feet to the right of P_2. Looking again (as must always be done) to see whether all the assumed loads are on the span, it is found that they are. Then

$$\text{abs. max. } M_2 = \frac{37{,}500\ (15 - 1.67)^2}{30} - 12{,}500 \cdot 5 = 159{,}725 \text{ foot-pounds.}$$

The first term in the second member will always be in the same form, *viz.*,

$$\frac{(\text{total load})\ (\text{distance of resultant from right support})^2}{\text{span}}.$$

It is now to be observed that since the max. moment on the beam will occur near the center, with as many heavy loads as near the center as possible, it may be that a greater moment than that just found will result by considering P_1 and P_2 only. In this case the center of the span will be 1.25 feet to the left of P_2 and max. $M_2 = \frac{25{,}000 \cdot 13.75^2}{30} = 157{,}550$ foot-pounds or, as it turns out, less than that previously found.

If P_2 were at the center, the moment at that point would be $25{,}000 \cdot \frac{12.5}{30} \cdot 15 = 156{,}250$ foot-pounds, so that the absolute max. M is less than 3 per cent greater.

Yet even this may make the difference between a beam of one size and one of another, and if the beam is designed under rigid specifications the use of the center moment might make the design technically inadmissible under the specifications, although perfectly safe.

CASE V. BEAM FIXED AT ONE END AND SIMPLY SUPPORTED AT THE OTHER

29. (*a*) **Effect of a Single Concentrated Load.**—The case of a beam with one end fixed is statically undetermined, and, to find the value of the negative moment at the fixed end, the condition must be imposed that the slope at that end is zero. This case is fully treated in Chap. XII, but, for completeness, a simple solution may here be given, so that the results may be studied and conclusions drawn.

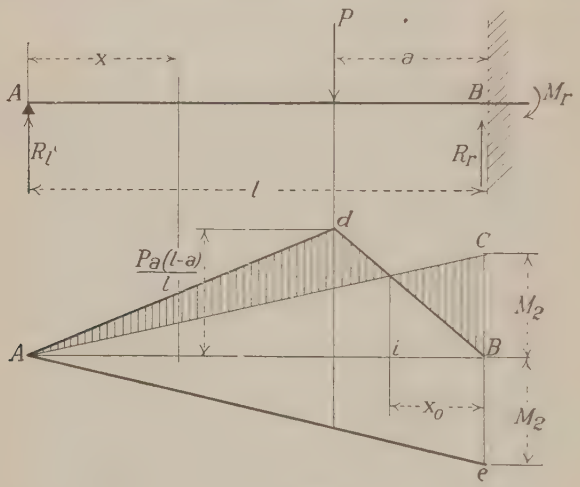


Fig. 254.

The only principle necessary is that of moment-areas. In Fig. 254, AdB is the moment line for the concentrated load if both ends were simply supported; AeB for a negative moment M_2 at B. Laying off $Bc = Be$, the actual moment area is the shaded area. Since the deflection of A from the tangent at B is zero, it follows that the moment, about A, of this area, is zero, or the moment of the areas AdB and AcB are equal, or

$$\frac{Pa(l-a)^3}{3l} + \frac{Pa^2(l-a)}{2l}\left(l - a + \frac{a}{3}\right) = \frac{M_2 l^2}{3}$$

from which

$$\text{(negative)} \quad M_2 = M_r = \frac{Pa(l-a)(2l-a)}{2l^2}. \tag{21}$$

Then, from $\Sigma M = 0$:

$$M_2 = -Pa + R_l l$$

$$R_l = -\frac{Pa(l-a)(2l-a)}{2l^3} + \frac{Pa}{l} = \frac{Pa^2(3l-a)}{2l^3}. \tag{22}$$

Then, from $\Sigma V = 0$:

$$R_r = P - R_l = \frac{P(2l^3 - 3a^2l + a^3)}{2l^3} = \frac{P(l-a)(2l^2 + 2al - a^2)}{2l^3}. \quad (23)$$

There is no moment at i, whose distance from B is

$$x_o = \frac{M_2}{R_r} = \frac{al(2l-a)}{2l^2 + 2al - a^2}. \quad (24)$$

From B to i, the top fibers of the beam will be in tension. The max. $+M$ will be at the load, and its value is

$$\text{Max. } +M \text{ (at load)} = \frac{Pa(l-a)}{l} - \frac{M_2(l-a)}{l} = \frac{Pa^2(l-a)(3l-a)}{2l^3}. \quad (25)$$

The max. $(-M)$ is M_2 (numerically).

At any point distant x from A,

$$M = R_l x;\ S = R_l;\ \text{for } x < (l-a)$$
$$M = R_l x - P(x - l + a);\ S = R_l - P;\ \text{for } x > (l-a).$$

To find where the load P must lie in order to cause the max. pos. moment to be the greatest possible, differentiate Eq. (25) and place $\frac{dM}{da} = 0$; or

$$\text{Max. max. } +M = 0.174Pl \quad (26)$$

which occurs for

$$a = \frac{l}{2}(3 - \sqrt{3}) = 0.634l. \quad (27)$$

Similarly, to find where P must lie in order that M_2 may be a maximum numerically, differentiate Eq. (21), and we find

$$\text{Max. max. } (-M) = -0.192Pl \quad (28)$$

which occurs for

$$a = l\left(1 - \frac{1}{\sqrt{3}}\right) = 0.423l. \quad (29)$$

Values of x_o, $+M$ (at load), and $-M_2$(at fixed end) are given in the following table; for various values of a:

$\frac{a}{l}$ =	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{x_o}{l}$ =	0.087	0.15	0.20	0.24	0.27	0.30	0.31	0.32	0.33
$-\frac{M_2}{Pl}$ =	0.086	0.144	0.178	0.192	0.187	0.168	0.136	0.096	0.049
$\frac{+M \text{ (at load)}}{Pl}$ =	0.013	0.045	0.085	0.125	0.156	0.173	0.169	0.141	0.085

The numerical effect of fixing the end B is to reduce the (positive) shear between A and the load, and to increase the (negative) shear between the load and B.

Remembering that for a beam simply supported at each end the max. $+M$ (for load at center) $= 0.25Pl$ it follows:

(1) *The positive moments on a beam simply supported are everywhere decreased by fixing one end, and for a distance x_o from the fixed end they are converted into negative moments. The max. value of x_o is $\frac{l}{3}$ (for $a = l$).*

(2) *The max.* $+M$ *for a concentrated load which may be anywhere on a beam with one end fixed is* [*for* $a = 0.634l$] *closely* 0.7 *of that for both ends supported* $\left[\text{for } a = \frac{l}{2}\right]$.

(3) *The max.* $-M$ (*which occurs at the fixed end*), *for a concentrated load which may lie anywhere on a beam with one end fixed is* [*for* $a = 0.423l$] *closely* 0.77 *of the max.* $+M$ *for both ends supported.*

The negative moments in this case are therefore very important, and must be provided for in the design.

These negative moments, however, arise solely from the resistance of the wall or other structure to which the beam is fastened at the fixed end; and since all materials yield somewhat—even if but slightly—under stress, it follows that *the negative moments are never as great as above computed, while the positive moments are greater;* in other words, a beam is never actually fixed at the end in such a manner that there is no yielding or rotation there. What the actual condition is will depend upon circumstances. If the beam is fixed by being riveted to another part of the structure, as, for instance, to a column, the yielding can in many cases be computed, and such cases will be hereafter treated. Sometimes the beam may be practically fixed, as by being imbedded in a solid mass of masonry resting on an unyielding foundation, while in other cases the negative moments may be considerably reduced. Only by the exercise of a trained and experienced judgment can the engineer decide doubtful cases; but in general the negative moments may be taken somewhat smaller, and the positive moments somewhat greater, than above deduced.

30. (*b*) **Uniformly Distributed Load.**—In this case the following results may be deduced from those just given, by making $P = wdx$, and integrating over the length of the beam (Fig. 255).

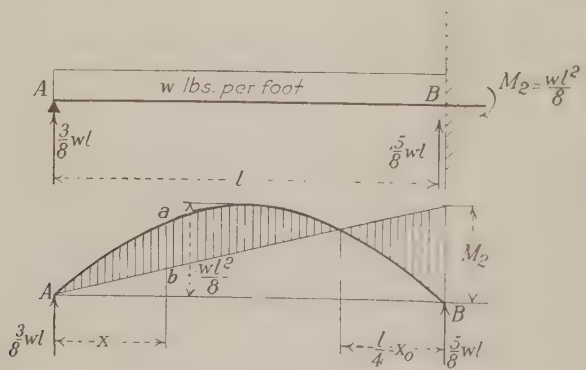


FIG. 255.

$$R_l = \tfrac{3}{8}wl \tag{30}$$

$$R_r = \tfrac{5}{8}wl \tag{31}$$

$$M_2 = -\frac{wl^2}{8} \text{ (negative moment)} \tag{32}$$

$$x_o = \frac{l}{4}. \tag{33}$$

At any distance x from A,

$$S = \frac{3}{8}wl - wx = \frac{w}{8}(3l - 8x) \tag{34}$$

$$M = \frac{3}{8}wlx - \frac{wx^2}{2} = \frac{wx}{8}(3l - 4x) \tag{35}$$

S is zero for $x = \frac{3}{8}l$
M is zero for $x = \frac{3}{4}l$
$+M$ is max. for $x = \frac{3}{8}l$
and max. $+M = \frac{9}{128}\,wl^2$.

The portion of the beam AC with a span of $\frac{3}{4}l = l_1$ is the same as a beam simply supported, and the max. M occurs at its center, and equals $\frac{wl_1^2}{8} = \frac{9}{128}\,wl^2$.

In this case max. $-M$ is numerically greater than max. $+M$. Bearing in mind the remarks at the end of the last paragraph, the negative moments, and the distance x_o will generally be smaller than the values given above, and the positive moments greater.

Problems

(1) Derive the formulae for the case in which the distributed load covers the left half of the span.

(2) Do the same when the uniform load covers any distance b from A to the right.

(3) Do the same when it covers the middle half of the span.

CASE VI. BEAM FIXED AT BOTH ENDS

31. (a) **Concentrated Load** (Fig. 256).—In Chap. XII this statically undetermined case is investigated, and from the conditions that at each end the slope is zero, the result is obtained,

$$\text{Negative } M_l = \frac{Pa^2(l - a)}{l^2} \tag{36}$$

$$\text{Negative } M_r = \frac{Pa(l - a)^2}{l^2}. \tag{37}$$

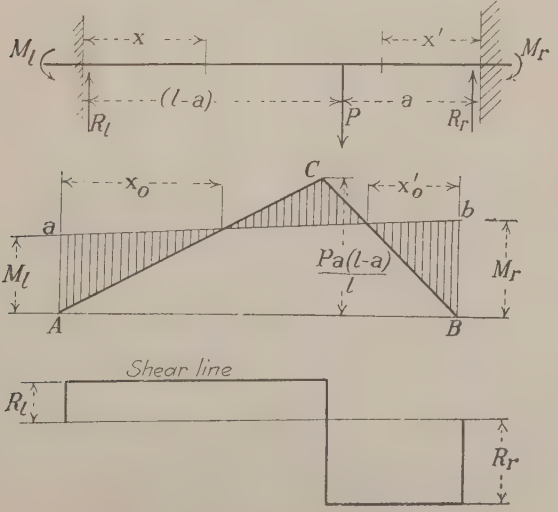


FIG. 256.—Moment line ab = zero line on z.

These results may also be obtained very swiftly, as follows: The shaded area is the moment area. Since the slope and deflection of B with reference to the tangent at A each equal zero, it follows that,

(1) The moment area between A and B equals zero, or the triangular area AcB equals the area $AabB$, or

$$\frac{Pal(l-a)}{2l} = \frac{(M_l + M_r)l}{2} \text{ or } Pa(l-a) = (M_l + M_r)l.$$

(2) The moment about A of the area AcB equals that of the area $AabB$, or

$$\frac{Pa(l-a)^3}{3l} + \frac{Pa^2(l-a)}{6l}(3l-2a) = \frac{M_l l^2}{6} + \frac{M_r l^2}{3}.$$

from which Eqs. (36) and (37) follow at once.

The reactions are

$$R_l = \frac{Pa}{l} - \frac{M_r}{l} + \frac{M_l}{l} = \frac{Pa^2}{l^3}(3l-2a) \tag{38}$$

$$R_r = \frac{P(l-a)}{l} + \frac{M_r}{l} - \frac{M_l}{l} = \frac{P(l-a)^2}{l^3}(l+2a). \tag{39}$$

Equation (39) must be obtainable from Eq. (38) by substituting $(l-a)$ for a. (Note the importance of recognizing these relations and checking results by means of them.)

The moment at any point is,

To the left of the load,

$$\text{Positive } M = R_l x - M_l = \frac{Pa^2}{l^2}\left[\frac{x(3l-2a)}{l} - (l-a)\right] \tag{40}$$

To the right of the load,

$$\text{Positive } M = R_r x_1 - M_r = \frac{P(l-a)^2}{l^2}\left[\frac{x'(l+2a)}{l} - a\right] \tag{41}$$

These moments are zero when

$$x = \frac{l(l-a)}{3l-2a} = x_o \tag{42}$$

$$x' = \frac{al}{l+2a} = x_o'. \tag{43}$$

The moment at the load is (by placing $x' = a$ in (41))

$$\text{Positive } M_1 = \frac{2Pa^2(l-a)^2}{l^3} \tag{44}$$

which is always numerically greater than one of the end negative moments and less than the other, except when the load is at the center of the span, when all three are equal.

In a beam of this kind, then, for a single concentrated load alone, the greatest moment is always one of the negative moments at the end. This may not be true, however, for a series of concentrated loads. The max. end moment is at the end nearest to the load.

The shear at any point to the left of the load is $+R_l$ and to the right it is $-R_r$. Loads to the right of any given point cause positive shear and loads to the left cause negative shear, and both are greatest when the load is just to one side of the point, as in beams simply supported at the ends.

From Eq. (36), by differentiation, it follows that *the max. negative M at one end occurs when the load is one-third of the span distant from that end.* Its value is

$$\text{Max. negative } M_l = \tfrac{4}{27}Pl \text{ (numerically)}. \tag{45}$$

Problem.—Endeavor to deduce this result from Eq. (37).

From Eq. (44), the max. moment at the load occurs when the load is at the center of the span, and

$$\text{Max. positive } M_1 = \tfrac{1}{8}Pl \tag{46}$$

or one-half that for a supported beam. This result follows directly from the fact that the moment area must be zero.

The max. negative moment in a case of this kind, for a single movable concentrated load, is therefore greater than the max. positive moment, in the ratio of 32 to 27, or greater by about 18.5 per cent. It is therefore fair to say that in this form of beam the negative moments are of greater importance than the positive moments. In some instances, however, they seem to have been little considered.

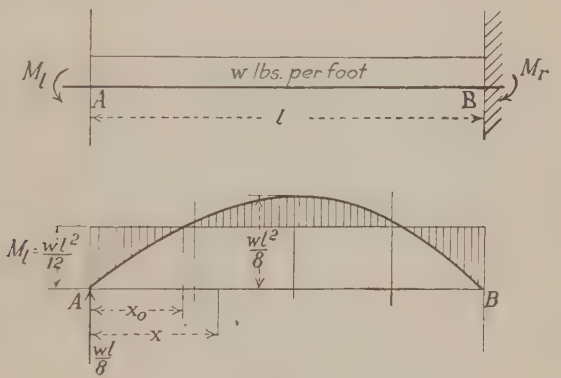


FIG. 257.

32. (*b*) **Uniformly Distributed Load** (Fig. 257).—From the fact that the moment area must be zero, the area of the rectangle must equal that of the parabola, or

$$M_r l = \frac{2}{3} l \cdot \frac{wl^2}{8}$$

$$\text{Max. negative } M_r = M_l = \frac{wl^2}{12}. \tag{47}$$

The max. positive M is at the center and

$$\text{Max. positive } M_c = \frac{wl^2}{24}. \tag{48}$$

The moment at any distance x from a support is

$$\text{Positive } M = \frac{wx(l-x)}{2} - \frac{wl^2}{12}.$$

This is zero when

$$x = x_o = \frac{l}{2}\left(1 - \frac{1}{\sqrt{3}}\right) = 0.203l. \tag{49}$$

About 0.6 of the span is exposed to positive moments and 0.4 to negative moments, but the latter are much greater than the former.

The reactions are each $\frac{wl}{2}$, and the shears the same as for a simply supported beam. The max. $+S$ for any point will occur when the load covers the portion of the beam to the right of the point.

33. (*c*) **Partial Distributed Load** (Fig. 258).—In Eq. (36), by replacing P by wdx_1, and a by x_1 the value of M_l will be that due to a load wdx_1, at a distance x_1

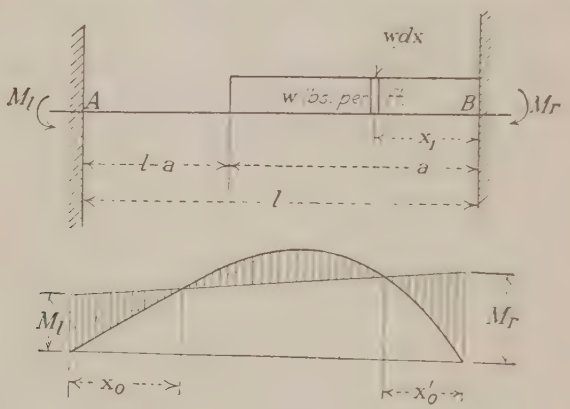


FIG. 258.

from B. Integrating this between the limits a and 0 will give the value of M_l desired. Thus

$$dM_l = \frac{wdx_1x_1^2(l - x_1)}{l^2}$$

$$\text{(negative) } M_l = \frac{w}{l^2}\int_o^a x_1^2(l - x_1)dx_1 = \frac{wa^3}{12l^2}(4l - 3a). \tag{50}$$

Similarly from Eq. (37),

$$\text{Negative } M_r = \frac{w}{l^2}\int_o^a x_1(l - x_1)^2dx_1 = \frac{wa^2}{12l^2}(6l^2 - 8la + 3a^2). \tag{51}$$

For $a = l$, these both reduce to Eq. (47), as they should.

From Eq. (38), to find max. positive shear at C load CB, and we find

$$\text{Max. } +S = R_l = \frac{w}{l^3}\int_o^a (3l - 2x_1)x_1^2dx_1 = \frac{wa^3}{2l^3}(2l - a). \tag{52}$$

For this loading,

$$R_r = \frac{w}{l^3}\int_o^a (l - x_1)^2(l + 2x_1)dx_1 = \frac{wa}{2l^3}(2l^3 - 2la^2 + a^3). \tag{53}$$

For $a = l$, both of these reduce to $\frac{wl}{2}$, and the sum of (52) and (53) gives wa, as must be the case

For maximum negative shear at a point C (Fig. 259), load the part of the beam to the left of the point, and find R_r. Thus from Eq. (39), making $P = wdx$, $l - a = x$, $a = l - x$, we have

$$dR_r = \frac{wx^2}{l^3}(3l - 2x)dx$$

$$R_r = \frac{w}{l^3}\int_0^{l-a} x^2(3l - 2x)dx = \frac{w}{2l^3}(l^4 - 2l^3a + 2la^3 - a^4). \qquad (54)$$

For $a = 0$, this reduces to $\frac{wl}{2}$, as it should.

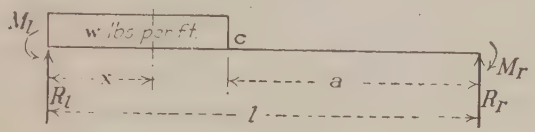


Fig. 259.

34. Use of Influence Lines with a System of Concentrated Loads. Load on Floor Beams.—The influence line for any function (shear, moment, reaction, etc.) being drawn, the ordinate to that line at any point represents the effect, on the function, of a unit load at that point. If a series of concentrated loads lie on the span, the effect of that system may be found by multiplying each load by the ordinate to the influence line at the point where the load acts, and adding the results. Suppose the influence line to be composed of two straight lines as in

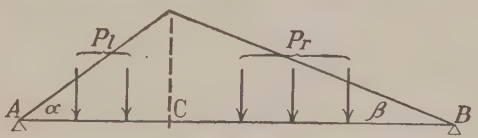


Fig. 260.

Fig. 260. This would represent the influence line for moment at C on a supported beam AB, but it might represent some other function. Let P_l be the total load on AC and P_r be the total load on CB, there being no load at the point C. Now if the entire system of loads be moved a distance dx to the left, the effect on the function will be *increased* by

$$P_r dx \tan \beta - P_l dx \tan \alpha$$

or it will really be increased if

$$\frac{P_r}{CB} > \frac{P_l}{AC}.$$

This is the same result which, in the case of moment, was proved analytically in Sec. 16, Eq. (12), but it is now evident that it is true, not only for moment on a supported beam, but for any function for which the influence line has the same form. Thus:

Whenever the influence line for any function is composed of two straight lines starting from zero at A and B, and meeting over C, then the max. value of the function,

for a series of concentrated loads, will occur for the same loading which would give max. M at C on a supported beam AB; i.e. when some load is at C, such that when that load is just to the right of C the average load per foot on CB is greater than the average load per foot on AC, and when it is just to the left of C the average load on AC is greater than that on CB.

An example of this principle is in finding the max. reaction on a floor beam from two adjacent stringers. Let AC and CB (Fig. 261) be two stringers, resting on floor beams at A, C, and B. The load on the floor beam at C comes entirely from the stringers AC and CB, and the influence line for that load is AcB. If the stringers are connected to the web of the floor beam, instead of resting on top of it, it is necessary to find, not only the max. load on the floor beam from both stringers, but the max. from each. The max. load from CB will be the max.

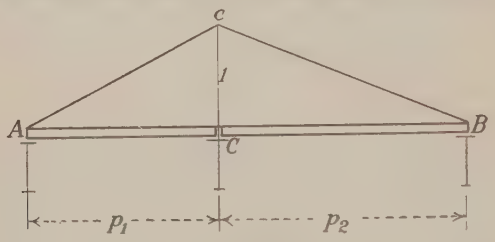


Fig. 261.

reaction at C on CB, and will always occur when a heavy load is at C, and other heavy loads on CB and as near to C as possible, irrespective of any loads on AC. Similarly for AC. The max. reaction on the floor beam from both stringers, however, will obviously not be the sum of the max. reaction on each stringer, but will occur with one heavy load at C and others on both AC and CB and as near to C as possible; and will be for the same loading that would give max. M at C on a span AB.

The load at C being found, if the bridge is a single-track railroad bridge, the floor beam is exposed to its own weight and to two symmetrical concentrated loads (Fig. 262). The moment, aside from that due to the dead load, increases uniformly from either support to one load, and is constant between the loads. If it is a double-track railroad bridge there are four equal and symmetrical concentrated loads and the dead weight.

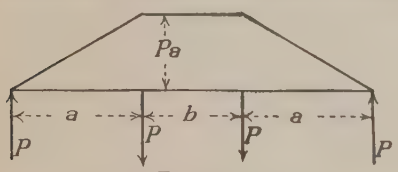


Fig. 262.

When the student has thoroughly mastered the principles in this chapter and that on Deflection and Slope, he will have laid the foundation for a knowledge of structures, both statically determined and statically undetermined; for the computation of stresses in the former depends upon moment and shear, and in the latter upon deflection and slope also.

Problems

1. State when the moment at a point between two floor beams will be less, numerically, than if there were no floor beams.

2. State when it will be less algebraically.

3. Draw the curves of shear and moment for a beam fixed at one end and free at the other, the load being uniformly distributed, and extending from the free end, but not covering the entire beam.

4. Do the same for a uniformly distributed load not extending to either end.

5. Do the same for a uniform load combined with a single concentrated load, in problems 3 and 4.

6. Draw the curves of shear and moment for a beam fixed at one end and free at the other.

(*a*) For a uniform load covering the whole beam, and a couple at the free end.

(*b*) For a uniform load covering part of the beam extending to the free end, and a couple at the free end.

(*c*) For a uniform load not extending to either end, and a couple at the free end.

(*d*) For each of the three cases above, combined with a single concentrated load.

7. Deduce Eqs. (2) and (3) directly from the influence lines.

8. If the max. moment at the center of a beam supported at each end, for a uniform load, is 100,000 foot-pounds, what is the moment at a point half way from

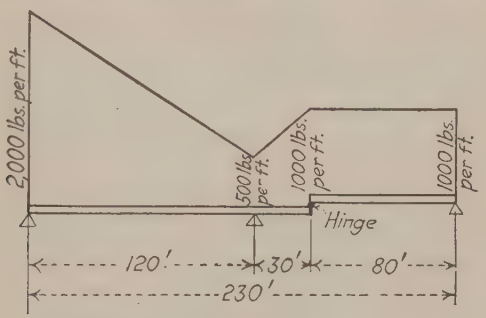


FIG. 263.

the support to the center? What at one quarter way? What at one-third? Do this by mental arithmetic alone.

9. If the max. moment for a uniform load at a point 20 feet from the left support, on a beam 60 feet long supported at each end, is 80,000 foot-pounds, what is the moment at a point 10 feet from the support? What at points 10 and 5 feet from the support? Do this *entirely* by mental arithmetic.

10. In Fig. 227 find the vertices of the parabolas $cf''e$ and $e'f''c'$.

11. Deduce Eqs. (30), (31), and (32), directly from the moment areas.

12. In Fig. 244 what is the curve joining all points such as b.

13. Find the max. M on a beam supported at each end, with a distributed load increasing uniformly from zero at each end to 2,000 pounds per foot at a point 10 feet from the left end of a span of 30 feet.

14. Draw the curves of moments and shears and find max. M and S in the following case (Fig. 263).

15. Prove the first of the three statements near the end of Art. 29—not from the table preceding, but algebraically.

CHAPTER XII

SLOPE AND DEFLECTION OF BEAMS IN PURE FLEXURE

1. General Principles.—The fundamental principles, methods, and formulae for finding the deflection of beams have been given in Chap. X. It was there shown that there are several methods, namely:

The Deflection May Be Found: (*a*) By starting with the expression, using the proper sign,

$$\frac{d^2y}{dx^2} = \pm \frac{M}{EI} \tag{1}$$

inserting the value of M, the loading being known, and integrating twice, determining the constants of integration from the conditions of the problem, which fix the slope or the deflection at certain points (see Chap. X, Art. 16);

(*b*) By the equation

$$d = \int_B^A \frac{Mxdx}{EI} \tag{1a}$$

in which, in Fig. 263*a*,

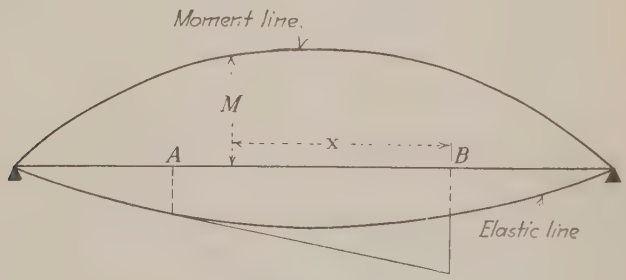


Fig. 263*a*.

d is the deflection of any point B from the tangent at A,

x is the distance of any point between A and B from B,

and the integration extends from A to B (see Chap. X, Art. 17);

(*c*) From the moment area, by means of Theorem II, Chap. X, Art 17, but working it out without integrating an expression; method (*b*) is merely an application of this method which allows a solution by integration;

(*d*) By means of the stresses produced by a unit load, using the equation (see Chap. X, Art. 18),

$$d = \int_o^l \frac{M_t M_s dx}{EI} \tag{2}$$

in which

d is the deflection at any desired point, B,

M_r is the moment at any point A, due to a unit load applied at the desired point B,

M_s is the moment at A due to the actual loads for which the deflection at B is desired.

The Slope May Be Found:

(a) As in the first method of finding deflection, starting with Eq. (1) and integrating once;

(b) By means of the moment area, and Theorem I of Chap. X, Art. 17;

(c) By means of the rotation produced by a unit couple applied at the desired point, B, using the equation (see Chap. X, Art. 20),

$$\alpha = \int_o^l \frac{M_r M_s dx}{EI} \tag{3}$$

in which

α is the slope at the desired point B,

M_r is the moment at any point A due to a unit couple applied at the desired point B,

M_s is the moment at A due to the external loads for which the slope (rotation) at B is desired.

Each method has its uses. In some cases one is the simplest, and in other cases another. Generally, but not always, the slope is found most easily by the second method, and the deflection by the third method; both being the method of moment areas.

In the following pages will be given a discussion of all the usual cases which may occur, for beams supported, in any manner, at one or two supports. The method depending upon the moment diagram, or line of moments, will generally be given first, since the value of the slope and deflection may by this method generally be written down at once, all that is required being some algebraical reduction. The elastic line will be drawn in its true general position, though not of course to scale, and slopes and deflections will be given numerically, so as to avoid all algebraical interpretation as to sign.

The method by the use of the calculus will then be given, in each case, so that the reader may have the opportunity to gain a complete understanding and mastery of both methods.

This discussion will necessarily involve consideration of the shear and moment diagrams.

These cases are all treated as if there were no floor beams. The results will generally be sufficiently accurate to apply in case there are floor beams; but if greater accuracy is desired, the only thing to do is to find the concentrated load at each floor beam and find separately the effect of each of these concentrations.

CASE I. BEAM FIXED AT ONE END (Fig. 264)

2. (a) With a Concentrated Load at the Free End.—

$$\text{Slope at distance } x \text{ from support} = \alpha = \frac{\text{area } abb'a'}{EI} = \frac{Px(2l - x)}{2EI}. \tag{4}$$

$$\text{Deflection at distance } x \text{ from support} = d = \frac{\text{moment about } b \text{ of } abb\,a'}{EI}$$

$$= \frac{Px^2(3l - x)}{6EI} \tag{5}$$

$$\text{Slope at free end} = \alpha_1 = \frac{Pl^2}{2EI} \tag{6}$$

$$\text{Deflection at free end} = d_1 = \frac{Pl^3}{3EI}. \tag{7}$$

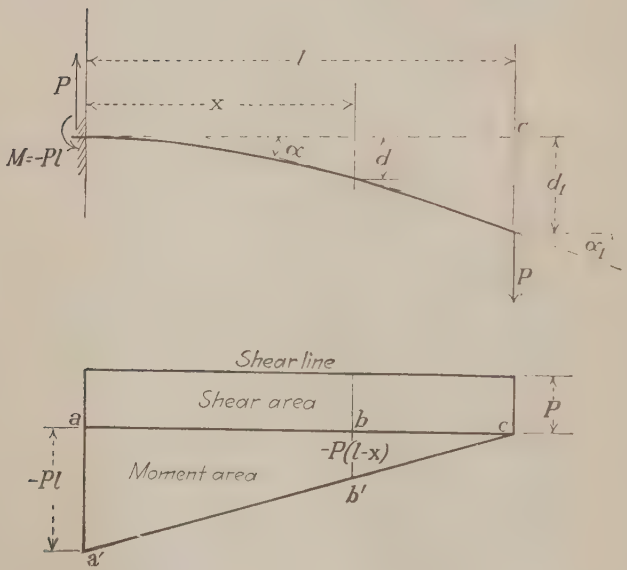


Fig. 264.

Using Eqs. (2) and (3):

for free end; $M_s = -P(l - x)$; $M_t = -(l - x)$; $M_r = -1$

$$\alpha_1 = \int_o^l \frac{P(l - x)dx}{EI} = \frac{Pl^2}{2EI}$$

$$d_1 = \int_o^l \frac{P(l - x)^2dx}{EI} = \frac{Pl^3}{3EI};$$

for point b_1 distant x_1 from support, $M_s = -P(l - x)$, x being now a variable point;

$M_t = -(x_1 - x)$ between a and b_1
$= 0$ between b_1 and c
$M_r = -1$ between a and b_1
$= 0$ between b_1 and c

$$\alpha = \int_o^{x_1} \frac{P(l - x)dx}{EI} = \frac{Px_1(2l - x_1)}{2EI}, \text{ as in (4)}$$

$$d = \int_o^{x_1} \frac{P(l - x)(x_1 - x)dx}{EI} = \frac{Px_1^2(3l - x_1)}{6EI}, \text{ as in (5).}$$

By Eq. (1a), the deflection at the free end, from the tangent at the fixed end, if x is the distance from the free end, is

$$\vartheta = \int_o^l \frac{Mxdx}{EI} = \int_o^l \frac{Px^2dx}{EI} = \frac{Pl^3}{3EI}.$$

Using Eq. (1); origin at support, x positive to right, y positive down.

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = \frac{P(l-x)}{EI}$$

$$\frac{dy}{dx} = \frac{Plx}{EI} - \frac{Px^2}{2EI} + C.$$

For $x = 0, \frac{dy}{dx} = 0$, $\quad \therefore C = 0$

$$\therefore \frac{dy}{dx} = \frac{Px(2l-x)}{2EI}, \text{ as in (4)}$$

$$y = \frac{Plx^2}{2EI} - \frac{Px^3}{6EI} + C_1.$$

For $x = 0, y = 0$, $\quad \therefore C_1 = 0$

$$\therefore y = \frac{Px^2(3l-x)}{6EI} \text{ as in (5).}$$

3. (*b*) **With a Concentrated Load Anywhere.**—Here the deflection and slope at the point where the load is applied are the same as if that point were the end of the beam. From that point to the free end the axis is straight. Therefore, if a is the distance from the load to the support

$$\text{Slope at load and at free end} = \alpha_1 = \frac{Pa^2}{2EI} \tag{8}$$

$$\text{Deflection at load} = d = \frac{Pa^3}{3EI} \tag{9}$$

$$\text{Deflection at free end} = d + \alpha_1(l-a) = \frac{Pa^2(3l-a)}{6EI}. \tag{10}$$

4. (*c*) **With a Series of Concentrated Loads.**—Add the effects of each load separately.

5. (*d*) **With a Uniform Load w per Foot.**—(1) Let the uniform load cover the entire length (Fig. 265). From the moment diagram:

$$\alpha = \frac{w[l^3 - (l-x)^3]}{6EI} = \frac{w(3l^2x - 3lx^2 + x^3)}{6EI} \tag{11}$$

$$\alpha_1 = \frac{wl^3}{6EI} \tag{12}$$

$$d = \frac{w}{6EI}\left[l^3\left(x - \frac{l}{4}\right) + (l-x)^3\left(\frac{l-x}{4}\right)\right]$$

$$= \frac{w}{24EI}[6l^2x^2 - 4lx^3 + x^4] \tag{13}$$

$$d_1 = \frac{wl^4}{8EI}. \tag{14}$$

From Eqs. (2) and (3):

For free end; $\quad M_s = -\frac{w(l-x)^2}{2}$; $M_t = -(l-x)$; $M_r = -1$

$$\alpha_1 = \int_o^l \frac{w(l-x)^2dx}{2EI} = \frac{wl^3}{6EI}, \text{ as in (12)}$$

$$d_1 = \int_o^l \frac{w(l-x)^3dx}{2EI} = \frac{wl^4}{8EI}, \text{ as in (14).}$$

From Eq. (1); origin at 0; x positive to right, y downward.

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = \frac{w(l-x)^2}{2EI}$$

$$\frac{dy}{dx} = -\frac{w(l-x)^3}{6EI} + C.$$

For $x = 0$, $\frac{dy}{dx} = 0$; $\quad \therefore C = \frac{wl^3}{6EI}$

$$\therefore \frac{dy}{dx} = \frac{w[l^3 - (l-x)^3]}{6EI}, \text{ as in (11)}$$

$$y = \frac{wl^3x}{6EI} + \frac{w(l-x)^4}{24EI} + C_1.$$

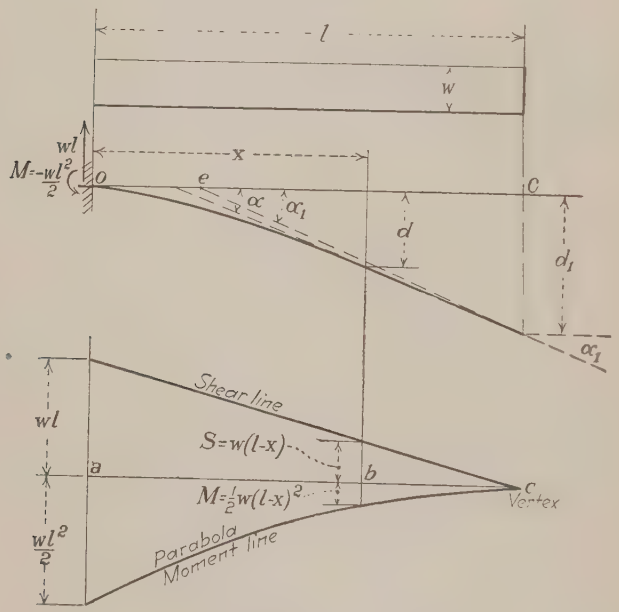


FIG. 265.

For $x = 0$, $y = 0$, $\quad \therefore C_1 = -\frac{wl^4}{24EI}$

$$\therefore y = \frac{w}{24EI}[6l^2x^2 - 4lx^3 + x^4], \text{ as in (13).}$$

To find the point e, where the tangent to the elastic line at the free end cuts Oc,

$$\alpha_1 = \tan \alpha_1 = \frac{d_1}{ec}$$

$$ec = \frac{d_1}{\alpha_1} = \frac{3}{4}l. \qquad (15)$$

(2) Let the uniform load cover only a portion of the length.

If the uniform load covers a distance $(l - a)$ at the free end of the beam (Fig. 266) the moment at the fixed end is $\frac{w(l^2 - a^2)}{2}$, and that at the left end of the load is $\frac{w(l-a)^2}{2}$.

From a' to b' the moment line is straight and from b' to c it is a parabola with vertex at c.

Hence, from the moment diagram,

$$EI\alpha_2 = \frac{wal(l-a)}{2} \tag{16}$$

$$EI\alpha_4 = \frac{w(l^3 - a^3)}{6} \tag{17}$$

$$EId_2 = \frac{wa^2(l-a)(3l+a)}{12} \tag{18}$$

$$EId_4 = \frac{w(3l^4 + a^4 - 4la^3)}{24}. \tag{19}$$

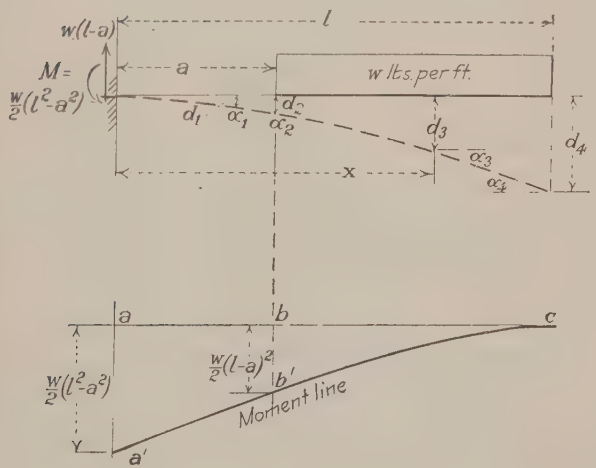


FIG. 266.

Problem.—Find the expressions for slope and deflection at any point in *ab* and at any point in *bc*.

Note.—The above results must check with those for a full uniform load or for a concentrated load. If $a = 0$, Eqs. (17) and (19) must reduce to Eqs. (12) and (14); and if $a = l - dx$, and all terms in the expanded expressions having higher powers of dx than the first are neglected, Eqs. (17) and (19) must reduce to Eqs. (6) and (7), with $P = wdx$.

If the partial load does not extend to the free end, the part of the beam from the support to the right end of the distributed load corresponds to the distance l in Fig. 266, and the axis to the right of this point is straight, with the same slope as at c.

6. (*e*) **With a Couple at the Free End, Having a Moment** M_0.—Let the couple have the direction indicated in Fig. 267. From the moment diagram:

$$\alpha = \frac{M_0 x}{EI} \tag{20}$$

$$\alpha_1 = \frac{M_0 l}{EI} \tag{21}$$

$$d = \frac{M_0 x^2}{2EI} \tag{22}$$

$$d_1 = \frac{M_0 l^2}{2EI}. \tag{23}$$

From Eq. (1):

$$\frac{d^2y}{dx^2} = \frac{M_0}{EI}; \frac{dy}{dx} = \frac{M_0 x}{EI} + C = \frac{M_0 x}{EI}$$

$$y = \frac{M_0 x^2}{2EI} + C_1 = \frac{M_0 x^2}{2EI}.$$

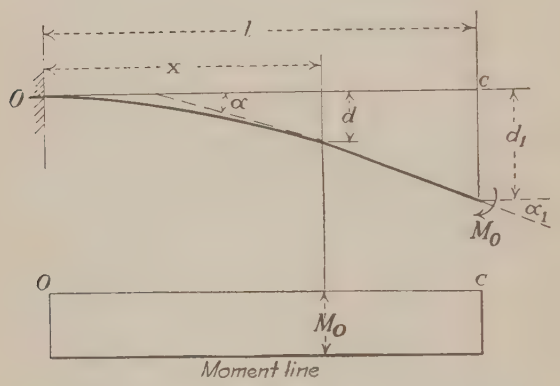


Fig. 267.

7. (*f*) **With a Couple** M_0 **Applied at Any Point.**—In this case the axis will be straight beyond the point of application of the moment, as in Case (*b*).

Problems.

1. Write expressions for slope and deflection at the free end for a uniform load over part of the beam and a couple applied anywhere.
2. Do the same for a couple and a concentrated load applied anywhere.

CASE II. BEAM SUPPORTED AT EACH END, WITHOUT FLOOR BEAMS

8. (*a*) **With a Concentrated Load at the Center** (Fig. 268).—From the moment diagram:

$$\text{Slope at end} = \alpha_1 = \frac{\text{area } abc}{EI} = \frac{Pl^2}{16EI} \tag{24}$$

$$\text{Slope at } b = \alpha = \frac{\text{area } bcc'b'}{EI} = \frac{P(l^2 - 4x^2)}{16EI} \tag{25}$$

Slope at center = 0

$$\text{Deflection at center} = d_1 = \frac{\text{moment of area } abc \text{ about } a}{EI}$$

$$= \frac{Pl^3}{48EI}. \tag{26}$$

$$\text{Deflection at } B = d = d_1 - \frac{\text{moment of area } bcc'b' \text{ about } b'}{EI}$$

$$= \frac{Px(3l^2 - 4x^2)}{48EI}. \tag{27}$$

Problem.—Deduce the same results by the other two methods.

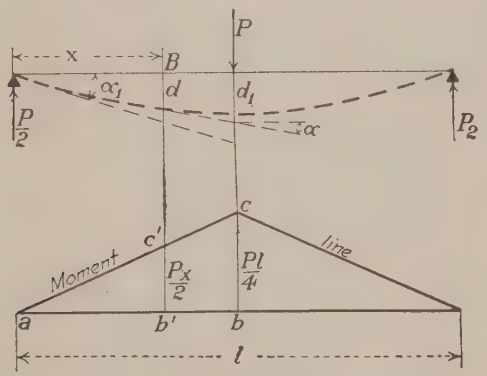


FIG. 268.

9. (*b*) **With a Concentrated Load Not at the Center** (Fig. 269).—Let the load be to the right of the center. In this case the elastic curve will not be horizontal at the load or at the center C, but at some intermediate point D, such that the moment of the area add' about a equals the moment of the area bdd' about b; or since

$$dd' = \frac{Pax_o}{l}.$$

x_o is found from the equation (cancelling P):

$$\frac{x_o^3 a}{3l} = (\text{moment of } aeb - \text{mom. of } eg'b - \text{mom. of } add') \text{ about } b$$

$$= \frac{al^2}{6} - \frac{a^3}{6} - \frac{x_o^2 a}{2l}\left(l - \frac{2}{3}x_o\right)$$

from which

$$x_o = \sqrt{\frac{l^2 - a^2}{3}}. \tag{28}$$

This may easily be constructed geometrically, as follows:

Lay off $am = l - a$; $an = \frac{l + a}{3}$; draw a semi-circle with mn as a diameter, and it will cut ab at d.

If $a = \frac{l}{2}$, $x_o = \frac{l}{2}$, as it should.

Having found where the beam is horizontal,

$$EI\alpha_1 = \text{area } add' = \frac{Px^2{}_o a}{2l} = \frac{Pa(l^2 - a^2)}{6l} \tag{29}$$

$$EI\alpha_2 = \text{area } dd'g'b = \frac{Pa(l-a)(2l-a)}{6l}. \tag{30}$$

These values each become the same as Eq. (24) if $a = \frac{l}{2}$.

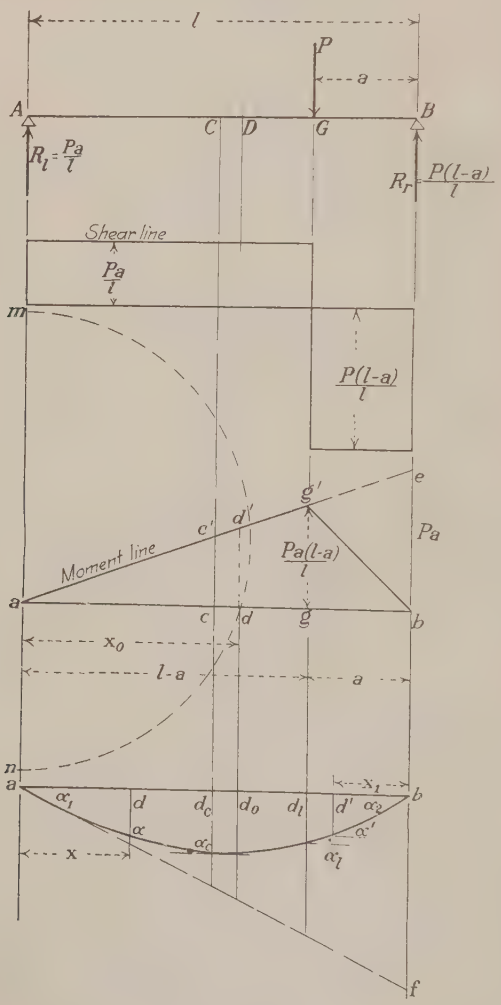


Fig. 269.

If, as assumed, $a < \frac{l}{2}$, then $\alpha_2 > \alpha_1$, or the slope is greatest at the end of the beam nearest the load, as would be expected.

The above slopes may also be obtained most easily as follows: in the lower figure, by the principle of slope,

$$\begin{aligned} bf = \alpha_1 l &= \text{moment of area } ag'b \text{ about } b \div EI \\ &= (\text{mom. of } aeb - \text{mom. of } g'eb) \div EI \\ \therefore EI\alpha_1 &= \frac{Pa(l^2 - a^2)}{6l}. \end{aligned} \tag{29}$$

The value of α_2 may be obtained from this by substituting $(l - a)$ for a.

$$\therefore EI\alpha_2 = \frac{P(l-a)a(2l-a)}{6l}. \tag{30}$$

The following results follow easily:

$$\text{The max. deflection (at } d) = d_o = \frac{\text{moment of area } add' \text{ about } a}{EI}.$$

$$\therefore EId_o = \frac{Pax_o^3}{3l} = \frac{Pa}{3l}\left(\frac{l^2 - a^2}{3}\right)^{3/2}. \tag{31}$$

$$\text{The deflection at load} = d_l = \alpha_2 \cdot a - \frac{\text{moment of } gg'b \text{ about } g}{EI}.$$

$$\therefore EI \cdot d_l = \frac{Pa^2(l-a)^2}{3l}. \tag{32}$$

At any point between a and g distant x from a

$$\text{Slope} = \alpha = \alpha_1 - \frac{\text{area to } x}{EI}$$

$$\therefore EI\alpha = \frac{Pa}{6l}(l^2 - a^2 - 3x^2) \tag{33}$$

$$\text{Deflection} = d = \alpha_1 x - \frac{\text{moment of area about } x}{EI}$$

$$\therefore EId = \frac{Pax}{6l}(l^2 - a^2 - x^2). \tag{34}$$

At the center of the span, the slope and deflection are given by the equations:

$$\text{Slope: } EI \cdot \alpha_c = \frac{Pa}{24l}(l^2 - 4a^2) \tag{35}$$

$$\text{Deflection: } EI \cdot d_c = \frac{Pa}{48}(3l^2 - 4a^2). \tag{36}$$

At any point between b and g, distant x_1 from b,

$$\text{Slope} = \alpha' = \alpha_2 - \frac{\text{area to } x_1}{EI} \text{ (downward to the left)}$$

$$\therefore EI\alpha' = \frac{P(l-a)}{6l}(2al - a^2 - 3x_1^2) \tag{37}$$

$$\text{Deflection} = d' = \alpha_2 x_1 - \frac{\text{moment of area about } x_1}{EI}$$

$$\therefore EId' = \frac{Px_1(l-a)}{6l}(2al - a^2 - x_1^2). \tag{38}$$

These equations hold respectively to the left and right of the load, wherever it may be placed, that is whether to the left or right of the center of the span.

If $x = l - a$, and $x_1 = a$, the values of α and α' become equal but of opposite sign, as should evidently be the case; while d and d' become both the same as d_l.

At the point of max. deflection the slope is zero, and if the value of x_o is substituted for x in Eq. (33) this is found to be the case.

At the load, the slope α_l is found by making $x_1 = a$ in Eq. (37) or $x = l - a$ in Eq. (33), and the results will be numerically equal but algebraically opposite. This slope will always be downward toward the center. Its value will be

$$EI\alpha_l = \frac{Pa(l-a)(l-2a)}{3l}. \tag{39}$$

By the use of Eq. (1), values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, and y must be obtained separately in AG and GB. The constants of integration are found from the conditions that slope and deflection, in both parts of the beam, must be identical at the load. The student should work this out, and the results must agree with those found by the easier method used above.

Equation (28) shows where the axis is horizontal. It is of interest to know how far this is from the center, and the ratio of this distance to half the span. This is given by the equation,

$$\frac{x_o - \frac{l}{2}}{\frac{l}{2}} = \frac{2x_o}{l} - 1 = \frac{2}{l}\sqrt{\frac{l^2 - a^2}{3}} - 1.$$

This is a max. when $a = 0$, when it equals $\frac{2}{\sqrt{3}} - 1 = 0.153$. Hence the point of max. deflection is never more than $0.076l$ from the center, and generally it is near enough to assume it at the center.

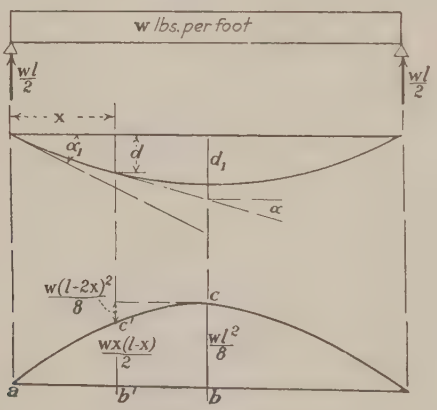


Fig. 270.

10. (*c*) **With a Uniformly Distributed Load** (Fig. 270).—From the moment diagram:

$$\text{Slope at end} = \alpha_1 = \frac{\text{area } abc}{EI}$$

$$\therefore EI\alpha_1 = \frac{wl^2}{8} \cdot \frac{l}{2} \cdot \frac{2}{3} = \frac{wl^3}{24}. \tag{40}$$

Slope at distance x from end $= \alpha = \dfrac{\text{area } bcc'b'}{EI}$

$$\therefore EI\alpha = \frac{wx(l-x)}{2}\left(\frac{l}{2}-x\right)+\frac{2}{3}\cdot\frac{w(l-2x)^2}{8}\cdot\left(\frac{l}{2}-x\right)$$

$$= \frac{w(l-2x)}{24}(l^2+2lx-2x^2)$$

$$= \frac{w}{24}(l^3-6lx^2+4x^3). \tag{41}$$

Deflection at the center $= d_1 = \dfrac{\text{moment of area } abc \text{ about } a}{EI}$

$$\therefore EI\cdot d_1 = \tfrac{5}{384} \text{ of } wl^4. \tag{42}$$

Deflection at any point distant x from a

$$= d = d_1 - \frac{\text{moment of } bcc'b' \text{ about } b'}{EI}$$

$$\therefore EI\cdot d = \frac{wx(l-x)}{24}(l^2+lx-x^2). \tag{43}$$

By using Eq. (1); taking origin at a, distances to the right and downward as positive,

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{wx(l-x)}{2EI}$$

$$\frac{dy}{dx} = -\frac{w}{EI}\left(\frac{lx^2}{4}-\frac{x^3}{6}\right)+C.$$

For $x = \dfrac{l}{2}$, $\dfrac{dy}{dx} = 0$,

$$\therefore C = \frac{wl^3}{24EI}$$

$$\therefore \frac{dy}{dx} = \frac{w}{24EI}(l-2x)(l^2+2lx-2x^2) \text{ as in Eq. (41)}$$

$$y = \frac{w}{24EI}(l^3x-2lx^3+x^4)+[C_1=0] \text{ as in Eq. (43).}$$

For $x = 0$, $y = 0$,

$$\therefore C_1 = 0.$$

11. (*d*) **With a Partial Uniform Load** (Fig. 271).—The simplest method of treating this somewhat difficult case, a method which also illustrates how the effect of a uniformly distributed load may always be obtained, by integration, from the effect of a single concentrated load, is as follows:

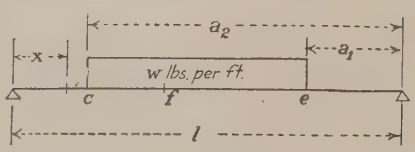


Fig. 271.

Equation (33) gives the slope at any point to the left of a concentrated load distant a from the right support. Hence to find the slope at any point to the left of the distributed load, a is a variable, and P is $w \cdot da$. Hence, *to the left of the distributed load*, at any point x from the support

$$EI\cdot d\alpha = \frac{wa\cdot da}{6l}(l^2-a^2-3x^2)$$

$$\therefore EI\cdot\alpha = \frac{w}{6l}\int_{a_1}^{a_2}(l^2a-a^3-3x^2a)da \text{ } (x \text{ is here constant})$$

$$= \frac{w(a_2^2-a_1^2)}{12l}\left(l^2-3x^2-\frac{a_2^2+a_1^2}{2}\right) \tag{44}$$

this slope being downward toward the right.

Similarly, for the deflection to the left of the load, from Eq. (34).

$$EI \cdot d = \frac{wx}{6l}\int_{a_1}^{a_2}(l^2a - a^3 - x^2a)da$$
$$= \frac{wx(a_2^2 - a_1^2)}{12l}\left(l^2 - x^2 - \frac{a_2^2 + a_1^2}{2}\right). \quad (45)$$

For any point to the right of the distributed load, Eq. (44) will give the slope *downward to the left* if for a_2 and a_1 are substituted respectively $(l - a_1)$ and $(l - a_2)$, and if x is the distance to the left of the right support. Equation (45) will, if changed in the same way, give the deflection.

The equations giving slope and deflection at c (by making $x = l - a_2$ in Eqs. (44) and (45)) will be

$$EI \cdot \alpha_c = \frac{w(a_2^2 - a_1^2)}{24l}(12la_2 - 4l^2 - 7a_2^2 - a_1^2) \quad (46)$$

$$EI \cdot d_c = \frac{w(l - a_2)(a_2^2 - a_1^2)}{24l}(4la_2 - 3a_2^2 - a_1^2). \quad (47)$$

Similarly, the slope and deflection at e are given numerically by substituting in Eqs. (46) and (47) for a_1 and a_2 respectively $(l - a_2)$ and $(l - a_1)$; thus

$$EI \cdot \alpha_e = \frac{w}{24l}(a_2 - a_1)(2l - a_2 - a_1)[2l(a_2 + a_1) - 7a_1^2 - a_2^2] \quad (48)$$

$$EI \cdot d_e = \frac{wa_1}{24l}(a_2 - a_1)(2l - a_2 - a_1)[2l(a_2 + a_1) - 3a_1^2 - a_2^2]. \quad (49)$$

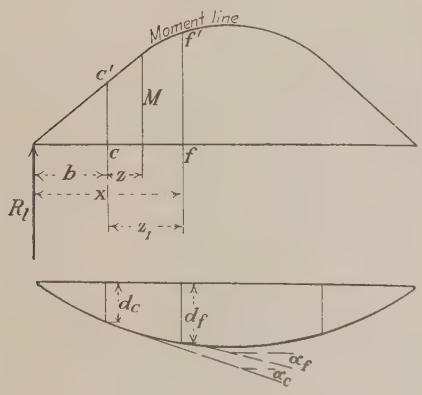


FIG. 272.

The deflection will be downward, but the slope given by Eq. (48) will be *downward to the left*, and hence opposite in sign to that given by Eq. (44).

For any point f between c and e, perhaps the simplest method, avoiding much algebraical manipulation is the following, referring to Fig. 272,

$$EI\alpha_f = EI\alpha_c - \text{area } cc'f'f.$$

The area may be found by integration, taking z as the variable.

$$M = R_l(b + z) - \frac{wz^2}{2}$$

$$\therefore \text{area } cc'f'f = \int_o^{z_1}\left[R_l(b + z) - \frac{wz^2}{2}\right]dz$$
$$= \frac{R_l(b + z_1)^2}{2} - \frac{R_lb^2}{2} - \frac{wz_1^3}{6}$$

Substituting for R_l its value $\frac{w(a_2^2 - a_1^2)}{2l}$ and for z_1 its value $(x - b) = (x - l + a_2)$, and for α_c its value from Eq. (46)

$$EI\alpha_f = \frac{w(a_2^2 - a_1^2)}{24l}[2l^2 - a_2^2 - a_1^2 - 6x^2] + \frac{w}{6}(x - l + a_2)^3. \quad (50)$$

Similarly the deflection at f will be given by the equation

$EI \cdot d_f = EI \cdot d_e + EI \cdot \alpha_c z_1 -$ (statical moment of area $cc'f'f$ about f).

The last term is

$$\int_o^{z_1} Mdz(z_1 - z) = \frac{R_lz_1^2}{6}(z_1 + 3b) - \frac{wz_1^4}{24}. \quad (51)$$

Substituting for R_l and z_1 their values, and the values of d_c and α_c from Eqs. (47) and (46) we find

$$EI \cdot d_f = \frac{w}{24}\left[(x - l + a_2)^4 + \frac{x(a_2^2 - a_1^2)}{l}(2l^2 - 2x^2 - a_2^2 - a_1^2)\right]. \quad (52)$$

For a full loading, that is, if $a_2 = l$ and $a_1 = 0$, Eqs. (50) and (52) reduce to Eqs. (41) and (43) respectively, as they should.

The max. deflection may be found by differentiating Eq. (52), placing the first derivative with respect to x equal to zero and solving for x. If the value of x comes between c and e the point is correctly found; if it falls to the left of c then the result must be discarded, and Eq. (45) used. The process is complicated, and rarely, if ever, needed. The writer would use an approximate method of trial if he found it necessary to solve this problem.

The principle of reciprocity of deflection (Chap. X, Art. 19) is often of use in the case of beams. The deflection at the center due to a load P at a point g (Fig. 269), will be the same as the deflection at g due to a load P at the center, as will be seen by comparing Eqs. (27) and (36).

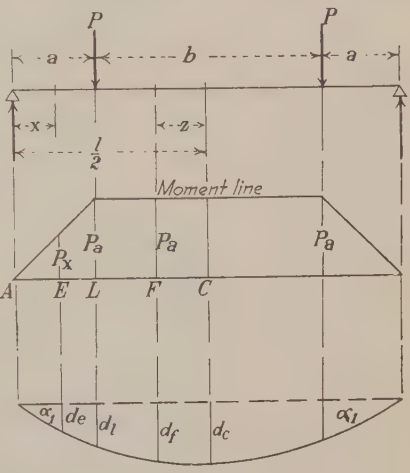


Fig. 273.

12. (*e*) **For a System of Concentrated Loads.**—In this case the effect of each load must be considered separately. Sometimes, in doing this, the principle of reciprocity of deflections is useful. (Note that this does not apply to slope.)

13. (*f*) **Two Symmetrical Loads** (Fig. 273).—This is a very common case, as, for instance, the floor-beam of a railroad bridge, which is loaded by equal and symmetrical loads at the stringer connections.

From the moment line the following results follow directly, and may be written at once:

$$\text{Slope at } C = 0$$

$$\text{Deflection at } C = d_c = \frac{1}{EI}\,(\text{moment of area above } AC \text{ about } A)$$

$$\therefore EI \cdot d_c = Pa\left[\frac{a^2}{3} + \frac{b}{2}\left(a + \frac{b}{4}\right)\right] \tag{53}$$

$$EI \cdot \alpha_f = Paz \tag{54}$$

$$EI \cdot \alpha_l = \frac{Pab}{2} \tag{55}$$

$$EI \cdot \alpha_1 = \frac{Pa}{2}(b + a) \tag{56}$$

$$EI \cdot \alpha_e = \frac{P}{2}[a(b + a) - x^2] \tag{57}$$

$$EI \cdot d_f = EI \cdot d_e - \frac{Paz^2}{2} \tag{58}$$

$$EI \cdot d_l = EI \cdot d_e - \frac{Pab^2}{8} \tag{59}$$

$$EI \cdot d_e = EI\alpha_1 \cdot x - \frac{Px^3}{6} = \frac{Px}{6}[3a(b + a) - x^2]. \tag{60}$$

Problem.—Deduce these results by starting with Eq. (1) and proceeding by integration.

14. (*g*) **With a Negative Moment at the Right End** (Fig. 274).—The elastic curve will be horizontal at some point b such that the moment of abc about a equals the moment of $bced$ about d, or

$$\frac{M_r x_o^3}{3l} = \frac{M_r x_o}{l} \cdot \frac{(l - x_o)^2}{2} + \frac{M_r (l - x_o)^3}{6l} \text{ or } x_o = \frac{l}{\sqrt{3}} \tag{61}$$

$$EI \cdot \alpha_1 = M_r \frac{x_o}{l} \cdot \frac{x_o}{2} = \frac{M_r l}{6} \text{ downward to the left as shown.} \tag{62}$$

At any point $x < x_o$;

$$EI \cdot \alpha = \frac{M_r l}{6} - \frac{M_r x^2}{2l} = \frac{M_r}{6l}(l^2 - 3x^2) \tag{63}$$

(downward to the left).

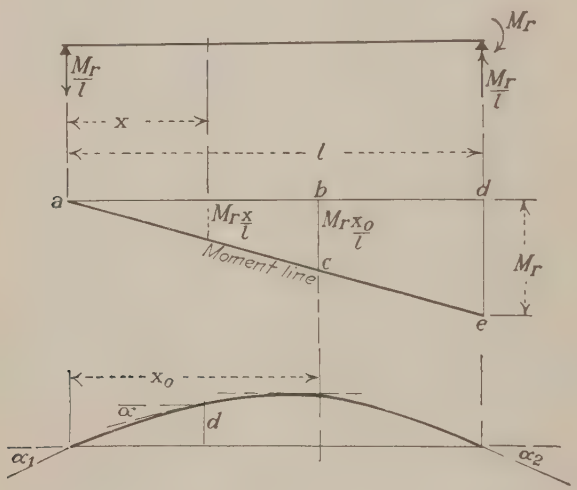


Fig. 274.

At any point $x > x_o$ the slope will be given by Eq. (63) if measured in the same direction, or downward to the left, but it will actually be downward to the right and equal numerically to $\frac{M_r}{6l}(3x^2 - l^2)$.

$$EI\alpha_2 = \frac{M_r l}{3}. \tag{64}$$

The max. deflection d_o will be at $x = x_o$, and will be upward.

$$EI \cdot d_o = \frac{M_r x_o}{l} \cdot \frac{x_o}{2} \cdot \frac{2x_o}{3} = \frac{M_r l^2}{9\sqrt{3}}. \tag{65}$$

At any point $x < x_o$,

$$EI \cdot d = EI \cdot \alpha_1 x - \frac{M_r x}{l} \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{M_r x}{6l}(l^2 - x^2) \text{ (upward).} \tag{66}$$

At any point $x > x_o$ Eq. (66) holds also.

Using Eq. (1), with origin at a, x positive to the right, and y positive upward:

$$\frac{d^2y}{dx^2} = +\frac{M}{EI} = -\frac{M_r x}{lEI}$$

$$EI\cdot\frac{dy}{dx} = -\frac{M_r x^2}{2l} + C$$

$$EI\cdot y = -\frac{M_r x^3}{6l} + Cx + [C_1 = 0]$$

$$\text{At } x = 0,\ y = 0,\ \therefore C_1 = 0$$

$$\text{At } x = l,\ y = 0,\ \therefore C = \frac{M_r l}{6}$$

$$\therefore EI\cdot\frac{dy}{dx} = \frac{M_r}{6l}(l^2 - 3x^2) \text{ as in (63)}$$

$$EI\cdot y = \frac{M_r x}{6l}(l^2 - x^2) \text{ as in (66)}$$

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{\sqrt{3}} \text{ as in (61).}$$

When $x < \frac{l}{\sqrt{3}}$, $\frac{dy}{dx}$ is positive.

When $x > \frac{l}{\sqrt{3}}$, $\frac{dy}{dx}$ is negative.

$$\text{Max. } y, \text{ for } x = \frac{l}{\sqrt{3}} = d_o = \frac{M_r l^2}{9\sqrt{3}EI}, \text{ as in (65).}$$

The procedure by the use of Eq. (1) is sometimes the simplest. Like most if not all analytical processes, however, it is less obvious, concrete, and tangible than a geometrical or direct process, consisting as it does in first inserting correct data in the mathematical machine, and then merely turning the crank. The student, however, should have complete command of all methods.

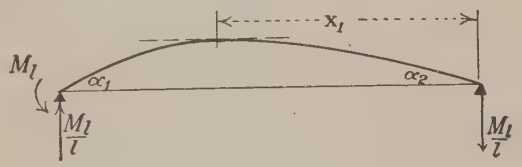


FIG. 275.

With a Negative Moment at the Left End (Fig. 275).—There will be a slope at the left end, downward to the left, α_1;

$$EI\cdot\alpha_1 = \frac{M_l l}{3}. \tag{64a}$$

and at the right end, downward to the right, α_2;

$$EI\cdot\alpha_2 = \frac{M_l l}{6}. \tag{62a}$$

The slope will be zero at $x_1 = \frac{l}{\sqrt{3}}$. (61a)

At any point on the span the slope will be a slope downward to the right equal algebraically to α, x being the distance from the right end,

$$EI\cdot\alpha = \frac{M_l}{6l}(l^2 - 3x^2). \tag{63a}$$

When $x > \frac{l}{\sqrt{3}}$ this becomes negative, or actually downward to the left.

At any point on the span the upward deflection will be

$$EI \cdot d = \frac{M_1 x_1}{6l}(l^2 - x^2). \tag{66a}$$

15. (*h*) **With a Moment (Couple) Applied Anywhere** (Fig. 276).—Refer to the remarks in §2 of Chap. XI with reference to the application of a moment or couple at a point. In this ideal case there is a sudden change in the moment at the point where M_2 is applied, and, with the sense of M_2 as indicated, the moment to the left is negative (tension in top fibers) and to the right positive (compression in top fibers).

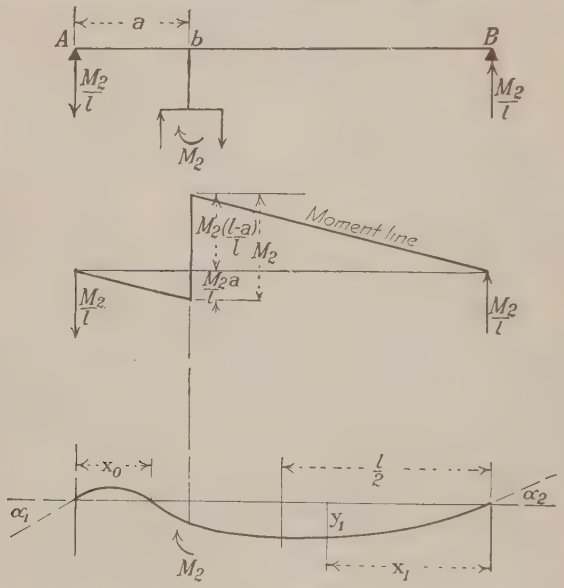


Fig. 276.

This case will be solved by using Eq. (1). Take the origin at A, x positive to the right, and y positive upward; then to the left of b

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{M_2 x}{l}.$$

A positive moment M_2 will cause the slope from A to b to decrease as x increases.

$$EI \cdot \frac{dy}{dx} = -\frac{M_2 x^2}{2l} + C$$

$$EI \cdot y = -\frac{M_2 x^3}{6l} + Cx + [C_1 = 0].$$

For $x = 0$; $y = 0$; $\therefore C_1 = 0$.

To the right of b, take the origin at B, x_1 positive to the left, and y_1 positive *downward*. The curve will be concave upward, with compression in upper fibers, and the slope will decrease as x_1 increases; hence

$$EI \cdot \frac{d^2y_1}{dx_1^2} = -\frac{M_2 x_1}{l}$$

$$EI \cdot \frac{dy_1}{dx_1} = -\frac{M_2 x_1^2}{2l} + C'$$

$$EI \cdot y_1 = -\frac{M_2 x_1^3}{6l} + C'x_1 + [C'' = 0].$$

For $x_1 = 0$, $y_1 = 0$; $\therefore C'' = 0$.

For $x = a$ and $x_1 = l - a$; $\frac{dy}{dx} = \frac{dy_1}{dx_1}$; and $y = -y_1$

$$\therefore -\frac{M_2 a^2}{2l} + C = -\frac{M_2(l-a)^2}{2l} + C'$$

$$-\frac{M_2 a^3}{6l} + Ca = \frac{M_2(l-a)^3}{6l} - C'(l-a)$$

$$\therefore C' = \frac{M_2}{6l}(l^2 - 3a^2)$$

$$C = \frac{M_2}{6l}(6al - 2l^2 - 3a^2).$$

Hence, finally,
To the left of the applied couple (y positive upward),

$$EI \cdot \frac{dy}{dx} = \frac{M_2}{6l}(6al - 2l^2 - 3a^2 - 3x^2) \tag{67}$$

when $\frac{dy}{dx}$ is positive the slope being upward to the right.

$$EI \cdot y = \frac{M_2 x}{6l}(6al - 2l^2 - 3a^2 - x^2). \tag{68}$$

To the right of the applied couple (y_1 positive downward)

$$EI \cdot \frac{dy_1}{dx_1} = \frac{M_2}{6l}(l^2 - 3a^2 - 3x_1^2) \tag{69}$$

when $\frac{dy_1}{dx_1}$ is positive the slope being downward to the left.

$$EI \cdot y_1 = \frac{M_2 x_1}{6l}(l^2 - 3a^2 - x_1^2). \tag{70}$$

At the applied couple ($x = a$ in Eqs. (67) and (68));

$$EI \cdot \frac{dy}{dx} = \frac{M_2}{6l}(6al - 2l^2 - 6a^2) \tag{71}$$

$$EI \cdot y = \frac{M_2 a}{3l}(3al - l^2 - 2a^2). \tag{72}$$

Equation (71) is always negative, since $3a(l-a) < l^2$; for the product $a(l-a)$ of the two parts of a whole, l, is a maximum when the two parts are equal, or $a = \frac{l}{2}$ in which case $3a(l-a)$ is $\frac{3}{4}l^2$. Hence the elastic line always slopes downward to the right at the applied couple if M_2 is right-handed, and *vice versa;* or, as must be true *a priori;* the applied couple must rotate the axis, at the point of application, in its own sense. When $a = l - a = \frac{l}{2}$, C must equal C'.

From Eq. (72), $y = 0$ when $a = l$ or $\frac{l}{2}$, both of which results are obvious, the latter from the principle of symmetry. Further, y is negative if $a < \frac{l}{2}$ and positive if $a > \frac{l}{2}$; *i.e.*, a right-handed moment M_2 lowers its point of application if applied to the left of the center, and raises it if to the right of the center; while a left-handed moment M_2 raises its point of application if to the left of the center and lowers it if to the right. Hence with M_2 right-handed and to the $\left\{ \begin{matrix} \text{left} \\ \text{right} \end{matrix} \right\}$ of the center, the point of zero deflection will be on the $\left\{ \begin{matrix} \text{left} \\ \text{right} \end{matrix} \right\}$ of the point of application, or always

in the short segment of the span; and with a left-handed M_2 this last result is also true.

At the center of the span $\left(x_1 = \frac{l}{2}\right.$ in Eqs. (69) and (70) assuming $a < \left.\frac{l}{2}\right)$

$$EI \cdot \alpha_c = EI \cdot \frac{dy_1}{dx_1} = \frac{M_2}{24l}(l^2 - 12a^2) \tag{73}$$

$$EIy_c = \frac{M_2}{48}(3l^2 - 12a^2) \tag{74}$$

since $a < \frac{l}{2}$, y_c is positive, or downward, if M_2 is right-handed. This follows from the result just found, that a positive M_2 lowers its point of application if to the left of the center, and the curve is below AB from the point of application to B. The slope at the center may be in either direction, depending upon the value of a, but will always be in the same direction as the slope at the load.

The curve will be horizontal, to the right of the couple (from Eq. (69)) when

$$x_1 = x_1' = \sqrt{\frac{l^2 - 3a^2}{3}}. \tag{75}$$

The deflection downward at this point will be (Eq. (70))

$$\text{Max. } EI \cdot y_1 = \frac{M_2}{3l}\left(\frac{l^2 - 3a^2}{3}\right)^{3/2}. \tag{76}$$

The curve will be horizontal, to the left of the couple (from Eq. (67)) when

$$x = x' = \sqrt{\frac{6al - 2l^2 - 3a^2}{3}}. \tag{77}$$

The deflection upward at this point will be (Eq. 68))

$$\text{Max. } EI \cdot y = \frac{M_2}{3l}\left(\frac{6al - 2l^2 - 3a^2}{3}\right)^{3/2}. \tag{78}$$

The deflection will be zero at the left of the couple, if $a < \frac{l}{2}$, that is to say in the shorter segment, when (Eq. (68))

$$x = x_o = \sqrt{6al - 2l^2 - 3a^2}. \tag{79}$$

The slope at the left end, α_1, ($x = 0$ in Eq. (67)), is

$$EI \cdot \alpha_1 = \frac{M_2}{6l}(6al - 2l^2 - 3a^2) \tag{80}$$

which is always positive (*i.e.*, upward to the right) if M_2 is positive.

The slope at the right end, α_2, ($x_1 = 0$ in Eq. (69)) is

$$EI \cdot \alpha_2 = \frac{M_2}{6l}(l^2 - 3a^2) \tag{81}$$

which is always positive (*i.e.*, downward to the left) if M_2 is positive.

The axis of the elastic line slopes in the same direction at each end, as is easily seen must be the case.

If $a = l$, Eqs. (81) and (64) agree, as they should.

16. (*i*) **With a Negative Couple at Each End** (Fig. 277).—This case may be deduced from Case (*g*), or independently.

The curve will be horizontal at some point b, distant x_o from A, such that the moment about A of the area $Acc'A'$ equals the moment about B of the area $cBB'c'$, or

$$\frac{M_r x_o^3}{3l} + \frac{M_l(l - x_o)x_o^2}{2l} + \frac{M_l x_o^3}{6l} = \frac{M_l(l - x_o)^3}{3l} + \frac{M_r x_o(l - x_o)^2}{2l} + \frac{M_r(l - x_o)^3}{6l}$$

$$3M_l x_o^2 - 3M_r x_o^2 - 6M_l l x_o + 2M_l l^2 + M_r l^2 = 0$$

from which

$$x_o = -\frac{M_l l}{M_r - M_l} + \sqrt{\frac{(2M_l + M_r)l^2}{3(M_r - M_l)} + \frac{M_l^2 l^2}{(M_r - M_l)^2}} \tag{82}$$

when $M_l = 0$, this reduces to Eq. (61).

When $M_l = M_r$, Eq. (82) reduces to $-\infty$, $+\infty$, or is indeterminate; but the equation preceding reduces to $x_o = \frac{l}{2}$, as it should.

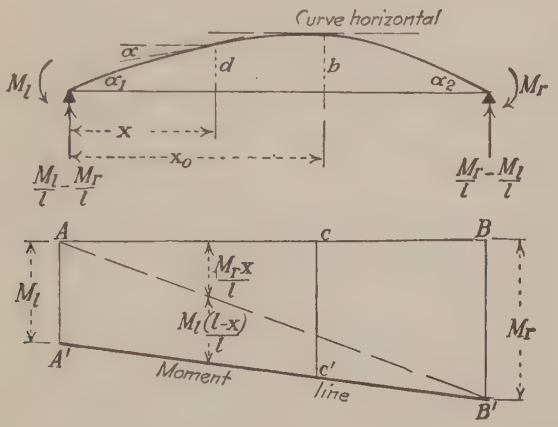


Fig. 277.

The remaining results may be derived most easily from Case (g); thus, numerically, with M_l and M_r both negative moments as shown, but used numerically in the following equations,

$$EI \cdot \alpha_1 \text{ (downward to the left)} = \frac{M_r l}{6} + \frac{M_l l}{3} \tag{83}$$

$$EI \cdot \alpha_2 \text{ (downward to the right)} = \frac{M_r l}{3} + \frac{M_l l}{6} \tag{84}$$

$$EI \cdot \alpha \text{ (downward to the left)} = \frac{M_r}{6l}(l^2 - 3x^2) + \frac{M_l}{6l}[3(l-x)^2 - l^2] \tag{85}$$

$$= \frac{M_r}{6l}(l^2 - 3x^2) + \frac{M_l}{6l}(2l^2 - 6lx + 3x^2) \tag{86}$$

$$EI \cdot d \text{ (upward)} = \frac{M_r x(l^2 - x^2)}{6l} + \frac{M_l(l-x)}{6l}[l^2 - (l-x)^2]$$

$$= \frac{M_r x(l^2 - x^2)}{6l} + \frac{M_l x(l-x)}{6l}(2l - x). \tag{87}$$

The max. deflection will be at x_o, and can be found by substituting in Eq. (87) the value of x_o from Eq. (82).

17. (j) **Beam Overhanging at One End** (Fig. 278).—The portion of the beam between the supports is precisely the same as Case II(g), the moment M_2 at the right end being a right-handed moment equal to $P_2 a_2$. The equations of that case apply directly and need not be repeated.

The portion of the beam which overhangs is the same as Case I(a), except that the beam is not fixed horizontally at the left end, but slopes downward to the right at an angle (Case II(g))

$$EI \cdot \alpha_2 = \frac{M_2 l}{3} = \frac{P_2 a_2 l}{3}.$$

The slopes in this part of the beam, as given by the equations of Case I(a) must therefore be increased by α_2, and the deflection by $\alpha_2 x$. Thus, the slope at the free end will be

$$\alpha_3 = \alpha_2 + \frac{P_2 l^2}{2EI},$$

or

$$EI \cdot \alpha_3 = \frac{P_2 l}{6}(3l + 2a_2) \tag{88}$$

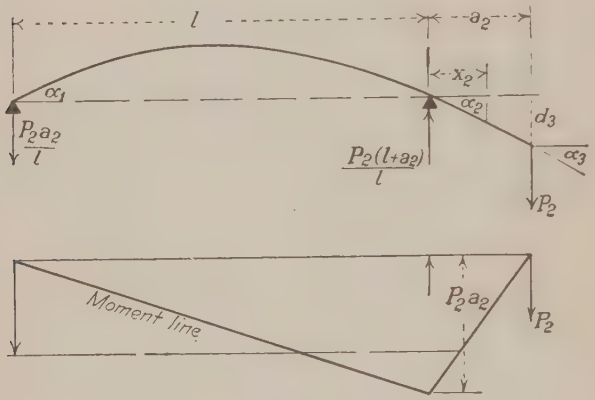


Fig. 278.

and the deflection,

$$EI \cdot d_3 = \frac{P_2 l}{3}(a_2^2 + l^2). \tag{89}$$

18. (k) **Beam Overhanging at Each End.**—*If the overhangs are equal, with equal concentrated loads at the ends,* this case is very simple (Fig. 279). The elastic line

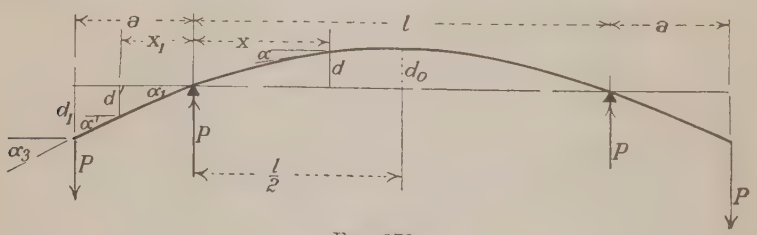


Fig. 279.

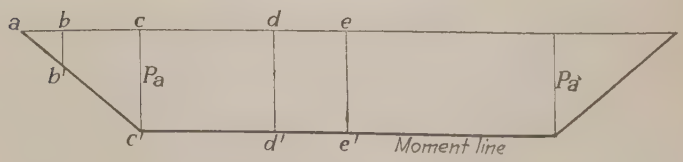


Fig. 279b.

will be horizontal at the center of l, and the slopes and deflections may be written down at once from the moment line. Thus

$$EI \cdot \alpha_1 = \text{area } cc'e'e = \frac{Pal}{2} \tag{90}$$

$$EI \cdot \alpha' = \text{area } bb'c'e'e = \frac{Pal}{2} + Pax_1 - \frac{Px_1^2}{2} \tag{91}$$

$$EI \cdot \alpha_3 = \text{area } ac'e'e = \frac{Pal}{2} + \frac{Pa^2}{2} = \frac{Pa(l+a)}{2} \tag{92}$$

$$EI \cdot \alpha = \text{area } dd'e'e = \frac{Pa}{2}(l - 2x) \tag{93}$$

$$EI \cdot d_o = \text{moment of area } cc'e'e \text{ about } c = \frac{Pal^2}{8} \tag{94}$$

$$EI \cdot d = d_o - \text{moment of area } dd'e'e' \text{ about } d = \frac{Pax(l-x)}{2} \tag{95}$$

$$EI \cdot d' = EI \cdot \alpha_1 x_1 + \text{moment of } bb'c'c \text{ about } b$$

$$= \frac{Pal}{2} \cdot x_1 + \frac{Pax_1^2}{2} - \frac{Px_1^3}{6} = \frac{Px_1}{6}(3al + 3ax_1 - x_1^2) \tag{96}$$

$$EI \cdot d_1 = \frac{Pa^2}{6}(3l + 2a). \tag{97}$$

With unequal overhangs and loads (Fig. 280) the case may be solved at once from Cases (*i*) and (*j*). If there is only a load at the right end, the case is the same as (*j*),

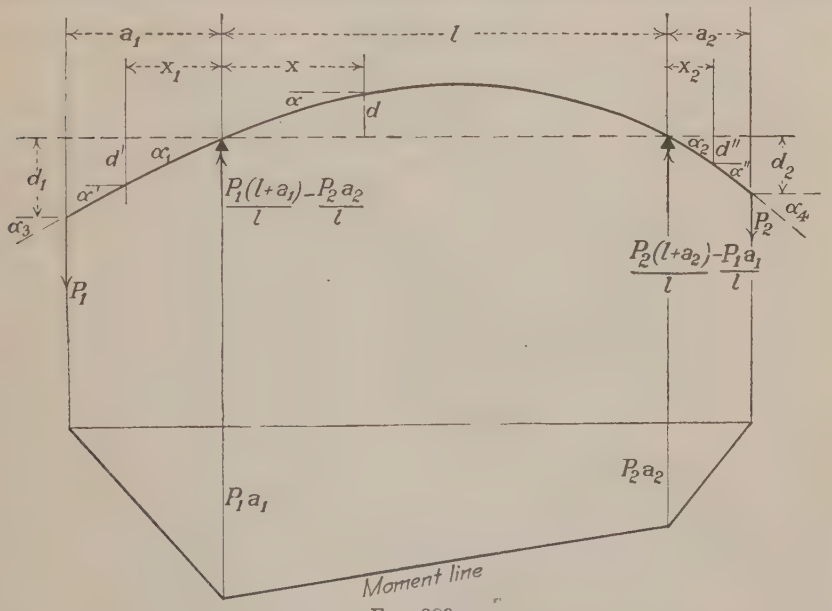


FIG. 280.

the beam under a_1 being straight and inclined downward to the left at an angle $\alpha_1 = \frac{P_2 a_2 l}{6EI}$.

If there are unequal loads at each end, then the case is the same as II(*i*), in which M_l is a left-handed moment $P_1 a_1$ and M_r is a right-handed moment $P_2 a_2$. The following results may be written down at once.

Under l:

$$EI \cdot \alpha \text{ (upward to the right)} = \frac{P_2 a_2}{6l}(l^2 - 3x^2) - \frac{P_1 a_1}{6l}[l^2 - 3(l-x)^2] \tag{98}$$

$$EI \cdot d \text{ (upward)} = \frac{P_2 a_2 x}{6l}(l^2 - x^2) + \frac{P_1 a_1 x(l-x)}{6l}(2l - x). \tag{99}$$

Under a_1:

$$EI \cdot \alpha_1 = \frac{l}{6}(P_2a_2 + 2P_1a_1) \tag{100}$$

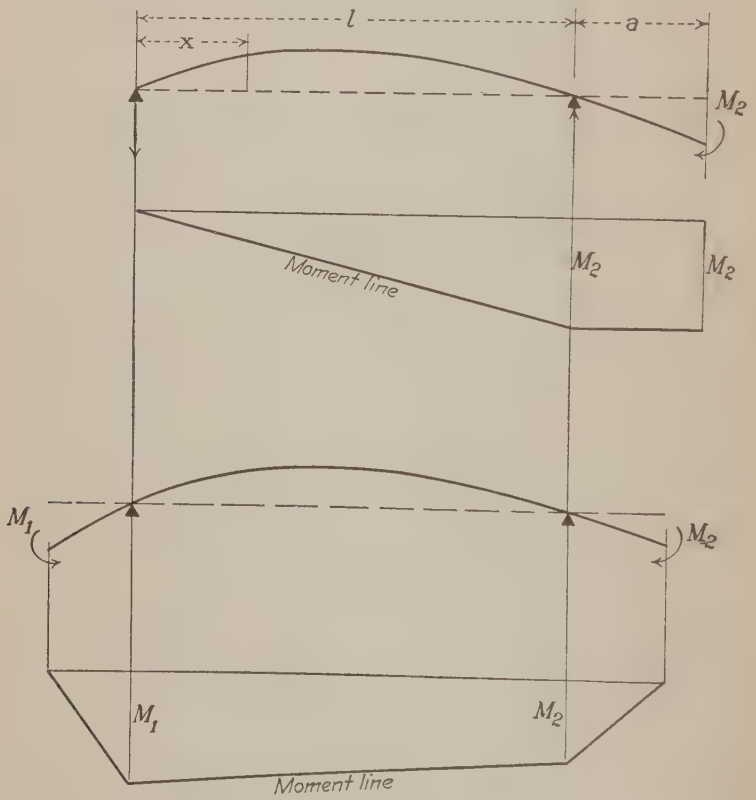
$$EI \cdot \alpha' = \frac{l}{6}(P_2a_2 + 2P_1a_1) + \frac{P_1x_1(2a_1 - x_1)}{2} \tag{101}$$

$$EI \cdot \alpha_3 = \frac{l}{6}(P_2a_2 + 2P_1a_1) + \frac{P_1a_1^2}{2} \tag{102}$$

$$EI \cdot d' = \frac{x_1}{6}[P_2a_2l + P_1a_1(2l + 3x_1) - P_1x_1^2] \tag{103}$$

$$EI \cdot d_1 = \frac{a_1}{6}[P_2a_2l + 2P_1a_1(l + a_1)] \tag{104}$$

and similarly under a_2.



FIGS. 281 and 282.

19. (*l*) **Overhanging Beam with Moment at End** (Fig. 281).—The part between the supports is the same as Case II(*g*) and the part overhanging is the same as Case I(*e*) except that the fixed end has an inclination $\frac{M_2l}{3EI}$.

If the beam overhangs at each end (Fig. 282), with a moment at each end, the case is similarly a combination of Cases I(*e*) and II(*i*).

20. (*m*) As an illustration of the ease with which deflection may be found by area moments, consider a beam loaded with equal loads at two points as shown

in Fig. 283. The moment line is shown and the deflection at the center (times EI) is the moment of half the total moment area about one end, or

$$EI \cdot \partial = \frac{Wl}{3} \cdot \frac{l}{3} \cdot \frac{1}{2} \cdot \frac{l}{3} \cdot \frac{2}{3} + \frac{Wl}{3} \cdot \frac{l}{6}\left(\frac{l}{3} + \frac{l}{12}\right)$$

$$= \frac{23}{648} Wl^3.$$

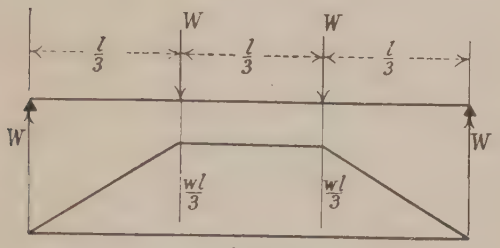


FIG. 283.

CASE III. BEAM SUPPORTED AT ONE END AND FIXED AT THE OTHER

21. A beam is termed "supported" at one end when there is a reaction but no moment at that end; that is to say, when that end is entirely free to rotate without restraint under the action of the applied loads.

If there is a moment at the end, in a direction which tends to restrain the rotation that would otherwise occur, the beam may be said to be more or less fixed at that end. The word *fixed*, however, when used without qualification, is understood to mean that the end of the beam is *prevented from rotating at all*, that is, that the axis remains in its original direction. Such beams are also called "built-in" beams, or beams "encastré."

A beam fixed at one or both ends, may be considered first as supported, and then a moment may be considered to be applied which will bring the axis back to its original direction.

These cases may therefore all be solved by means of the results already derived.

22. (a) **With a Single Concentrated Load.**—Let a concentrated load P act anywhere, its distance from the (right-hand) fixed support being a (Fig. 284). The moment diagram for the concentrated load alone, with both ends simply supported, is bda. That for the moment M_2 alone is bae, the moments due to M_2 being all negative. By laying off $ac = ae$ and drawing ac, the line bda becomes the moment line referred to bc as a zero line; *i.e.*, the vertical distance between bc and bda at any point is algebraically the value of the moment at that point. The point of inflexion, where the moment is really zero, is i. Unless the line bc lies above bd,[1] the point of inflexion is found from the condition that $ig = gi'$ or

$$\frac{Px_1(l-a)}{l} = \frac{M_2(l-x_1)}{l}$$

[1] If the line bc should lie above bd it would mean that the reaction at b were downward. This would of course be impossible if the moment at a is simply caused by the resistance of the wall to the tendency of the beam to rotate, for this resistance could clearly never be so great as to lift the beam from the support at b. If the moment at a were due to some outside action exerted on the beam, by applied forces or by an adjacent span, and not simply a passive resistance, the beam might be lifted from the support b.

or, since x_1 becomes x_o,

$$x_o = \frac{M_2 l}{P(l-a) + M_2}. \tag{105}$$

Equation (30) gives the slope at a, due to P alone on a supported beam *downward* to the *left*, as $\frac{Pa(l-a)(2l-a)}{6lEI}$; and Eq. (64) gives the slope at a due to the moment M_2 alone on a supported beam, *upward* to the *left*, as $\frac{M_2 l}{3EI}$. Since the actual slope at a is zero

$$\frac{Pa(l-a)(2l-a)}{6l} = \frac{M_2 l}{3},$$

or

$$M_2 = \frac{Pa(l-a)(2l-a)}{2l^2}. \tag{106}$$

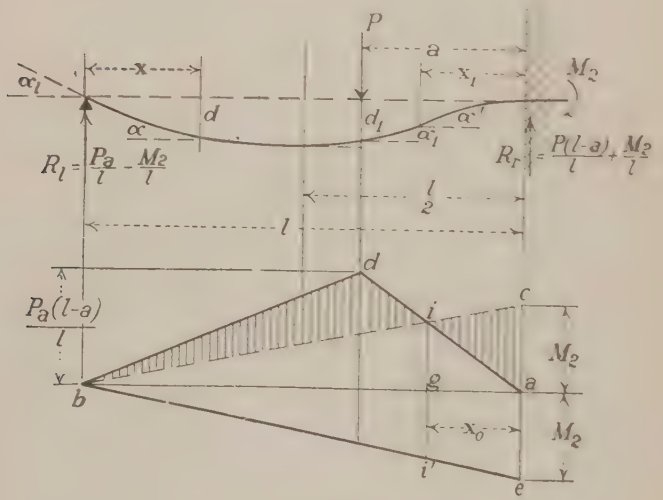


FIG. 284.

Substituting this in Eq. (105),

$$x_o = \frac{al(2l-a)}{2l^2 + a(2l-a)}. \tag{107}$$

The value of the reactions will be

$$R_l = \frac{Pa^2(3l-a)}{2l^3} \tag{108}$$

$$R_r = \frac{P(l-a)(2l^2+2al-a^2)}{2l^3}. \tag{109}$$

At any point between the load P and the left support, Eq. (33) gives the slope due to P, *downward* to the right, as $\frac{Pa(l^2-a^2-3x^2)}{6lEI}$; and Eq. (63) gives the slope at any point of the beam, due to M_2, *upward* to the right, as $\frac{M_2(l^2-3x^2)}{6lEI}$.

Hence the actual slope to the left of P, at a distance x from b, downward to the right, is

$$EI \cdot \alpha = \frac{Pa(l^2 - a^2 - 3x^2) - M_2(l^2 - 3x^2)}{6l}.$$

Substituting for M_2 its value

$$EI \cdot \alpha = \frac{Pa^2[(l - a)(l^2 - x^2) - 2lx^2]}{4l^3}. \tag{110}$$

The slope at the left support ($x = 0$) is given by

$$EI \cdot \alpha_l = \frac{Pa^2(l - a)}{4l}. \tag{111}$$

At any point between P and the right support, distant x_1 from the latter, the slope due to P alone, downward to the left, is by Eq. (37), $\frac{P(l-a)(2al - a^2 - 3x_1^2)}{6lEI}$, and Eq. (63) gives the slope due to M_2, in the same direction, but for x it is necessary to substitute $(l - x)$, hence the real slope, upward to the right, will be given by

$$EI \cdot \alpha' = \frac{P(l - a)(2al - a^2 - 3x_1^2) + M_2[l^2 - 3(l - x_1)^2]}{6l}.$$

Substituting for M_2 its value

$$EI \cdot \alpha' = \frac{Px_1(l - a)}{4l^3}[a(2l - a)(2l - x_1) - 2l^2x_1]. \tag{112}$$

The slope *at the load* (α_1) downward to the left, is found by making $x_1 = a$ in Eq. (112), or

$$\begin{aligned} EI \cdot \alpha_1 &= \frac{Pa^2(l - a)}{4l^3}[2l^2 + a^2 - 4al]. \\ &= \frac{Pa^2(l - a)}{4l^3}[2l(l - 2a) + a^2]. \end{aligned} \tag{113}$$

The same result, with opposite sign, will be obtained by making $x = l - a$ in Eq. (110). The slope will be zero (beam horizontal) at the load when

$$a = l(2 - \sqrt{2}) = 0.586l. \tag{114}$$

When a is less than this value the point of max. deflection and zero slope will be to the left of the load, and *vice versa.*

Between the left support and P_1 from Eqs. (34) and (66),

$$\begin{aligned} EI \cdot d &= \frac{Pax}{6l}(l^2 - a^2 - x^2) - \frac{M_2x}{6l}(l^2 - x^2) \\ &= \frac{Pax}{12l^3}[3al(l^2 - al - x^2) + a^2x^2] \\ &= \frac{Pa^2x}{12l^3}[3l^2(l - a) - x^2(3l - a)]. \end{aligned} \tag{115}$$

Between the right support and P from Eqs. (38) and (66), substituting in (66) $l - x_1$ for x,

$$EI \cdot d' = \frac{Px_1^2(l - a)}{12l^3}[3al(2l - a) - x_1(2l^2 + 2al - a^2)]. \tag{116}$$

If $a < 0.586l$, the max. deflection is to the left of P, and placing the first derivative from Eq. (115) equal to zero, it is found to be at a distance from the left support of

$$x_o = l\sqrt{\frac{l-a}{3l-a}}, \tag{117}$$

and the value of this deflection is (from Eq. (115) with $x = x_o$)

$$\text{Max. } EI \cdot d_o = \frac{Pa^2(l-a)}{6}\sqrt{\frac{l-a}{3l-a}}. \tag{118}$$

The deflection at the load ($x = l - a$ in Eq. (115) or $x_1 = a$ in Eq. (116)) is

$$EI \cdot d_1 = \frac{Pa^3(l-a)^2}{12l^3}(4l-a). \tag{119}$$

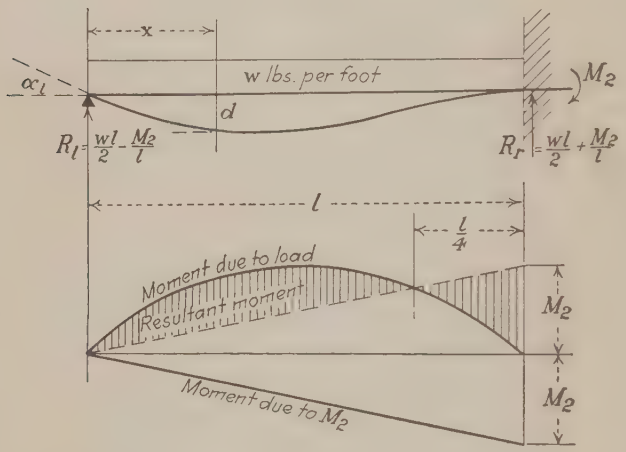


Fig. 285.

23. (*b*) **With a Uniformly Distributed Load over the Entire Span** (Fig. 285).—From Eq. (41), the slope, *downward to the right*, due to the distributed load alone on a supported beam, at a distance x from the left end, is

$$EI \cdot \alpha = \frac{w}{24}(l^3 - 6lx^2 + 4x^3)$$

and the *downward* deflection is, from Eq. (43),

$$EI \cdot d = \frac{wx(l-x)}{24}(l^2 + lx - x^2).$$

The effect of the moment M_2 alone, on a supported beam, is to cause a slope, *upward to the right*, by Eq. (63), of

$$EI \cdot \alpha = \frac{M_2}{6l}(l^2 - 3x^2)$$

and a deflection, *upward*, by Eq. (66), of

$$EI \cdot d = \frac{M_2 x}{6l}(l^2 - x^2).$$

The slope at the right end is ($x = l$).

Due to distributed load: $EI \cdot \alpha = \frac{wl^3}{24}$ downward to the left:

Due to moment M_2: $EI \cdot \alpha = \frac{M_2 l}{3}$ downward to the right.

Hence, since the actual slope at the right end is zero

$$\frac{M_2 l}{3} = \frac{wl^3}{24}$$

$$M_2 = \frac{wl^2}{8} \tag{120}$$

$$R_l = \tfrac{3}{8}wl \tag{121}$$

$$R_r = \tfrac{5}{8}wl \tag{122}$$

also,

$$EI \cdot \alpha = \frac{w(l^3 - 6lx^2 + 4x^3)}{24} - \frac{M_2(l^2 - 3x^2)}{6l}$$

$$= \frac{w}{48}(l^3 - 9lx^2 + 8x^3) \tag{123}$$

$$EI \cdot d = \frac{wx(l - x)(l^2 + lx - x^2)}{24} - \frac{M_2 x(l^2 - x^2)}{6l}$$

$$= \frac{wx(l - x)}{48}(l^2 + lx - 2x^2). \tag{124}$$

The max. deflection ($\alpha = 0$) will occur at the point where $l^3 - 9lx^2 + 8x^3 = 0$, or

$$x = 0.41l \text{ (approx.)}. \tag{125}$$

The max. deflection will be,

$$\text{Max. } EI \cdot d_o = 0.0054\, wl^4. \tag{126}$$

Problem.—Deduce the results of Cases (*a*) and (*b*) by considering this case as that of a beam fixed at the right end.

24. (*c*) **Partial Uniform Load.**—This case may be solved by considering it as a combination of two cases previously treated. Thus for a load extending from the

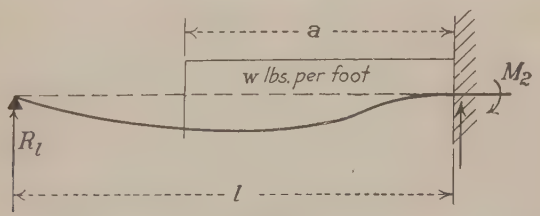


FIG. 286.

fixed support for a distance a (Fig. 286), it is a combination of Cases I(a) and I(d). The deflection at the left end, which must be zero, will give the expression, from Eqs. (14), (12), and (7),

$$0 = \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l - a) - \frac{R_l l^3}{3EI}$$

$$\therefore R_l = \frac{wa^3(4l - a)}{8l^3} \tag{127}$$

$$M_2 = \frac{wa^2}{2} - R_l l = \frac{wa^2}{8l^2}(4l^2 - 4al + a^2). \tag{128}$$

From Eqs. (4) and (11) the equation for slope under the uniform load may be obtained, by making the proper substitutions; and similarly for other cases.

CASE IV. BEAM FIXED AT BOTH ENDS

Both ends are supposed rigidly fixed, so that the rotation there is zero. With downward loads, the moment at each end will be negative.

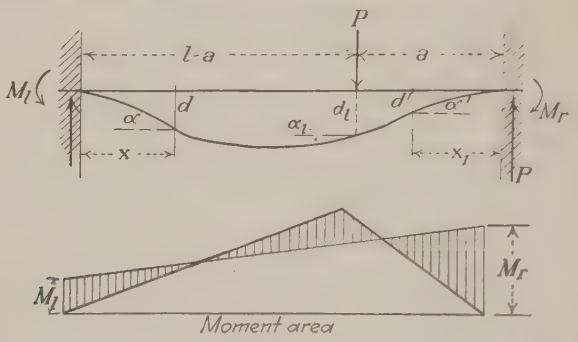


FIG. 287.

25. (*a*) **With a Single Concentrated Load** (Fig. 287).—From Eqs. (29) and (83) (using the values of M_r and M_l numerically),

$$\frac{Pa(l^2 - a^2)}{6l} = \frac{M_r l}{6} + \frac{M_l l}{3}. \tag{129}$$

From Eqs. (30) and (84)

$$\frac{Pa(l - a)(2l - a)}{6l} = \frac{M_r l}{3} + \frac{M_l l}{6}. \tag{130}$$

From which the negative moments at the ends are,

$$\text{(negative)}\quad M_l = \frac{Pa^2(l - a)}{l^2} \tag{131}$$

$$\text{(negative)}\quad M_r = \frac{Pa(l - a)^2}{l^2}. \tag{132}$$

The value of M_r may be obtained from the value of M_l by substituting $(l - a)$ for a.

From Eqs. (33) and (86) slope between M_l and P (downward to the right) $= \alpha$;

$$EI \cdot \alpha = \frac{Pa(l^2 - a^2 - 3x^2)}{6l} - \frac{M_r}{6l}(l^2 - 3x^2) - \frac{M_l}{6l}(2l^2 - 6lx + 3x^2)$$

$$= \frac{Pa^2x}{2l^3}[2l(l - a) - x(3l - 2a)]. \tag{133}$$

Slope between P and $M_r = \alpha'$ (downward to the left);

$$EI \cdot \alpha' = \frac{P(l - a)}{6l}(2al - a^2 - 3x_1^2) + \frac{M_r}{6l}(l^2 - 3x^2) - \frac{M_l}{6l}(l^2 - 3x_1^2)$$

$$= \frac{P(l - a)(2al - a^2 - 3x_1^2)}{6l} + \frac{M_r}{6l}[l^2 - 3(l - x_1)^2] - \frac{M_l}{6l}(l^2 - 3x_1^2)$$

$$= \frac{P(l - a)^2x_1}{2l^3}[2al - x_1(l + 2a)]. \tag{134}$$

Equation (134) is obtained also from (133) by substituting x_1 for x, and $(l - a)$ for a.

Deflection between M_l and P (downward) = d; from Eqs. (34) and (87),

$$EI \cdot d = \frac{Pax(l^2 - a^2 - x^2)}{6l} - \frac{M_r x(l^2 - x^2)}{6l} - \frac{M_l x(l - x)(2l - x)}{6l}$$

$$= \frac{Pax(l^2 - a^2 - x^2)}{6l} - \frac{Pa(l - a)^2}{l^2} \cdot \frac{x(l^2 - x^2)}{6l} - \frac{Pa^2(l-a)}{l^2} \cdot \frac{x(l-x)(2l - x)}{6l}$$

$$= \frac{Pa^2x^2}{6l^3}[3l(l - a) - x(3l - 2a)]. \tag{135}$$

At the center of the span, with $x = \frac{l}{2}$ in (133) and (135),

$$EI \cdot \alpha_c = \frac{Pa^2}{8l}(l - 2a) \tag{133a}$$

$$EI \cdot d_c = \frac{Pa^2(3l - 4a)}{48}. \tag{135a}$$

Deflection between P and M_r (downward) = d'; substitute in (135) x_1 for x, and $(l - a)$ for a:

$$EI \cdot d' = \frac{P(l - a)^2x_1^2}{6l^3}[3al - x_1(l + 2a)]. \tag{136}$$

The slope at the load, α_l, is given by ($x_1 = a$ in (134))

$$EI \cdot \alpha_l = \frac{P(l - a)^2a^2}{2l^3}(l - 2a). \tag{137}$$

If $a < \frac{l}{2}$, the slope is downward toward the center of the beam.

The deflection at the load, d_l (from (136))

$$EI \cdot d_l = \frac{P(l - a)^3a^3}{3l^3}. \tag{138}$$

The slope at the center $\left(a < \frac{l}{2}, x = \frac{l}{2} \text{ in (133)}\right)$

$$EI \cdot \alpha_c = \frac{Pa^2}{8l}(l - 2a). \tag{139}$$

This is positive if $a < \frac{l}{2}$ hence, *at the center the slope is downward toward the load.* The max. deflection therefore occurs between the load and the center. To find its position put (133) = 0, and we find

$$x = \frac{2l(l - a)}{3l - 2a}. \tag{140}$$

The max. deflection occurs at this point, and is (135)

$$EI \cdot d_{\max.} = \frac{2Pa^2(l - a)^3}{3(3l - 2a)^2}. \tag{141}$$

If the load is at the center $\left(a = \frac{l}{2}\right)$ the following are the results:

$$EI \cdot \alpha = \frac{Px(l - 2x)}{8} \tag{142}$$

$$EI \cdot d = \frac{Px^2(3l - 4x)}{48} \tag{143}$$

$$EI \cdot d_l = \frac{Pl^3}{192} \tag{144}$$

or 0.25 of its value for a beam supported at the ends.[1]

26. (*b*) **With a Uniformly Distributed Load** (Fig. 288).—Here the negative moments $M_l = M_r = M_o$. Some results are obtained most easily from the diagram.

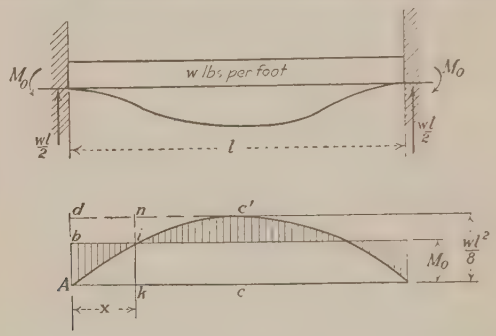


FIG. 288.

Since the slope is zero at the ends and at the center, the moment area between center and one end equals zero, *i.e.*,

$$M_o l = \frac{2}{3} l \cdot \frac{wl^2}{8}; \quad \therefore M_o = \frac{wl^2}{12}. \tag{145}$$

The point of inflexion, where $M = 0$, and the elastic line changes its direction of curvature, is at k.

$$ik = \frac{2}{3} cc'; \quad \therefore in = \frac{1}{3} \cdot Ad$$

$$\therefore \frac{in}{Ad} = \frac{1}{3} = \frac{\overline{nc'}^2}{\overline{Ac}^2}; \quad \therefore nc' = \frac{Ac}{\sqrt{3}} = \frac{l}{2\sqrt{3}} = 0.289l \tag{146}$$

or

$$x = 0.211l.$$

The remaining results are obtained most easily from Eq. (1). Taking A as the origin, x positive to the right and y downward

$$EI \cdot \frac{d^2y}{dx^2} = \frac{wl^2}{12} - \frac{wlx}{2} + \frac{wx^2}{2}$$

$$EI \cdot \frac{dy}{dx} = \frac{wl^2x}{12} - \frac{wlx^2}{4} + \frac{wx^3}{6} + [C = 0]$$

$$EI \cdot y = \frac{wl^2x^2}{24} - \frac{wlx^3}{12} + \frac{wx^4}{24} + [C' = 0].$$

[1] If the student will attempt to work this problem through by the conventional method, starting with Eq. (1), he will perhaps arrive at the conclusion of the writer, that the above is a simpler method of dealing with the problem. He may also be able to judge whether it is better to use mathematical results in strictly abstract algebraical form or to see the concrete numerical character of results as he goes along. He should demonstrate Eqs. (139), (140), and (141) directly from the moment diagram.

Between the center and A the slope is at all points downward to the right and the deflection downward.

$$EI \cdot \alpha = \frac{wx}{12}(l^2 - 3lx + 2x^2) \tag{147}$$

$$EI \cdot d = \frac{wx^2}{24}(l^2 - 2lx + x^2) \tag{148}$$

By placing $\frac{d^2y}{dx^2} = 0$ the result in Eq. 146 will be obtained.

The max. slope, at k, is

$$EI \cdot \alpha_{\max.} = 0.008wl^3. \tag{149}$$

If this value should ever be required for numerical use, it should be worked out to more significant figures.

The max. deflection, at the center $\left(x = \frac{l}{2} \text{ in (148)}\right)$

$$EI \cdot d_1 = \frac{wl^4}{384} \tag{150}$$

or 0.2 of that for a beam simply supported.

27. (*c*) **With Two Symmetrical and Equal Loads** (Fig. 289).—As in case (*b*), the beam is horizontal at the ends and center. Hence if $M_0 = Aa'$

$$M_0 l = Pa(a + b)$$

$$M_0 = \frac{Pa(a+b)}{l} = \frac{Pa(l-a)}{l}. \tag{151}$$

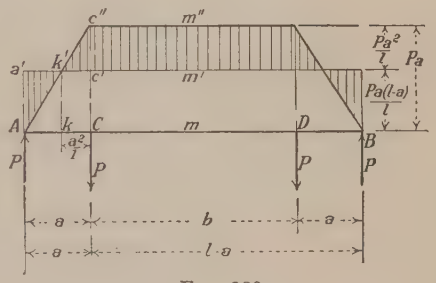


Fig. 289.

The point of inflexion, x_0 from A, is found from similar triangles:

$$\frac{Pa(l-a)}{l} : Pa :: x_0 : a$$

$$x_0 = \frac{a(l-a)}{l}. \tag{152}$$

The distance from k to C is $a - x_0 = \frac{a^2}{l}$.

The positive moment between C and D is

$$Pa - M_0 = \frac{Pa^2}{l}.$$

All slopes will be downward toward the center, and all deflections downward. The slope at any point to the right of k will be the negative area $Aa'k'$ minus the positive area from k' to the point; or if the point is to the left of k it will be the negative area from A to the point. At the center m, the slope will be zero; the area $Aa'm'm$ being equal to $Ac''m''m$, or $Aa'k' = k'c''m''m'$.

In AC:

$$EI \cdot \alpha = M_0 x - \frac{Pax^2}{2a} = \frac{Px}{2l}(2al - 2a^2 - lx). \tag{153}$$

In CD:

$$EI \cdot \alpha = M_0 x - \frac{Pa^2}{2} - Pa(x - a) = \frac{Pa^2}{2l}(l - 2x). \tag{154}$$

The deflection will be the moment of the moment area, about the point taken, since the axis is horizontal at the ends, or

In AC:

$$EI \cdot d = M_0 x \cdot \frac{x}{2} - Pa \cdot \frac{x}{a} \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{M_0 x^2}{2} - \frac{Px^3}{6}$$

$$= P\left(\frac{ax^2}{2} - \frac{a^2x^2}{2l} - \frac{x^3}{6}\right) = \frac{Px^2}{2}\left(a - \frac{a^2}{l} - \frac{x}{3}\right). \qquad (155)$$

In CD:

$$EI \cdot d = M_0 \frac{x^2}{2} - Pa \cdot \frac{a}{2}\left(x - \frac{2a}{3}\right) - \frac{Pa(x-a)^2}{2}$$

$$= Pa^2\left(\frac{x}{2} - \frac{x^2}{2l} - \frac{a}{6}\right). \qquad (156)$$

For $x = a$, the values of the slope and deflection check with each other.

The same results may be obtained by the use of Eq. (1). With origin at A, x and y positive to the right and downward respectively, M_0 being the (negative) moment at A.

In AC:

$$EI \cdot \frac{d^2y}{dx^2} = M_0 - Px$$

$$EI \cdot \frac{dy}{dx} = M_0 x - \frac{Px^2}{2} + [C = 0] \qquad (157)$$

which is the same as (153).

$$EI \cdot y = \frac{M_0 x^2}{2} - \frac{Px^3}{6} + [C_1 = 0] \qquad (158)$$

which is the same as (155).

In CD:

$$EI \cdot \frac{d^2y}{dx^2} = M_0 - Pa$$

$$EI \cdot \frac{dy}{dx} = M_0 x - Pax + C_2 \qquad (159)$$

which is the same as (154) when the value below of C_2 is inserted.

$$EI \cdot y = M_0 \frac{x^2}{2} - Pa\frac{x^2}{2} + C_2 x + C_3. \qquad (160)$$

For $x = a$, the values in (157) and (159) and in (158) and (160), must agree, and for $x = \frac{l}{2}$, (159) must reduce to zero, or

$$M_0 \frac{l}{2} - \frac{Pal}{2} + C_2 = 0; \therefore C_2 = \frac{Pal}{2} - M_0 \frac{l}{2}.$$

Equating (157) and (159) for $x = a$, the result is the value of M_0 in Eq. (151). Equating (158) and (160) gives $C_3 = -\frac{Pa^3}{6}$, hence Eq. (160) becomes

$$EI \cdot y = Pa^2\left(\frac{x}{2} - \frac{x^2}{2l} - \frac{a}{6}\right) \text{or the same as (156).}$$

This case affords a very good example of the two methods that may be employed.

It is apparent that a beam with one or both ends fixed is more rigid than the same beam if simply supported, *i.e.*, it has less deflection under a given load. Generally, also it is stronger, *i.e.*, the same load produces a smaller max. M, although in the case of one end fixed and uniform load the max. M is the same as if the beam were simply supported, but occurs at the support, and is negative, instead of being at the center and positive. Hence the importance of carefully considering the negative moments at and near the supports of beam with fixed ends.

28. Numerical Solution of Slope and Deflection.—If a number of loads act upon a beam, two methods of procedure are possible. The first is to find the effect of each load separately and add the results. This makes possible the determination of slope and deflection at any given point, but does not show where the max. occurs, nor clearly show the law of change within any given distance. The second method is to formulate separate equations for the different sections of

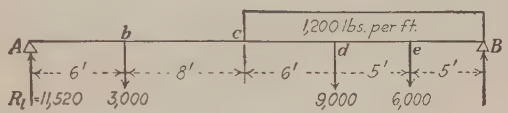


FIG. 290.

the beam in which the law of shear differs. As an example of this second method, take the following case (Fig. 290).

Taking origin at A, x and y positive to right and downward, the following relations are obvious:

$$\text{In } Ab\text{: } EI \cdot \frac{d^2y}{dx^2} = -11{,}520x$$

$$EI \cdot \frac{dy}{dx} = -5{,}760x^2 + C_1$$

$$EI \cdot y = -1{,}920x^3 + C_1x + [C_2 = 0]$$

$$\text{In } bc\text{: } EI \cdot \frac{d^2y}{dx^2} = -M = -11{,}520x + 3{,}000(x - 6)$$

$$= -8{,}520x - 18{,}000$$

$$EI \cdot \frac{dy}{dx} = -4{,}260x^2 - 18{,}000x + C_3$$

$$EI \cdot y = -1{,}420x^3 - 9{,}000x^2 + C_3x + C_4$$

$$\text{In } cd\text{: } EI \cdot \frac{d^2y}{dx^2} = -11{,}520x + 3{,}000(x - 6) + \frac{1{,}200(x - 14)^2}{2}$$

$$= 600x^2 - 25{,}320x + 99{,}600$$

$$EI \cdot \frac{dy}{dx} = 200x^3 - 12{,}660x^2 + 99{,}600x + C_5$$

$$EI \cdot y = 50x^4 - 4{,}220x^3 + 49{,}800x^2 + C_5x + C_6$$

$$\text{In } de\text{: } EI \cdot \frac{d^2y}{dx^2} = -11{,}520x + 3{,}000(x - 6) + 9{,}000(x - 20) + \frac{1{,}200(x - 14)^2}{2}$$

$$= 600x^2 - 16{,}320x - 80{,}400$$

$$EI \cdot \frac{dy}{dx} = 200x^3 - 8{,}160x^2 - 80{,}400x + C_7$$

$$EI \cdot y = 50x^4 - 2{,}720x^3 - 40{,}200x^2 + C_7x + C_8.$$

In *cf*: $EI \cdot \frac{d^2y}{dx^2} = -11{,}520x + 3{,}000(x - 6) + 9{,}000(x - 20) + 6{,}000(x - 25) + \frac{1{,}200(x - 14)^2}{2}$

$$= 600x^2 - 10{,}320x - 230{,}400$$

$$EI \cdot \frac{dy}{dx} = 200x^3 - 5{,}160x^2 - 230{,}400x + C_9$$

$$EI \cdot y = 50x^4 - 1{,}720x^3 - 115{,}200x^2 + C_9x + C_{10}.$$

The constants are found by placing the values of $\frac{dy}{dx}$ and of y at the points, b, c, d, and e, obtained from the equations for adjoining sections, equal to each other, and also y at A and $B = 0$. If there are n sections of the beam (5 in the above case), there are $2n$ constants to be determined. There are $n - 1$ points with 2 conditions each, and two end points with one condition each, or $2n$ conditions to determine the $2n$ constants. If the case is one that is statically indeterminate as regards the outer forces (as the beam fixed at both ends), there will be additional equations sufficient to solve the case (as the slope at each end = 0).

Solving the above case, the following values result:

$$C_1 = 1{,}334{,}207.$$
$$C_2 = 0.$$
$$C_3 = 1{,}388{,}207.$$
$$C_4 = -108{,}000.$$
$$C_5 = 839{,}407.$$
$$C_6 = 1{,}812{,}800.$$
$$C_7 = 2{,}639{,}407.$$
$$C_8 = -10{,}187{,}200.$$
$$C_9 = 4{,}514{,}407.$$
$$C_{10} = -25{,}812{,}200.$$

Inserting these values, the numerical equations resulting will give equations for slope and deflection in each section of the beam.

The student will find the working out of the above case a good exercise. He should also determine the point of maximum deflection, and its value.

29. Variable Moment of Inertia.—A similar method may be used if the moment of inertia is variable. The divisions of the span will come at points where I changes, and, as in the previous paragraph, at concentrated loads and points where distributed load begins or changes. The effect of variations in I may also be allowed for by reducing all to the same I by the use of a multiplier. This may be best shown by an illustration.

Let it be required to find slope and deflection in the following case (Fig. 291) of a beam fixed at one end and loaded with 500 pounds per foot. The moment of inertia in (feet)^4 is as shown, in four sections of the beam.

The moment at $A = 500 \times \frac{68^2}{2} = 1{,}156{,}000$ foot-pounds

The moment at $b = 500 \times \frac{64^2}{2} = 1{,}024{,}000$ foot-pounds

The moment at $c = 500 \times \frac{58^2}{2} = 841{,}000$ foot-pounds

The moment at $d = 500 \times \frac{20^2}{2} = 100{,}000$ foot-pounds

The area BAa' = ⅓ × 1,156,000 × 68 = 26,202,667 pound-feet
The area Bbb' = ⅓ × 1,024,000 × 64 = 21,845,333 pound-feet
The area Bcc' = ⅓ × 841,000 × 58 = 16,259,333 pound-feet
The area Bdd' = ⅓ × 100,000 × 20 = 666,667 pound-feet = A_4
The area $Aa'bb'$ = 4,357,333 pound-feet = A_1
The area $bb'cc'$ = 5,586,000 pound-feet = A_2
The area $cc'dd'$ = 15,592,667 pound-feet = A_3.

Since the change of slope in any distance dx equals the moment area in that distance divided by EI, clearly the slope at the end is given by the equation, E being constant

$$E\alpha = \Sigma\frac{A}{I} = \frac{A_1}{I_1} + \frac{A_2}{I_2} + \frac{A_3}{I_3} + \frac{A_4}{I_4} = \frac{4,357,333}{4.85} + \frac{5,586,000}{3.81} + \frac{15,592,667}{2.81} + \frac{666,667}{1.85}$$
$$= 8,273,912.$$

Since α is a ratio of two distances, the dimension of each member of this equation is $\frac{\text{force}}{\text{distance}^2}$.

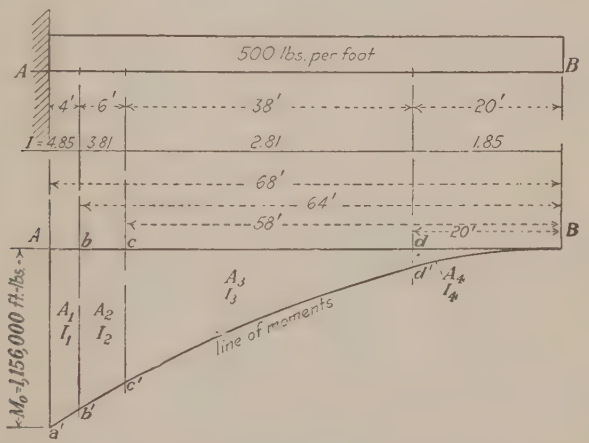


FIG. 291.

To find the numerical value of the slope, since the moments are in foot-pounds and distances in feet, E must be taken in the same units. Assuming it as 28,000,000 pounds per square inch, this equals 4,032,000,000 pounds per square foot; hence $\alpha = \frac{8,273,912}{E} = 0.00205$, that is to say about 2 per thousand.

Another way of treating this is by reducing all moment areas to a common value of I, which should be taken as the maximum, or I_1, or

$$E\alpha = \sum\frac{A}{I} = \frac{A_1}{I_1} + \frac{A_2}{I_1}\left(\frac{I_1}{I_2}\right) + \frac{A_3}{I_1}\left(\frac{I_1}{I_3}\right) + \frac{A_4}{I_1}\left(\frac{I_1}{I_4}\right).$$

Instead of A_2, therefore, use $A_2\frac{I_1}{I_2}$ = 7,110,787

Instead of A_3, therefore, use $A_3\frac{I_1}{I_3}$ = 26,912,610

Instead of A_4, therefore, use $A_4\frac{I_1}{I_4}$ = 1,747,747

A_1 = 4,357,333

Total..... 40,128,477

$$\alpha = \frac{40,128,477}{EI_1} = 0.00205, \text{ as before.}$$

To find deflection, take moments about B of the areas multiplied by their proper ratios. First find distances of centers of gravity of the areas from B, which may be termed x_1, x_2, x_3, and x_4.

$x_1 = (26{,}202{,}667 \times \frac{3}{4} \cdot 68 - 21{,}845{,}333 \cdot \frac{3}{4} \cdot 64) \div (26{,}202{,}667 - 21{,}845{,}333) = 66.04.$

Similarly, to the nearest hundredth

$$x_2 = 61.10$$

$$x_3 = 45.30$$

$$x_4 = \tfrac{3}{4} \cdot 20 = 15$$

$$EI_1 \cdot d = 4{,}357{,}333\ (66.04) + 7{,}110{,}787\ (61.10) + 26{,}912{,}610\ (45.30) + 1{,}747{,}747\ (15)$$

$$d = 0.1006 \text{ foot} = 1.207 \text{ inches.}$$

The rule for dealing with a case of variable moment of inertia is clearly this: Divide the moment area into sections in which the moment of inertia is constant. Weight each area by multiplying it by the ratio of the max. I to its I. Use the weighted areas as in the usual case, using for I the max. I: If Eq. (1) is used, and the deflection and slope are found by integration, divide each integral, using the proper I for each part. Thus, if $\int_{x=b}^{x=a} \frac{f(x)}{EI}$ is to be found, but c is an intermediate point between a and b, the value of I between b and c being I_1 and between c and a I_2, divide the integral and write

$$\int_{x=b}^{x=c} \frac{f(x)}{EI_1} + \int_{x=c}^{x=a} \frac{f(x)}{EI_2}.$$

These principles are easy of application.

30. Approximations in Finding Deflection with Variable I. Elastic and Inelastic Deflection. Remarks upon Accuracy.—It is sometimes desirable to make rapid calculations even if only approximate. In such cases a trained judgment will enable the engineer to make sometimes approximations which are surprisingly close. For instance, in the above case, by merely looking at the figure it might be conjectured that the average I would be about 2.8; but since the value of I near the support is considerably greater than this, and this has the effect of much reducing the deflection near the support, and, by reducing the slope there also, reducing the deflection near the end by more than would correspond to the average I, it might be guessed that the end deflection would correspond to a value of I of a little over 3, say 3.25, using this with the formula $d = \frac{wl^4}{8EI}$, the result is $d = 0.102$ foot $= 1.224$ inches, or very close to the truth. As a matter of fact, the average I is 2.74, and using this value the result is 0.121, or 21 per cent in excess of the truth. The reader should reflect upon the reason why the actual deflection is less than would correspond to the average I.

These remarks must not be interpreted as suggesting or approving the use of inaccurate methods or as encouraging slip-shod habits. But, after all, in almost all practical affairs, accuracy is only relative. Many people, if they can only accumulate sufficient computations, tables, and statistics, may easily deceive themselves and others into thinking that they have attained exactness; forgetting that no amount of elaborate computations or statistics, if founded upon inaccurate premises, wrongly interpreted, or omitting some elements of the problem, may lead to totally incorrect results. This holds true not only in engineering questions but in economic and financial questions, such as valuation. Some people seem to stand in awe of figures or anything mathematical, thinking that

because the mathematical machine is in itself exact, results deduced mathematically must be exact also. So in computing deflection, some engineers will carry the results of the flexural formulae to extreme accuracy, but will entirely neglect the deflection due to shear; forgetting also the fact that it is only the *elastic* deflection that is computed anyway, and that other elements, such as yielding of connections, deformation of rivets, play of pins, will cause certain *inelastic* deflections that are beyond the reach of accurate computation, and forgetting also the inherent inaccuracies in the flexural formulae themselves. Nothing is truer than that one man's guess may be worth more, and may be nearer the truth, than another man's elaborate computations. The student, therefore, should avoid the rigidity of mind that comes from the habitual use of a mathematical machine that is incapable of error if based on correct data and taking account of all elements of the problem, and should accustom himself to performing approximate computations, if only to check more exact ones. Such a habit, also, will accustom him to form quick judgments. How many young engineers, for instance, can look at a boulder and tell its weight within any reasonable limit?—or can look at a railroad cut and tell anywhere near how many yards came out of it?—or can ride over a railroad and tell with any approach to reason what it is worth? Such power of rapid judgment may sometimes be extremely useful, and can be cultivated to an astonishing degree.

31. Deflection of Beams of Uniform Strength.—In the chapter on Flexure, it has been shown that it is possible to give a beam such a shape that the maximum normal stress on every section, for a given loading, computed by the formula

$$f = \frac{Mv}{I};$$

will be constant. To find the deflection of such a beam, starting with the fundamental formula

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{f}{Ev} \tag{161}$$

this formula becomes, *for a rectangular beam where the depth is constant, h,*

$$\frac{d^2y}{dx^2} = \frac{f}{Ev} = \frac{2f}{Eh} = \text{constant} = C$$

$$\frac{dy}{dx} = Cx + C_1$$

$$y = \frac{Cx^2}{2} + C_1x + C_2. \tag{162}$$

This can easily be applied to special cases. Thus, for a beam supported at the ends and loaded at the middle, taking the origin at the center of the deflected beam (Fig. 292)

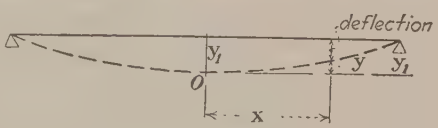


Fig. 292.

for $x = 0$, $\quad \frac{dy}{dx} = 0; \therefore C_1 = 0$

for $x = 0$, $\quad y = 0; \therefore C_2 = 0$

for $x = \frac{l}{2}$, $\quad y = y_1 = \text{deflection at the center} = \frac{Cx^2}{2} = \frac{fl^2}{4Eh}$

for $x = x$, $\quad y = \frac{fx^2}{Eh} = \frac{fl^2}{4Eh} - \text{deflection at that point}$

or, deflection at any distance x from the center $= \frac{fl^2}{4Eh} - \frac{fx^2}{Eh} = \frac{f}{4Eh}(l^2 - 4x^2)$. (163)

For the case where the breadth is constant, $v = \frac{h}{2}$ is a variable, the law of variation depending upon the loading. For any given case, the value of v can be substituted in Eq. (161), and the values of $\frac{dy}{dx}$ and of y may be found. Thus, for the

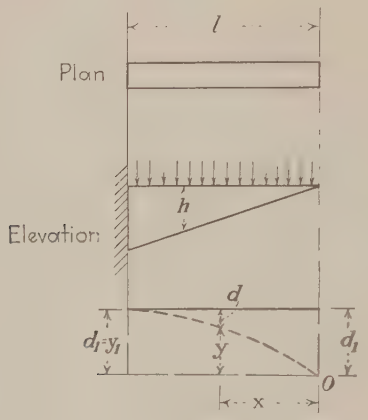


FIG. 293.

beam fixed at one end and loaded uniformly (Fig. 293), the origin being at the free end, h varies as x, or $v = cx$; the fundamental equation here is, having regard to signs, and calling a moment positive when it causes tension at the top,

$$\frac{d^2y}{dx^2} = -\frac{f}{Ev} = -\frac{f}{Ecx} = \frac{C}{x}$$

$$\frac{dy}{dx} = C \log x + C_1$$

$$y = Cx \log x - Cx + C_1x + C_2$$

for $x = 0$; $\quad y = 0; \therefore C_2 = 0$

for $x = l$; $\quad \frac{dy}{dx} = 0; \therefore C_1 = -C \log l = \frac{f \log l}{Ec}$

for $x = l$; $\quad y = y_1 = d_1 = Cl(\log l - 1) + C_1 l$

for $x = x$; $\quad d = d_1 - y = Cl(\log l - 1) + C_1 l - Cx(\log x - 1) - C_1 x$

$$= C[l(\log l - 1) - x(\log x - 1)] + C_1(l - x)$$

$$= \frac{f}{Ec}[l - x - x(\log l - \log x)] \qquad (161)$$

This subject, however, is not important, and need not be pursued further.

32. General Case of Deflection of a Beam Supported at Each End.—Let Fig. 294 represent a beam supported at each end, with a moment and shear at each end, as shown, a uniform load, and a concentrated load. Let s_1 and s_2 be the slopes at the ends in each case downward as shown, and let d be the *elevation* of the left end *above* the right end when deformed. There may be shears (not shown) on the ends, as well as moments, and V_1 and V_2 are not the reactions, but are the shears on sections just inside the supports. The beam was originally horizontal. The moment diagrams are shown for each load and

for the moments; they might be combined into one, but it is better to keep them separate. Then by the moment-area method,

$$D_2 \cdot EI = \frac{Pa \cdot l}{2} \cdot \frac{l}{3} - \frac{Pa \cdot a}{2} \cdot \frac{a}{3} + \frac{2}{3} l \cdot \frac{wl^2}{8} \cdot \frac{l}{2} - M_1 l \cdot \frac{l}{2} - \frac{(M_2 - M_1)l}{2} \cdot \frac{l}{3}$$

$$= \frac{Pa}{6}(l^2 - a^2) + \frac{wl^4}{24} - \frac{M_1 l^2}{3} - \frac{M_2 l^2}{6} \tag{162}$$

$$D_2 = s_1 l + d \tag{163}$$

$$D_1 EI = \frac{Pa(l-a)}{6}(2l - a) + \frac{wl^4}{24} - \frac{M_1 l^2}{6} - \frac{M_2 l^2}{3} \tag{164}$$

$$D_1 = s_2 l - d \tag{165}$$

$s_1 + s_2$ = change of slope from one end to the other

$$= \frac{Pa(l-a)}{2} + \frac{wl^3}{12} + \frac{M_2 l}{2} + \frac{M_1 l}{2} \tag{166}$$

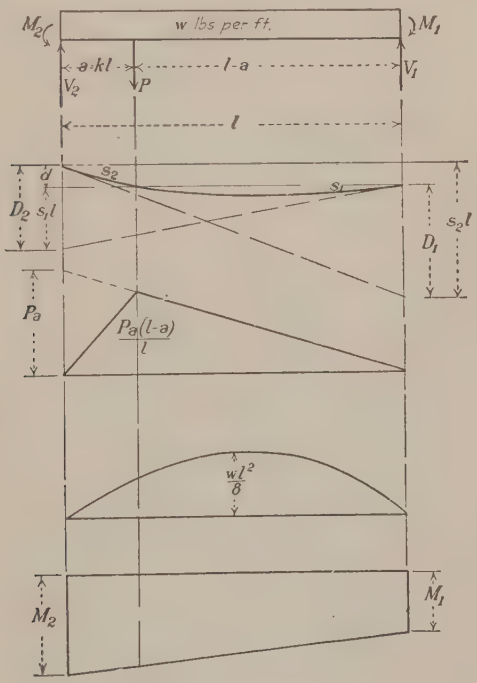


FIG. 294.

By statics,

$$V_1 + V_2 = wl + P \tag{167}$$

$$M_1 - M_2 + Pa - V_1 l + \frac{wl^2}{2} = 0. \tag{168}$$

Thus there are seven equations connecting the nine quantities M_1, M_2, V_1, V_2, D_1, D_2, s_1, s_2, and d.

From Eqs. (162) and (164):

$$\frac{6EI}{l^2}\cdot D_2 = P\left(a - \frac{a^3}{l^2}\right) + \frac{wl^2}{4} - 2M_1 - M_2 = Pl(k - k^3) + \frac{wl^2}{4} - 2M_1 - M_2 \tag{169}$$

$$\frac{6EI}{l^2}\cdot D_1 = P\left(2a - \frac{3a^2}{l} + \frac{a^3}{l^2}\right) + \frac{wl^2}{4} - M_1 - 2M_2 = Pl(2k - 3k^2 + k^3) + \frac{wl^2}{4} - M_1 - 2M_2 \tag{170}$$

or, calling $$P(k - k^3)l + \frac{wl^2}{4} = C_2,$$

and $$P(2k - 3k^2 + k^3)l + \frac{wl^2}{4} = C_1;$$

$$\frac{6EI}{l^2}\cdot D_1 = C_1 - M_1 - 2M_2 = \frac{6EI}{l^2}(s_2 l - d) \tag{171}$$

$$\frac{6EI}{l^2}\cdot D_2 = C_2 - 2M_1 - M_2 = \frac{6EI}{l^2}(s_1 l + d). \tag{172}$$

From Eqs. (171) and (172)

$$M_1 = \frac{2C_2 - C_1}{3} - \frac{2EI}{l}\left(\frac{3d}{l} + 2s_1 - s_2\right) \tag{173}$$

$$M_2 = \frac{2C_1 - C_2}{3} + \frac{2EI}{l}\left(\frac{3d}{l} + s_1 - 2s_2\right). \tag{174}$$

The quantities C_1 and C_2 depend on the loading alone, $\frac{3d}{l}$ depends on a change of level of one support.

For a beam fixed at both ends, with ends on the same level,

$$M_1 = Pl(k^2 - k^3) + \frac{wl^2}{12} \tag{175}$$

$$M_2 = Pl(k - 2k^2 + k^3) + \frac{wl^2}{12} \tag{176}$$

which agrees with the results of Art. 25 of this chapter; for, by Eq. (145), the moment at either end due to the uniform load is a negative moment $\frac{wl^2}{12}$; and in Eqs. (131) and (132), if $(l - a)$ is made $= kl$ and $a = l(1 - k)$, we find

$$M_l = M_2 = \frac{Pl^2(1 - k)^2 kl}{l^2} = Pl(k - 2k^2 + k^3) \text{ as in Eq. (176)}$$

$$M_r = M_1 = \frac{Pl(1 - k)k^2 l^2}{l^2} = Pl(k^2 - k^3) \text{ as in Eq. (175).}$$

CHAPTER XIII

COMBINED STRESSES

1. In previous chapters, the simple stresses of tension, compression, shearing, and torsion have been considered, and also flexure, which involves several of these. It remains to consider other combinations, namely:

(*a*) Tension, combined with compression at right angles to the tension.
(*b*) Tension combined with torsion.
(*c*) Compression combined with torsion.
(*d*) Tension combined with flexure.
(*e*) Compression combined with flexure.
(*f*) Transverse shearing combined with torsion.
(*g*) Torsion combined with flexure.
(*h*) Torsion combined with flexure and tension.
(*i*) Torsion combined with flexure and compression.

The other possible combinations, such as shearing and flexure, have already been covered.

All these cases are fully covered by the principles explained in Chap. IX, on the distribution of a force on a section, and in Chap. VI on the relations between stresses at a point. All that is necessary in this chapter is to refer to certain practical cases and to deduce certain formulae from those that have already been given.

2. (*a*) **Tension and Compression at Right Angles.**—If on any given plane there is pure tension, and on a plane at right angles pure compression, then these are the principal stresses, and the case has been fully treated in Chap. V. The intensity and direction of the stress on any other plane may be found by the methods there given. If f_t and f_c are the intensities of the tension and compression respectively, the elongation per linear unit in the direction of the tension will be $e_t = \frac{f_t}{E} + \frac{f_c}{mE}$ and the compression in the direction of the compression will be $e_c = \frac{f_c}{E} + \frac{f_t}{mE}$, in which E is the modulus of elasticity, which is assumed the same for tension and compression, and $\frac{1}{m}$ is Poisson's ratio. If the moduli of elasticity are not the same,

$$\left.\begin{aligned} e_t &= \frac{f_t}{E_t} + \frac{f_c}{mE_c} \\ e_c &= \frac{f_c}{E_c} + \frac{f_t}{mE_t} \end{aligned}\right\} \qquad (1)$$

Expressed in terms of the principal stresses at a point, n_1 and n_2, if these are taken algebraically, *i.e.*, if tension is positive and compression negative, and if a stretch is taken as positive and a contraction negative, then if E is the same for

tension and compression, the deformations (e_1 and e_2) per unit length along n_1 and n_2 will be given by

$$\left.\begin{aligned} Ee_1 &= n_1 - \frac{n_2}{m} \\ Ee_2 &= n_2 - \frac{n_1}{m} \end{aligned}\right\} \tag{1a}$$

The strain, e_3, in a direction perpendicular to the plane of n_1 and n_2 will be given by

$$Ee_3 = -\frac{n_1 + n_2}{m}.$$

If the stress is not parallel to one plane, but there are three principal stresses, n_1, n_2, and n_3, and if these are taken algebraically, then if the body is isotropic the elongations will be given by the equations:

$$\left.\begin{aligned} Ee_1 &= n_1 - \frac{n_2}{m} - \frac{n_3}{m} \\ Ee_2 &= n_2 - \frac{n_1}{m} - \frac{n_3}{m} \\ Ee_3 &= n_3 - \frac{n_1}{m} - \frac{n_2}{m} \end{aligned}\right\} \tag{1b}$$

3. (*b* and *c*) **Tension or Compression and Torsion.**—It has been shown that torsion is pure shearing on a cross-section, in a direction perpendicular to a radius from the center, and with an intensity increasing with the distance from the center. If with this is combined a direct axial tension T, then on a cross-section we have simply at any point a tensile stress of $\frac{T}{A}$ and a shear of the intensity of the torsional shear at the point considered.

The case is similar if there is axial compression.

4. (*d*) **Tension Combined with Flexure.**—It has been shown in Chap. IX how to find the distribution of stress over a section when the resultant force on that section acts at any given point. The difficulty in the case of tension and flexure

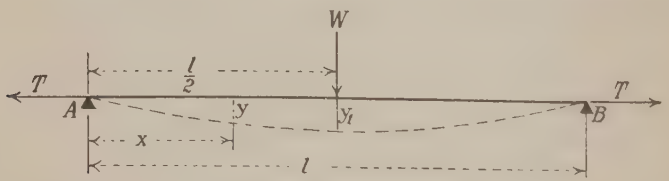


FIG. 295.—Hinged ends.

is to find that point. Suppose a beam, supported at the ends, to be exposed to transverse loads, and also to a tension T acting in the line joining the ends (Fig. 295). The transverse loads, acting alone, would deflect the beam as indicated, and the deflection at any point may be computed by the formulae of Chap. XII. The tension T then acts upon a deflected beam in which there is a positive moment at each point. But the tension itself has a lever arm about each point of the axis, and produces a negative moment at each point, causing tension in the upper fibers, and *reducing* the deflection. There will be equilibrium when

the deflections due to transverse load have been reduced, but the amount of this reduction is unknown.

It might be thought that this problem could be solved accurately by starting with the general equation

$$\frac{d^2y}{dx^2} = \pm \frac{M}{EI} \tag{2}$$

and replacing M by the actual moment, due to both transverse loads and T, in terms of the unknown y. Thus, taking the origin at A, distances to the right and downward as positive, we should have, for a load at the center,

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$M = \frac{Wx}{2} - Ty \tag{3}$$

$$EI\frac{d^2y}{dx^2} = -\left(\frac{Wx}{2} - Ty\right). \tag{4}$$

By integrating this, the equation of the elastic line would be found, and hence the exact point of application of the resultant on any section could be found.

But it must be remembered that the Eq. (2) involves certain assumptions (see Chap. X), among which is this, that $dx = ds$, or in other words, that *the deflections are zero*. This is a legitimate assumption in case the deflections are due to transverse loads, M being due to those loads alone; it is less legitimate in cases like the present, in which the moment M is due (partly at least) to the deflections themselves. Here it may be said that the deflections are due (partly at least) to the deflections themselves, and it is obviously in some degree erroneous to assume those deflections zero, and on the basis of that assumption to find the deflections. Nevertheless this method will be given for the benefit of those interested, and afterward approximate solutions will be indicated.

Hinged Ends; Load at the Center.—The fundamental equation is (Fig. 295)

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{Wx}{2} + Ty. \tag{4}$$

The general integral of this is, calling $q = \sqrt{\frac{T}{EI}}$,

$$y = C_1e^{qx} + C_2e^{-qx} + \frac{W}{2T}x \tag{5}$$

in which C_1 and C_2 are constants.

For $x = 0$, $y = 0$; hence $C_1 + C_2 = 0$; $C_1 = -C_2$.

For $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1e^{qx} - qC_2e^{-qx} + \frac{W}{2T}. \tag{6}$$

Hence

$$0 = qC_1e^{\frac{ql}{2}} - qC_2e^{-\frac{ql}{2}} + \frac{W}{2T}$$

$$-\frac{W}{2T} = qC_1\left(e^{\frac{ql}{2}} + e^{-\frac{ql}{2}}\right).$$

Calling the quantity in parenthesis u,

$$C_1 = -\frac{W}{2Tqu} = -C_2$$

$$y = \frac{W}{2T}\left(x - \frac{e^{qx} - e^{-qx}}{qe^{\frac{ql}{2}} + qe^{-\frac{ql}{2}}}\right) \tag{7}$$

which is the equation of the elastic line.

The maximum deflection, at the center, for $x = \frac{l}{2}$, will be

$$y_1 = \frac{W}{2T}\left(\frac{l}{2} - \frac{1}{q}\,\frac{e^{\frac{ql}{2}} - e^{-\frac{ql}{2}}}{e^{\frac{ql}{2}} + e^{-\frac{ql}{2}}}\right). \tag{8}$$

The maximum fiber stress is therefore

$$f = \frac{T}{A} + \frac{Mv}{I} - \frac{Ty_1 v}{I} \tag{9}$$

v is positive *below* the center of gravity axis

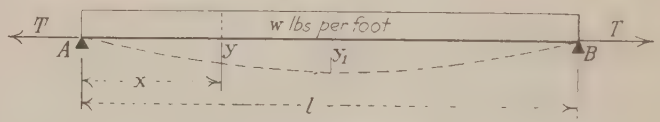


FIG. 296.—Hinged ends.

Hinged Ends; Uniform Load.—The fundamental equation is (Fig. 296)

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{wx(l - x)}{2} + Ty. \tag{10}$$

The general integral of this is, calling $q = \sqrt{\frac{T}{EI}}$; $\frac{wl}{2} = W$;

$$y = C_1 e^{qx} + C_2 e^{-qx} + \frac{Wx}{T} - \frac{wx^2}{2T} - \frac{w}{q^2T} \tag{11}$$

in which C_1 and C_2 are constants.

For $x = 0$; $y = 0$; hence

$$0 = C_1 + C_2 - \frac{w}{q^2T}$$

$$C_2 = \frac{w}{q^2T} - C_1.$$

For $x = \frac{l}{2}$; $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1e^{qx} - qC_2e^{-qx} + \frac{W}{T} - \frac{wx}{T}. \tag{12}$$

Hence

$$0 = qC_1e^{\frac{ql}{2}} - qC_2e^{-\frac{ql}{2}}$$

$$C_1 = C_2 \cdot \frac{1}{e^{ql}}$$

$$C_2 = \frac{we^{ql}}{q^2T(1 + e^{ql})} \tag{12a}$$

$$C_1 = \frac{w}{q^2T(1 + e^{ql})} \tag{12b}$$

$$y = \frac{w}{q^2T}\left[\frac{e^{qx} + e^{ql-qx}}{1 + e^{ql}} - 1\right] + \frac{Wx}{T} - \frac{wx^2}{2T} \tag{13}$$

which is the equation of the elastic line.

The maximum deflection, at the center, for $x = \frac{l}{2}$, will be

$$y_1 = \frac{w}{q^2T}\left[\frac{2e^{\frac{ql}{2}}}{1 + e^{ql}} - 1\right] + \frac{Wl}{4T}. \tag{14}$$

The maximum fiber stress is then given by Eq. (9).

Fixed End; Load at the Center.—The fundamental equation is (Fig. 297), M_1 being the *negative* moment at the end,

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{Wx}{2} + Ty + M_1. \tag{15}$$

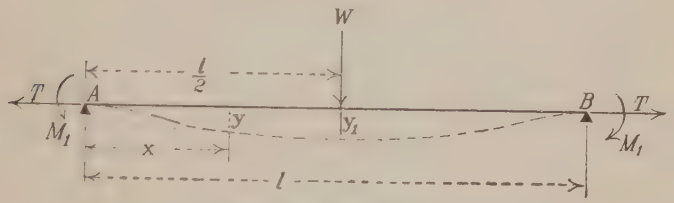


FIG. 297.—Fixed ends.

The general integral of this is, calling $q = \sqrt{\frac{T}{EI}}$

$$y = C_1e^{qx} + C_2e^{-qx} - \frac{M_1}{T} + \frac{Wx}{2T} \tag{16}$$

in which C_1 and C_2 are constants.

For $x = 0$; $y = 0$; hence

$$0 = C_1 + C_2 - \frac{M_1}{T}; \; C_1 = \frac{M_1}{T} - C_2.$$

For $x = \frac{l}{2}$; $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1e^{qx} - qC_2e^{-qx} + \frac{W}{2T}. \tag{17}$$

Hence

$$0 = qC_1e^{\frac{ql}{2}} - qC_2e^{-\frac{ql}{2}} + \frac{W}{2T}.$$

From these equations we find

$$y = \frac{M_1}{T}(e^{qx} - 1) + \frac{Wx}{2T} - (e^{qx} - e^{-qx})\frac{\frac{W}{2T} + \frac{qM_1}{T}e^{\frac{ql}{2}}}{q\left(e^{\frac{ql}{2}} + e^{-\frac{ql}{2}}\right)}. \tag{18}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, will be

$$y_1 = \frac{M_1}{T}\left(e^{\frac{ql}{2}} - 1\right) + \frac{Wl}{4T} - \left(e^{\frac{ql}{2}} - e^{-\frac{ql}{2}}\right)\frac{\frac{W}{2T} + \frac{qM_1}{T}e^{\frac{ql}{2}}}{q\left(e^{\frac{ql}{2}} + e^{-\frac{ql}{2}}\right)}. \tag{19}$$

The maximum fiber stress at the center is then given by Eq. (9);

at the ends; $$f = \frac{T}{A} - \frac{M_1v}{I}. \tag{20}$$

Remember that M_1 is the left-handed (negative) moment at the ends, so that the last term is really subtractive as indicated (v is positive below c. g.).

Fixed Ends; with a Distributed Uniform Load.—The fundamental equation is:

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{wx(l-x)}{2} + Ty + M_1. \tag{21}$$

The general integral of this is, calling $q = \sqrt{\frac{T}{EI}}$; $\frac{wl}{2} = W$;

$$y = C_1 e^{qx} + C_2 e^{-qx} - \frac{M_1}{T} + \frac{Wx}{T} - \frac{wx^2}{2T} - \frac{w}{q^2T} \tag{22}$$

in which C_1 and C_2 are constants.

For $x = 0$; $y = 0$; hence

$$0 = C_1 + C_2 - \frac{M_1}{T} - \frac{w}{q^2T}.$$

For $x = \frac{l}{2}$; $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = qC_1e^{qx} - qC_2e^{-qx} + \frac{W}{T} - \frac{wx}{T}. \tag{23}$$

Hence

$$0 = qC_1e^{\frac{ql}{2}} - qC_2e^{-\frac{ql}{2}}.$$

From these equations we find the equation of the elastic line.

$$y = \left(\frac{M_1}{T} + \frac{w}{q^2T}\right)\left(\frac{e^{qx} + e^{ql-qx}}{1 + e^{ql}} - 1\right) + \frac{Wx}{T} - \frac{wx^2}{2T}. \tag{24}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, is

$$y_1 = \left(\frac{M_1}{T} + \frac{w}{q^2T}\right)\left(\frac{2e^{\frac{ql}{2}}}{1 + e^{ql}} - 1\right) + \frac{Wl}{4T}. \tag{25}$$

The maximum stress is then given by Eqs. (9) and (20).

APPROXIMATE SOLUTIONS

(1) It is obvious that if the maximum deflection, d_b, *due to the transverse loads alone,* is computed, the real lever arm of T will be *less than this.* The effect of T is to partly straighten out the piece, and to *reduce* the deflection. The direct stress due to T alone increases the tensile stress in the bottom (convex) fiber, and decreases the compressive stress in the top (concave) fiber; the moment of T decreases the tensile stress in the bottom fiber and decreases the compressive stress in the top fiber. Hence, wherever there is tension below (positive M) it is *on the safe side* to neglect the *moment* of T. Thus, if M is the moment due to the transverse loads alone, the expression

$$f_t = \frac{T}{A} + \frac{Mv}{I} \tag{26}$$

will be too large; also, the expression

$$f_t = \frac{T}{A} + \frac{Mv}{I} - \frac{Td_bv}{I} \tag{27}$$

will be too small. If d_b were the *true deflection,* the last expression would be correct.

At and near the ends of a beam fixed at the ends there will be tension at the top; here

$$f_t = \frac{T}{A} + \frac{Mv}{I} - \frac{Td_bv}{I}$$

will be too large, while

$$f_t = \frac{T}{A} + \frac{Mv}{I}$$

will be too small, except at the very end, where there is no deflection, and it will be correct.

(2) (*a*) *Tension Piece with Hinged Ends.*—As a preliminary to this solution, let us find the relation between the maximum stress *due to bending*, and the deflection. Take first the case of a *beam with a concentrated load W at the center;* here,

$$d_b = \frac{Wl^3}{48EI}; f_b = \frac{Wl}{4} \cdot \frac{v}{I}; \therefore d_b = \frac{f_b l^2}{12Ev}.$$

In the case of a beam loaded uniformly with w pounds per foot,

$$d_b = \frac{5wl^4}{384EI}; f_b = \frac{wl^2}{8} \cdot \frac{v}{I}; \therefore d_b = \frac{5f_b l^2}{48Ev}.$$

In either case, it is very nearly accurate to write

$$d_b = \frac{f_b l^2}{10Ev}. \tag{28}$$

This is the relation between d_b and f_b, the deflection and maximum fiber stress, respectively, in bending, no matter what is the cause of the bending. Although this relation is obtained only from two specific cases of transverse loading, nevertheless the deflection depends solely upon the fiber stresses in bending, and it may reasonably be assumed that the relation holds generally. From Eq. (28),

$$f_b = \frac{10Evd_b}{l^2}.$$

Now in Eq. (27), the last two terms represent f_b, the stress due to bending, hence

$$f_b = \frac{Mv}{I} - \frac{Tdv}{I}$$

where d is the true deflection, for which value, as we have seen, Eq. (27) is correct. Substituting the value of d,

$$f_b = \frac{Mv}{I} - \frac{Tv}{I} \cdot \frac{f_b l^2}{10Ev}$$

$$f_b = \frac{Mv}{I + \frac{Tl^2}{10E}}$$

$$f_t = \frac{T}{A} + \frac{Mv}{I + \frac{Tl^2}{10E}}. \tag{29}$$

Example.—Let a horizontal steel eye-bar 6 inches × 1 inch × 20 feet, weighing 20.4 pounds per foot, have an axial tension of 60,000 pounds.

Here $A = 6; T = 60{,}000; I = \frac{1}{12} \cdot 6^3 = 18.$

By Eq. (26);

$$f = \frac{P}{A} + \frac{My}{I} = 10{,}000 + \frac{20.4 \times 20 \times 20 \times 12 \times 3}{8 \times 18} = 12{,}040$$

pounds per square inch.

This, as shown, is too large.

$$d_b = \frac{5 \times 20.4 \times 20 \times 20 \times 20 \times 20 \times 12 \times 12 \times 12 \times 12}{384 \times 30{,}000{,}000 \times 18 \times 12} = \frac{81.6}{600}$$

$$= 0.136 \text{ inch.}$$

By Eq. (27);

$$f = 10{,}000 + 2{,}040 - \frac{60{,}000 \times 0.136 \times 3}{18} = 10{,}680 \text{ pounds per square inch.}$$

This, as shown, is too small.

By Eq. (29); $\frac{Tl^2}{10EI} = \frac{60{,}000 \times 20 \times 20 \times 12 \times 12}{10 \times 30{,}000{,}000 \times 18} = \frac{16}{25}$

$$f = 10{,}000 + \frac{2{,}040}{1 + \frac{16}{25}} = 11{,}244.$$

This lies between the other two values, as it should, and it is quite accurate enough.

Using Eqs. (14) and (9):

$$q = \sqrt{\frac{60{,}000}{30{,}000{,}000 \times 18}} = \frac{1}{94.87}; \frac{ql}{2} = \frac{120}{94.87} = 1.265;$$

$$e^{1.265} = 3.543; e^{2.53} = 12.553$$

$$W = \frac{wl}{2} = 20.4 \times 10 = 204$$

$$y_1 = \frac{20.4 \times 9{,}000}{12 \times 60{,}000}\left[\frac{2 \times 3.543}{1 + 12.553} - 1\right] + \frac{204 \times 20 \times 12}{4 \times 60{,}000} = +0.082$$

$$f = 10{,}000 + 2{,}040 - \frac{60{,}000 \times 0.082 \times 3}{18} = 11{,}220.$$

All this shows that the use of Eq. (14) is unnecessary, unless in special cases

(*b*) *Tension Piece with Fixed Ends.*—For a concentrated load at the center

$$d_b = \frac{Wl^3}{192EI}; f_b = \frac{Wl}{8} \cdot \frac{v}{I}; d_b = \frac{f_b l^2}{24Ev}.$$

For a uniform load;

$$d_b = \frac{wl^4}{384EI}; f_b = \frac{wl^2}{24} \cdot \frac{v}{I}; d_b = \frac{f_b l^2}{16Ev}.$$

As a fair average, take in general

$$d_b = \frac{f_b l^2}{20Ev} = d. \tag{30}$$

This is the true deflection in terms of the true f_b.

Then $$f_b = \frac{20Evd_b}{l^2}.$$

In Eq. (27), the last two terms represent f_b, or

$$f_b = \frac{Mv}{I} - \frac{Tdv}{I}$$

and this is correct if d is the true deflection. M is the moment due to the transverse loads, including the effect of the end moment M. Hence,

$$f_b = \frac{Mv}{I} - \frac{Tf_b l^2}{20EI}$$

$$f_b = \frac{Mv}{I + \frac{Tl^2}{20E}}$$

$$f = \frac{T}{A} + \frac{Mv}{I + \frac{Tl^2}{20E}}. \tag{31}$$

Example.—Take the same bar as in the previous case, but suppose it riveted at the ends, perhaps made of two bars 6 inches × ½ inch, one on each side of a gusset plate at the joint.

By Eq. (26); $f = \frac{P}{A} + \frac{Mv}{I} = 10{,}000 + \frac{1}{144} \cdot wl^2 = 10{,}680$ pounds per square inch, at the center of the span.

This, as shown, is too large.

$$d_b = \frac{0.136}{5} = 0.0272 \text{ inch.}$$

By Eq. (27); $f = 10{,}680 - \frac{1{,}360}{5} = 10{,}408$ pounds per square inch.

This, as shown, is too small.

By Eq. (31); $$\frac{Tl^2}{20EI} = \frac{24}{75}$$

$$f = 10{,}000 + \frac{680}{1 + \frac{8}{25}} = 10{,}515.$$

This lies between the other two values, as it should.

By the more theoretical equations, 25, 9, and 20;

$$q = \frac{1}{94.87};\ \frac{ql}{2} = 1.265;\ e^{\frac{ql}{2}} = 3.543;\ e^{ql} = 12.553;$$

$W = 204$; M_1, taken the same as for no T, or $\frac{wl^2}{12}$;

$$y_1 = \left(\frac{20.4 \times 20 \times 20 \times 12}{12 \times 60{,}000} + \frac{20.4 \times 9{,}000}{12 \times 60{,}000}\right)\left(\frac{2 \times 3.543}{1 + 12.553} - 1\right) + \frac{204 \times 20 \times 12}{4 \times 60{,}000}$$

$$= 0.0175 \text{ inch}$$

$$f = 10{,}680 - \frac{60{,}000 \times 0.0175 \times 3}{18} = 10{,}505.$$

This again shows that the use of the more exact formulae is unnecessary.

At the end of the span the stress is larger, or $f = 10{,}000 + \frac{wl^2}{12} \times \frac{3}{18} = 11{,}360.$

It should be observed, however, that this example has not been solved correctly in any case, since the value of M_1 will not be the same for a piece with fixed ends exposed also to direct tension as it would be if there were no direct tension. The tension tends to straighten out the piece, reduce the slope at the ends, and so reduce the value of M_1 necessary to fix the ends. This may be computed; thus, with hinged ends, the slope at the end is given by Eq. (12), making $x = 0$, or

$$\frac{dy}{dx} = qC_1 - qC_2 + \frac{W}{T}.$$

The values of C_1 and C_2 are given by Eqs. (12*a*) and (12*b*), hence

$$\alpha_1 = \frac{dy}{dx} = \frac{w}{qT}\left(\frac{1 - e^{ql}}{1 + e^{ql}}\right) + \frac{W}{T}.$$

The moment M_1 at each end necessary to rotate the end back to the horizontal is given by

$$\alpha_1 = \frac{M_1 l}{2EI}.$$

Hence

$$M_1 = \frac{2EI}{l} \cdot \frac{w}{qT}\left(\frac{1 - e^{ql}}{1 + e^{ql}}\right) + \frac{2WEI}{lT} = \frac{2w}{lq^3}\left(\frac{1 - e^{ql}}{1 + e^{ql}}\right) + \frac{2W}{q^2 l}$$

$$= \frac{2 \times 20.4 \times 9{,}000 \times 94.87}{12 \times 20 \times 12}\left(\frac{1 - 12.553}{1 + 12.553}\right) + \frac{2 \times 204 \times 9{,}000}{20 \times 12} = 4{,}988 \text{ inch pounds}$$

Using now this new value of M_1 in Eq. (25) $\left(\text{instead of } \frac{wl^2}{12}\right)$,

$$y_1 = 0.0427 \text{ inch.}$$

Thus y_1 is very much greater than before. Using the new value, by Eq. (9). At the center;

$$f = \frac{T}{A} + \frac{\left(\frac{wl^2}{8} - M_1\right)v}{I} - \frac{Ty_1 v}{I}$$

$$= 10{,}000 + \left(\frac{20.4 \times 20 \times 20 \times 12}{8} - 4{,}988\right)\frac{3}{18} - \frac{60{,}000 \times 0.0427 \times 3}{18}$$

$$= 10{,}780 \text{ pounds per square inch.}$$

At the ends;

$$f = 10{,}000 + \frac{4{,}988 \times 3}{18} = 10{,}832 \text{ pounds per square inch.}$$

Thus it is seen that the corrected value of M_1 should be used if the more exact theoretical formulae are to be used. Even Eq. (31) gives results a little small.

5. (*e*) **Compression Combined with Flexure.**—This case is similar to the last, except that the effect of the axial compression is to *increase* the deflection, while the effect of the tension was to *decrease* it. The deflection due to the transverse loads alone being d_b, as before, it is obvious that

$$f = \frac{C}{A} + \frac{Mv}{I} \text{ will be too small} \tag{32}$$

$$f = \frac{C}{A} + \frac{Mv}{I} + \frac{Cd_b v}{I} \text{ will be too small.} \tag{33}$$

The true deflection, d, will be greater than d_b, and

$$f = \frac{C}{A} + \frac{Mv}{I} + \frac{Cdv}{I} \text{ will be correct.} \tag{34}$$

Using the relations in Eqs. (28) and (30), we have as the approximate formula: *For hinged ends:*

$$f = \frac{C}{A} + \frac{Mv}{I} + \frac{Cdv}{I} = \frac{C}{A} + \frac{10Evd}{l^2}$$

from which

$$d = \frac{\frac{Mv}{I}}{\frac{10Ev}{l^2} - \frac{Cv}{I}}$$

$$f = \frac{C}{A} + \frac{Mv}{I - \frac{Cl^2}{10E}} \tag{35}$$

For fixed ends:

at the center;

$$f = \frac{C}{A} + \frac{Mv}{I - \frac{Cl^2}{20E}}. \tag{36}$$

From Eq. (35) it is to be noted that f becomes infinite if $I = \frac{Cl^2}{10E}$ or if

$$C = \frac{10EI}{l^2}$$

which is almost exactly Euler's formula $\left(C = \frac{\pi^2 EI}{l^2}\right)$, since $\pi^2 = 9.87$. This formula, then, like Euler's, implies that any compressive force greater than that given by Euler's formula would bend the piece sufficiently to break it.

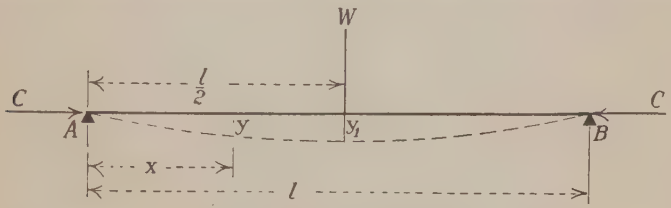


FIG. 298.—Hinged ends.

As in the previous case, the more exact theoretical formulae may be given for the benefit of those who wish to pursue the mathematical aspect of the subject farther.

Hinged Ends; Load at the Center.—The fundamental equation is (Fig. 298)

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{Wx}{2} - Cy. \tag{37}$$

The general integral of this is, calling $q^2 = \frac{C}{EI}$,

$$y = C_1 \sin qx + C_2 \cos qx - \frac{Wx}{2C} \tag{38}$$

in which C_1 and C_2 are constants.

For $x = 0$, $y = 0$; hence

$$0 = C_2.$$

For $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1 \cos qx - qC_2 \sin qx - \frac{W}{2C} \tag{39}$$

$$0 = qC_1 \cos \frac{ql}{2} - qC_2 \sin \frac{ql}{2} - \frac{W}{2C}$$

$$C_1 = \frac{W}{2Cq \cos \frac{ql}{2}}.$$

Hence the equation of the elastic line is

$$y = \frac{W \sin qx}{2Cq \cos \frac{ql}{2}} - \frac{Wx}{2C}. \tag{40}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, is

$$y_1 = \frac{W}{2C}\left(\frac{\tan \frac{ql}{2}}{q} - x\right). \tag{41}$$

And the maximum stress intensity is

$$f = \frac{C}{A} + \frac{Wl \cdot v}{4I} + \frac{Cy_1 \cdot v}{I}. \tag{42}$$

Hinged Ends; Uniform Load.—The fundamental equation is

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{wx(l-x)}{2} - Cy \tag{43}$$

The general integral of this is, calling $\frac{C}{EI} = q^2$;

$$y = C_1 \sin qx + C_2 \cos qx - \frac{wlx}{2C} + \frac{wx^2}{2C} - \frac{w}{q^2C} \tag{44}$$

in which C_1 and C_2 are constants.

For $x = 0$, $y = 0$; hence

$$0 = C_2 - \frac{w}{q^2C}; \; C_2 = \frac{w}{q^2C}.$$

For $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1 \cos qx - qC_2 \sin qx - \frac{wl}{2C} + \frac{wx}{C} \tag{45}$$

$$0 = qC_1 \cos \frac{ql}{2} - qC_2 \sin \frac{ql}{2}$$

$$C_1 = C_2 \tan \frac{ql}{2} = \frac{w}{q^2C} \tan \frac{ql}{2}.$$

Hence the equation of the elastic line is

$$y = \frac{w}{q^2C}\left(\tan \frac{ql}{2} \sin qx + \cos qx\right) - \frac{wx(l-x)}{2C} - \frac{w}{q^2C}. \tag{46}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, is

$$y_1 = \frac{w}{q^2C}\left(\frac{1}{\cos \frac{ql}{2}} - 1\right) - \frac{wl^2}{8C}. \tag{47}$$

The maximum fiber stress, at the center, is

$$f = \frac{C}{A} + \frac{wl^2 \cdot v}{8I} + \frac{Cy_1 v}{I}. \tag{48}$$

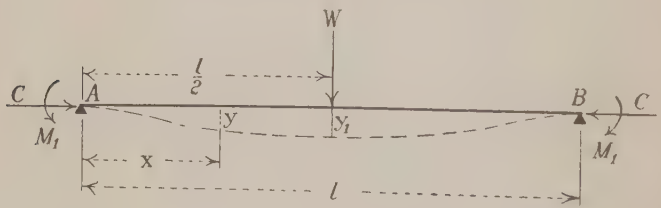


FIG. 299.—Fixed ends.

Fixed Ends; Load at the Center.—The fundamental equation is (Fig. 299)

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{Wx}{2} - Cy + M_1. \tag{49}$$

The general integral of this is, calling $\frac{C}{EI} = q^2$;

$$y = C_1 \sin qx + C_2 \cos qx + \frac{M_1}{C} - \frac{Wx}{2C} \tag{50}$$

in which C_1 and C_2 are constants.

For $x = 0$; $y = 0$; hence

$$0 = C_2 + \frac{M_1}{C}; \; C_2 = -\frac{M_1}{C}.$$

For $x = \frac{l}{2}$; $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1 \cos qx - qC_2 \sin qx - \frac{W}{2C} \tag{51}$$

$$0 = qC_1 \cos \frac{ql}{2} - qC_2 \sin \frac{ql}{2} - \frac{W}{2C}$$

$$C_1 = \frac{M_1 \tan \frac{ql}{2}}{C} + \frac{W}{2qC \cos \frac{ql}{2}}.$$

Hence the equation of the elastic line is

$$y = -\frac{M_1 \tan \frac{ql}{2} \sin qx}{C} + \frac{W \sin qx}{2qC \cos \frac{ql}{2}} - \frac{M_1}{C} \cos qx + \frac{M_1}{C} - \frac{Wx}{2C}. \tag{52}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, is

$$y_1 = -\frac{M_1}{C}\left(\frac{1}{\cos \frac{ql}{2}} - 1\right) + \frac{W \tan \cdot \frac{ql}{2}}{2qC} - \frac{Wl}{4C}. \tag{53}$$

The maximum fiber stress is then, at the center

$$f = \frac{C}{A} + \left(\frac{Wl}{4} - M_1\right)\frac{v}{I} + \frac{Cy_1 v}{I} \tag{54}$$

and at the ends

$$f = \frac{C}{A} + \frac{M_1 v}{I}. \tag{55}$$

Fixed Ends; Uniform Load.—The fundamental equation is

$$EI \cdot \frac{d^2y}{dx^2} = -\frac{wx(l - x)}{2} - Cy + M_1. \tag{56}$$

The general integral of this is, calling $\frac{C}{EI} = q^2$;

$$y = C_1 \sin qx + C_2 \cos qx + \frac{M_1}{C} - \frac{wlx}{2C} + \frac{wx^2}{2C} - \frac{w}{q^2C} \tag{57}$$

in which C_1 and C_2 are constants.

For $x = 0$, $y = 0$; hence

$$0 = C_2 + \frac{M_1}{C} - \frac{w}{q^2C}$$

$$C_2 = \frac{w}{q^2C} - \frac{M_1}{C}$$

For $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; hence we must find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = qC_1 \cos qx - qC_2 \sin qx - \frac{wl}{2C} + \frac{wx}{C} \tag{58}$$

$$0 = qC_1 \cos \frac{ql}{2} - qC_2 \sin \frac{ql}{2}$$

$$C_1 = C_2 \tan \frac{ql}{2} = \left(\frac{w}{q^2C} - \frac{M_1}{C}\right) \tan \frac{ql}{2}$$

Hence the equation of the elastic line is

$$y = \frac{w}{q^2C}(\tan \frac{ql}{2} \sin qx + \cos qx - 1) - \frac{M_1}{C}(\tan \frac{ql}{2} \sin qx + \cos qx - 1) - \frac{wx(l - x)}{2C}. \tag{59}$$

The maximum deflection, at the center, for $x = \frac{l}{2}$, is

$$y_1 = \left(\frac{w}{q^2C} - \frac{M_1}{C}\right)\left(\frac{1 - \cos \frac{ql}{2}}{\cos \frac{ql}{2}}\right) - \frac{wl^2}{8C}. \tag{60}$$

The maximum fiber stress is then,

at the center $$f = \frac{C}{A} + \left(\frac{wl^2}{8} - M_1\right)\frac{v}{I} + \frac{Cy_1v}{I} \tag{61}$$

at the ends $$f = \frac{C}{A} + \frac{M_1v}{I}. \tag{55}$$

6. Example of Compression and Flexure.—Assume a horizontal column 30 feet long, with hinged ends, made of two 15 inch channels, 33.9 pounds per foot each, with a total area of 19.8 square inches. The moment of inertia is 625.2. Assume that it carries a total transverse load of 300 pounds per foot, and a compression of 158,400 pounds.

By Eq. (32): $f = \frac{158{,}400}{19.8} + \frac{300 \times 30 \times 30 \times 12 \times 7.5}{8 \times 625.2} = 12{,}860.$

$$d_b = \frac{5 \times 300 \times 30 \times 30 \times 30 \times 30 \times 12 \times 12 \times 12 \times 12}{384 \times 12 \times 30{,}000{,}000 \times 625.2} = 0.292$$

By Eq. (33): $f = 12{,}860 + \frac{158{,}400 \times 0.292 \times 7.5}{625.2} = 13{,}414.$

Both values are clearly too small.

By Eq. (35): $\frac{Cl^2}{10E} = \frac{158{,}400 \times 30 \times 30 \times 12 \times 12}{10 \times 30{,}000{,}000} = 68.429$

$$f = 8{,}000 + \frac{300 \times 30 \times 30 \times 12 \times 7.5}{8(625.2 - 68.429)} = 13{,}336.$$

By Eqs. (47) and (48): $q = \sqrt{\frac{158{,}400}{30{,}000{,}000 \times 625.2}} = \frac{1}{344.1}$

$\frac{ql}{2} = 0.5231$ radians $= 29°56'$

$$\cos\frac{ql}{2} = 0.8666$$

$$y_1 = \frac{300 \times 118{,}409}{12 \times 158{,}400}\left(\frac{1}{0.8666} - 1\right) - \frac{300 \times 30 \times 30 \times 12 \times 12}{12 \times 8 \times 158{,}400}$$

$$= 0.321$$

$$f = 8{,}000 + \frac{7.5}{625.2}\left(\frac{300 \times 30 \times 30 \times 12}{8} + 158{,}400 \times 0.321\right)$$

$$= 13{,}470.$$

This indicates that Eq. (33) is very close, and it is much easier to compute.

7. Torsion Combined with Flexure.—The case of most practical importance where torsion is combined with other stresses is the combination of *torsion and flexure*, as for instance, a shaft, which is never exposed to pure torsion. If a shaft is driven by a belt passing over a pulley, the tension of the belt is different on the two sides (Fig. 300) or $P_1 > P_2$. The resultant, P, of P_1 and P_2 then acts

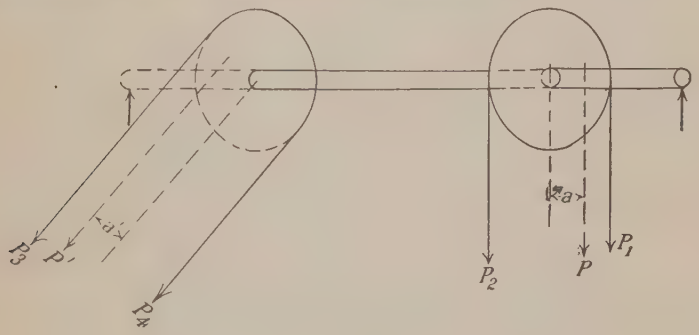


FIG. 300.

at some point between the two, at a distance a from the axis, and tends to rotate the shaft with accelerated rotation. The power is taken off the shaft at another pulley driving another belt leading to the machine to be operated, and when the motion is uniform the two belts exert equal and opposite twisting moments or torque, T. If the horsepower transmitted is H, and if n is the number of revolutions per minute, then

$$2\pi anP = 33{,}000H,$$

$$T = Pa = \frac{33{,}000H}{2\pi n} \tag{62}$$

dimensions being in feet.

The two forces, P and P', together with the reactions of the supports, from hangers or otherwise, constitute transverse loads on the shaft, not all lying in a plane. There are other transverse loads, such as the weight of the shaft, pulleys, etc., the thrust of cranks, etc. Whatever these may be, at any section of the shaft there are, in general, bending moments in two planes at right angles, containing the axis. These two moments, at any section, may be compounded, and the resultant bending moment, M_b, may be found, acting in some known plane AA' in Fig. 301, which shows the section of the shaft. This will cause a normal stress intensity at a distance r from the center of the shaft, equal to

$$f_t = \frac{M_b r}{I}. \tag{63}$$

The torque or twisting moment T, which will here be called M_t, will cause at the same point a shear intensity, acting in the plane of the section, at right angles to the radius, equal to

$$f_s = \frac{2rM_t}{\pi R^4} = \frac{rM_t}{J} = \frac{rM_t}{2I} \tag{64}$$

by Eq. (6) of Chap. VIII, J being the polar moment of inertia, which for a circular shaft equals twice the moment of inertia about a diameter. At this same point there will be a flexural shear acting on the plane of the section in the direction AA' (Fig. 301), whose intensity, as shown in Chap. X, cannot be accurately found, but which may be taken as equal to

$$f_s' = \frac{QS}{Ib} \tag{65}$$

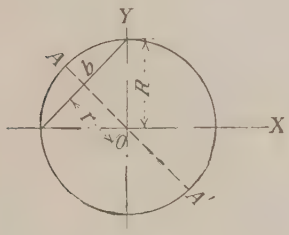


FIG. 301.

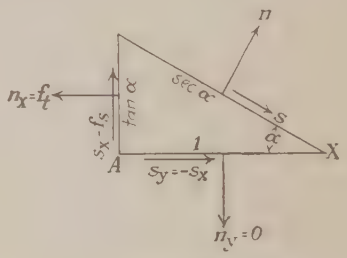


FIG. 302.

in which b is the breadth at the point in question, perpendicular to AA'. At the circumference of the shaft,

$$f_t = \frac{M_b R}{I}; f_s = \frac{M_t R}{2I} = \frac{2M_t}{\pi R^3} = \frac{2M_t}{AR}; f_s' = 0.$$

Hence, looking at the shaft in the direction AA', Fig. 301, a triangular particle has the unit stresses shown in Fig. 302; and by Eq. (13) of Chap. V the principal stresses are

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{f_t}{2} \pm \tfrac{1}{2}\sqrt{f_t^2 + 4f_s^2} = \frac{R}{2I}(M_b \pm \sqrt{M_b^2 + M_t^2}) \tag{66}$$

$$= \frac{2}{\pi R^3}(M_b \pm \sqrt{M_b^2 + M_t^2}).$$

From Eq. (10) of Chap. V, the angle α, which one principal stress makes with Ax, is given by the equation

$$\tan 2\alpha = \frac{2s_x}{f_t} = \frac{M_t}{M_b}. \tag{67}$$

It is obvious that n_1 will have the same sign as f_t, and n_2 the opposite sign. hence if A is the side of the shaft where n_x is tension, the principal stresses will act as shown in Fig. 303. On the opposite side, where n_x is compression, n_1 will be compression and n_2 tension. One of the two values of α will always lie between the values $\pm 45°$. When $M_t = 0$, AX and AY are the principal planes, and when $M_b = 0$, the principal planes are at 45° with AX. The max. intensity thus occurs on an oblique section of the shaft.

The greatest value of n_1 (intensity of direct stress) will occur at the outside of the section of the shaft where

$$M_b + \sqrt{M_b^2 + M_t^2}$$

is greatest.

The greatest intensity of shearing stress will occur at this same section, on planes making angles of 45° with the principal stresses, and by Eqs. (18) and (19) of Chap. V, its value will be

$$\text{Max. shear} = \frac{n_1 - n_2}{2} = \frac{2}{\pi R^3}\sqrt{M_b^2 + M_t^2}. \tag{68}$$

Principal Strains.—These are given by Eq. (1*a*), or inserting the values of n_1 and n_2, and assuming $m = 4$,

$$Ee_1 = \frac{R}{2I}\left(\frac{m-1}{m}M_b + \frac{m+1}{m}\sqrt{M_b^2 + M_t^2}\right) = \frac{R}{8I}\left(3M_b + 5\sqrt{M_b^2 + M_t^2}\right) \tag{69}$$

$$Ee_2 = \frac{R}{2I}\left(\frac{m-1}{m}M_b - \frac{m+1}{m}\sqrt{M_b^2 + M_t^2}\right) = \frac{R}{8I}\left(3M_b - 5\sqrt{M_b^2 + M_t^2}\right). \tag{70}$$

The values of Ee_1 and Ee_2 are not actual intensities of stress, but what were termed in Chap. IV, Art. 8, *ideal stress intensities*. They should be used in design-

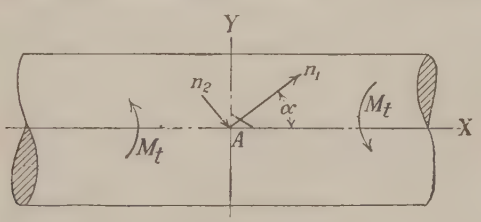


FIG. 303.

ing if the criterion for safe strength is considered to be *strain intensity* rather than *stress intensity* (see Chap. XXIII).

8. Torsion Combined with Direct Tension.—If a bar, in addition to a twisting moment, is exposed to a direct tension, T, causing a tensile stress intensity of $\frac{T}{A}$ on each unit area, the situation will be as shown in Fig. 302 except that the value of n_x will be

$$n_x = f_t = \frac{T}{A}.$$

Hence, by the formulae of the previous article,

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{f_t}{2} \pm \frac{1}{2}\sqrt{f_t^2 + 4f_s^2} = \frac{T}{2A} \pm \frac{1}{2}\sqrt{\frac{T^2}{A^2} + \frac{16M_t^2}{A^2R^2}}$$

$$= \frac{1}{2A}\left(T \pm \sqrt{T^2 + \frac{16M_t^2}{R^2}}\right) \tag{71}$$

$$\tan 2\alpha = \frac{2s_x}{f_t} = \frac{4M_t}{TR}. \tag{72}$$

$$\text{Max. shear} = \frac{1}{2A}\sqrt{T^2 + \frac{16M_t^2}{R^2}} \tag{73}$$

$$\left.\begin{matrix} Ee_1 \\ Ee_2 \end{matrix}\right\} = \frac{1}{2A}\left(\frac{m-1}{m}T \pm \frac{m+1}{m}\sqrt{T^2 + \frac{16M_t^2}{R^2}}\right) \tag{74}$$

for $m = 4$:

$$\left.\begin{matrix} Ee_1 \\ Ee_2 \end{matrix}\right\} = \frac{1}{8A}\left(3T \pm 5\sqrt{T^2 + \frac{16M_t^2}{R^2}}\right). \tag{75}$$

9. Torsion Combined with Direct Compression.—The results of the last article will apply to this case by making T negative. In Eq. (71), T is positive and n_1 and n_2 are the algebraic values of the principal stresses, tension being positive; thus n_1 is positive (tension) and n_2 is negative (compression); similarly e_1 or e_2 is an elongation when positive and a contraction when negative, and e_1 is actually always positive and e_2 negative.

If T becomes a compression, or negative, the maximum principal stress, n_1, will be compression, and n_2 will be tension; while e_1 will be a contraction and e_2 an extension.

10. Torsion Combined with Flexure and Direct Tension.—This case is similar to the preceding ones, and Fig. 302 again applies. Here

$$s_x = f_s = \frac{RM_t}{2I} \tag{64}$$

$$n_x = f_t = \frac{T}{A} + \frac{M_bR}{I}. \tag{76}$$

By substituting these values in Eq. (66), the values of n_1 and n_2 may be found, and similarly values for Ee_1 and Ee_2 may be obtained similar to Eqs. (69) and (70).

For direct compression, T becomes negative.

11. Torsion combined with transverse shear alone would simply mean that at any point of the section there would be a shear due to the torsion acting tangentially, and a shear due to the transverse shearing force; the resultant of these would be the actual shear at the point.

12. Torsion Combined with Other Stresses, for Bars Not Circular in Section. In such cases, the torsional stress intensity at any point must be determined or assumed, bearing in mind the facts stated in Art. 14 of Chap. VIII; and the direct stress intensity, whether arising from flexure or from an axial force, or both, must be computed, and Fig. 303 will then apply, by using the values ascertained for n_x and s_x. It is not worth while to endeavor to develop general formulae.

13. Professor Hancock[1] held round bars in torsion and then applied tension, and found the elastic limit lowered. If held in torsion to ⅓, ⅔, and the full torsion elastic limit, he found the tension elastic limit lowered as follows: for nickel steel with tension elastic limit of 56,000, this was lowered respectively 7 per cent, 21 per cent, 63 per cent; for carbon steel with tension elastic limit of 34,000, this was lowered respectively 6 per cent, 30 per cent, 54 per cent. These results are close to what would be anticipated. For instance, assuming the shearing elastic limit at 0.8 of that for tension, or 27,200 pounds for the carbon steel, a torsion

[1] *Trans. A.S.T.M.*, 1905.

equal to this would give a tension of the same amount on a plane at 45° with the axis at the outer fiber. A tension of 100 − 54 = 46 per cent of 34,000, or 15,640, would give on a 45° plane one-half this, or 7,820, which added to 27,200 is 35,020. If the exact shearing elastic limit were known and the principal stresses accurately found, the results would still be close, probably within the errors of observation.

14. The Hook.—A common example of combined tension and flexure is the hook (Fig. 304). Here, if the curvature of the hook is not considered in the formula for flexure, that is, if planar distribution is assumed,

$$\text{Max. } f = \frac{W}{A} + \frac{Wd \cdot v_1}{I} \text{ on the inside fiber.} \tag{77}$$

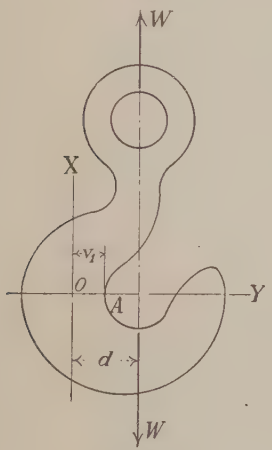


FIG. 304.

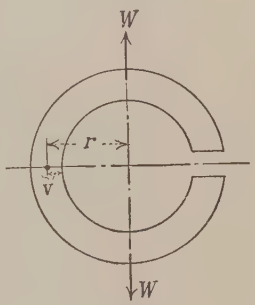


FIG. 305.

The same equation will hold for an open link (Fig. 305), d being equal to r.

In Chap. XX the stresses in curved pieces are discussed, and it is shown that if the assumption is made that plane sections remain plane, the distribution is not planar, but that the stress intensity at a distance v_1 from the axis through the center of gravity is given by the equation

$$\text{Max. } f = \frac{W}{A} + \frac{Wd}{Ar} + \frac{Wdrv_1}{I_o(r + v_1)} \tag{78}$$

in which

$$I_o = \int \frac{v^2 r dA}{r + v}.$$

For $r = \infty$, Eqs. (77) and (78) agree. For hooks, the value of r is small, and Eq. (78) would be much more accurate if founded on correct assumptions. But the assumption that plane sections remain plane may be less accurate than that of planar distribution which is the basis of Eq. (77), whether for straight or curved bars, so that it cannot yet be proved conclusively that Eq. (78) is better than Eq. 77.

15. The Bussey Bridge Hangers.—In 1887 a railroad bridge near Boston, Mass., known as the Bussey bridge, collapsed as a passenger train was passing over

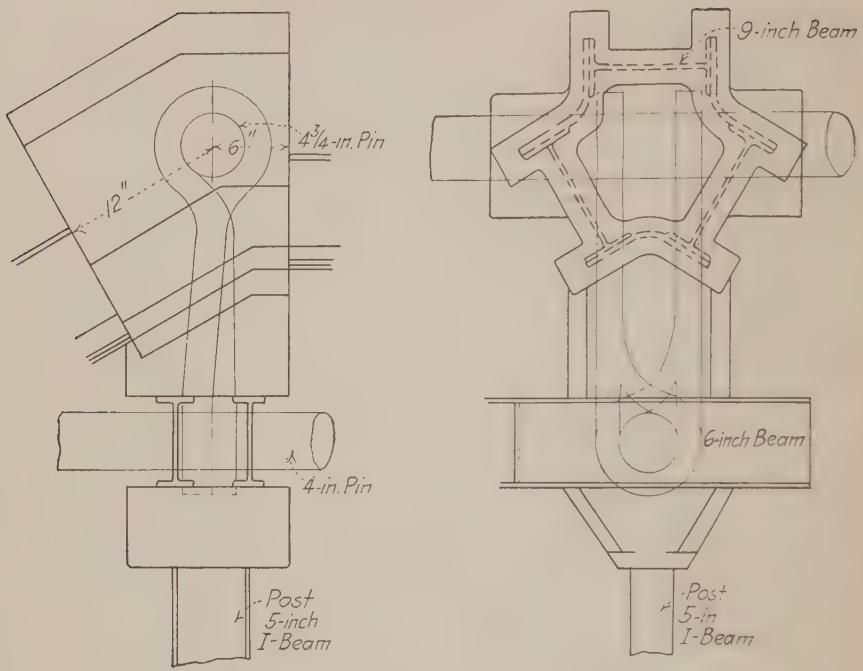


FIG. 306.—Longitudinal and transverse views of hangers.

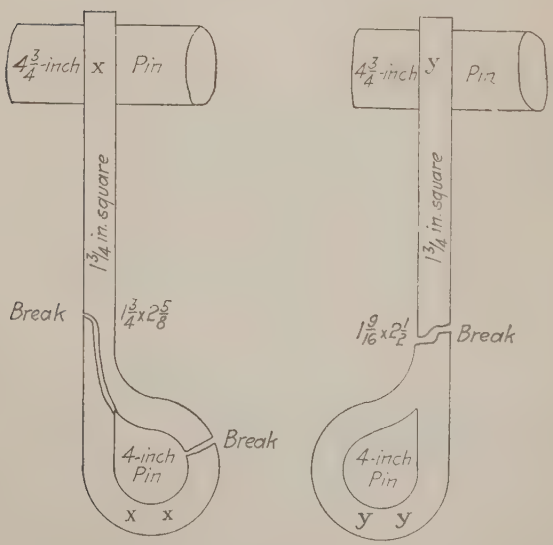


FIG. 307.—Longitudinal and transverse views of broken hangers.

it; 23 persons were killed and over 100 injured. The cause of the disaster was the use of eccentric hangers, similar to hooks. The bridge was a skew deck bridge of peculiar construction. The top chord of one truss was composed of three I-beams, with bent plates riveted to the flanges, forming a hollow column. The members of the chord abutted at their ends against cast iron joint blocks, through which extended pins, to which the diagonals were connected. The vertical posts abutted also against the cast iron joint blocks. The construction of the hip joint is shown in Figs. 306–307. The floor beam consisted of two I-beams, with a pin through each end, this pin being of course parallel with the length of the trusses, while the pins of the truss were of course at right angles to the trusses. The floor beams were suspended from the truss pins by forged and welded wrought iron hangers, two at each joint, the upper eye encircling the truss pins, and the lower eye the floor beam pins.

Part of the load on the floor beam at the hip joint could be carried by an I-beam post, but this was weak, and not intended to carry the load at this point, which was all supposed to be carried by the hangers to the truss joint. Allowing for what the I-beam post could carry, there was in any event a large load carried by the hangers, each of the two carrying one-half, like a hook. This load was sufficient to cause an extreme fiber stress, computed by Eq. (77), well above the elastic limit of the material, and the hangers failed by repeated stress, the part which failed having stood for 11 years, all the time exposed to a heavy passenger traffic. One or both of the hangers which failed showed signs of defective or burned material at the fractured section, and the weld was itself an element of weakness; but the case was an example of bad construction, the failure of which could have been predicted with certainty by a competent engineer.

At the time of this accident the railroad companies of Massachusetts, and of other states as well, did not have complete plans of their bridges, and many of them had no expert bridge engineer in charge of them. The accident resulted in the passage of an Act by the Legislature requiring the companies to file plans of all bridges with the Board of Railroad Commissioners, and to have periodical inspections made by the companies and reports submitted to said Board, which was also authorized to employ a Consulting Engineer to examine such plans and reports and to make such personal inspections as he might consider necessary. The writer occupied this position for 27 years, and found many bridges which required strengthening or rebuilding. Most states now have such an engineer connected with their railroad or public utility commissions.

The report of the investigation of this accident was published by the Railroad Commission in a special volume, in 1887.

16. Work of Combined Stresses.—To find the work done for the case of flexure and direct stress, the stress on any element of area dA, if M is the actual moment, and T the direct stress, will be

$$f \cdot dA = \frac{T dA}{A} + \frac{Mv}{I} \cdot dA.$$

The work done by this force on a slice dx in length, integrated over the area A, will give the total work done on the slice, and this integrated over the length will give the total work, or, for gradual application,

$$W = \int\int \frac{f^2 dA}{2E}\, dx = \frac{1}{2E}\int_o^l dx \int_{-y_1} dA \left(\frac{T^2}{A^2} + \frac{M^2 v^2}{I^2} + \frac{2TMv}{AI}\right)$$
$$= \frac{1}{2E}\int_o^l \left(\frac{T^2}{A} + \frac{M^2}{I}\right) dx = \int_o^l \frac{M^2 dx}{2EI} + \frac{T^2 l}{2EA}. \qquad (79)$$

The last expression is found from the fact that v is the distance from an axis through the center of gravity, so that $\int v dA = 0$. The above expression for W is the sum of the work due to direct stress alone and that due to the bending moment alone. This also results from the fact that the direct stress might be supposed to act alone first, and then the flexure might act, each doing its work as if independent.

Hence, *the work done by combined forces is the same as the sum of the work done by each force separately.* This holds true for all cases.

17. Instances of combined stresses are very common. Flexure and direct stress occurs in every member of a structure that is not vertical, and in all columns which bend laterally, or are eccentrically loaded; also in inclined beams in roofs or buildings. Torsion seldom occurs alone, but is generally combined with flexure and often with direct stress. A knowledge of this chapter is therefore important for the structural engineer.

18. Remarks Regarding Combined Stresses.—Specifications relating to combined stresses generally state that the combined stresses must not exceed the limits allowed for axial stresses alone. This is indefinite unless the conditions to be assumed are also prescribed. Thus a top chord of a truss is not infrequently exposed to transverse loads in addition to its own weight, and the case is one of compression combined with flexure. Sometimes the ties of a railroad track rest directly on the top chord of a deck truss, with no stringers or floor beams. When the truss deflects, the top chord is held more or less rigidly at the joints, but it is never in the ideal condition of fixed ends which is assumed in the equations that have been given; and of course it is not with hinged ends, for although the truss may be pin-connected, the top chord panel sections are continuous. The precise condition varies with the load, and can never be known. When the live load is on two adjacent panels the chord at the joint between is more nearly fixed than when the live load is on one panel only. In the specifications which the author prepared for the Massachusetts Railroad Commission, he attempted to make the matter at least more definite, if not correct, by the following provision:

"In case a member of a riveted structure or a member continuous over the joints is exposed to a transverse load in addition to a direct stress, the maximum fiber stress shall generally be computed as follows:

"(*a*) In a tension piece the direct stress on the net section per square inch shall be added to the maximum bending stress on the net section, computed by assuming the bending moment to be three-quarters of the maximum bending moment which would occur if the piece were simply supported at the joints.

"(*b*) In a compression piece the compressive stress per square inch (f) given by the formula

$$f = \frac{P}{A}\left(1 + \frac{1}{15{,}000}\,\frac{l^2}{r^2}\right)$$

P being the total compression and A the gross area, shall be added to the fiber stress due to bending, computed on the net section, as above described.

"In either case the bending moment at the joints shall be considered the same as the maximum computed as above described, but opposite in direction."

This specification is of course inaccurate, as any such specification must be, but it is definite. If specifications are to be used and followed, they should be definite even if wrong, for an error may be corrected, while an ambiguity cannot be interpreted, unless one party has power to decide on the interpretation, in which case it is only fair that he should state his interpretation and make the specifications definite, in order that the other party to the contract may know what to expect. Many engineers, however, prefer to have this matter vague, perhaps because they are not clear on it themselves, but do not like to say so.

CHAPTER XIV

BEAMS OVER MORE THAN TWO SUPPORTS

1. The previous chapters have treated of beams supported at two points, or of beams fixed at one or both ends. A beam may be supported at only one end, provided there is an outer moment applied at that end sufficient to maintain equilibrium (Fig. 308); or such a beam may be considered to extend into the supporting wall, and to have another point of support to the left, with a downward reaction there sufficient to produce the requisite moment at the face of the wall A; or it may be considered to be continuously supported by a downward reaction inside the wall. In any case, on a vertical section just outside the wall the shear is P and the moment Pl in the case shown.

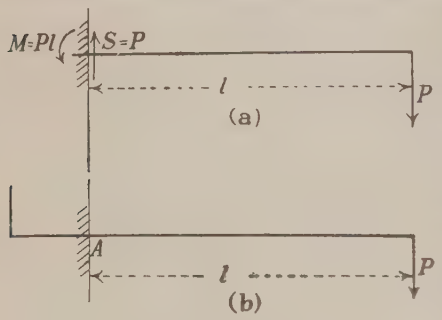


FIG. 308.

2. Beams frequently are supported at more than two points. If a straight beam is laid over several supports, all on the same level, there will be no reactions if there is no load (not even dead load). The same is true if a slightly curved beam is laid over supports which fit the curve of the entirely unloaded beam. If, however, a straight beam is laid over supports not on a level, or not in a straight line, there will be reactions even under no load at all, if the beam is to touch each support.

The usual case is a straight beam on level supports (Fig. 309) the beam and supports all in a plane, and the loads acting in the same plane, which is taken as that of the paper. This case is statically undetermined as regards the outer forces; that is, the reactions cannot be found by the principles of statics alone. The beam is supposed to touch each support, and the reaction may be upward or downward.

FIG. 309.

Let there be n supports and $n - 1$ spans. There will in the general case be three unknown quantities for each support; namely, a point of application and two component forces. The points of application are generally fixed by pins or otherwise, or are assumed; also all but one support has rollers on a horizontal surface or a surface of known inclination, so that in all there are $n + 1$ unknown quantities, which must be found in order to determine the reactions. Since there are but three statical equations, the reactions are statically determined only when

$n = 2$, that is for two supports. In other cases there are $n - 2$ conditions that must be found from elastic considerations, that is, from conditions of slope or deflection. These conditions are easily seen to be the following: considering the beam as supported on the two end supports, and loaded by the given loads and by the $n - 2$ intermediate reactions, the end reactions and all shears, moments, slopes, and deflections may be determined in terms of the known loads and the $n - 2$ intermediate reactions, and the $n - 2$ conditions desired are that at each of the intermediate $n - 2$ supports *the deflection is zero;* or, if the supports are not on a level or in a straight line, the deflection at each intermediate support is a known quantity.

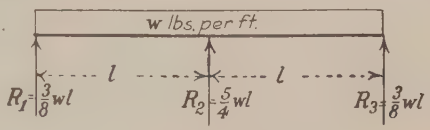


FIG. 310.

3. Example of Two Equal Spans with Uniform Load.—In the case of two equal spans and three supports (Fig. 310) loaded with a uniformly distributed load of w pounds per foot, the downward deflection at the center of the span $2l$, under the load w, would be

$$\frac{5}{384}\frac{w(2l)^4}{EI} = \frac{5}{24}\frac{wl^4}{EI}$$

while the upward deflection due to R_2 would be

$$\frac{1}{48}\frac{R_2(2l)^3}{EI} = \frac{1}{6}\frac{R_2 l^3}{EI}.$$

Equating these, $R_2 = \frac{5}{4}\ wl$; whence $R_1 = R_3 = \frac{3}{8}\ wl$. The moment at the center support is

$$M_2 = \frac{3}{8}wl^2 - \frac{1}{2}wl^2 = -\frac{wl^2}{8}. \tag{1}$$

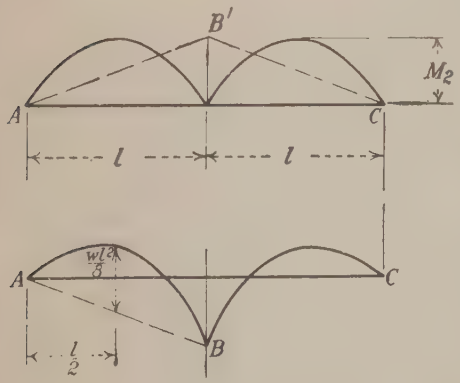


FIG. 311.

The curve of moments will therefore be found if in Fig. 311 we draw the two parabolas for moment on *simple spans*, and superpose on it the effect of the negative moment M_2; so that the moment at any point is shown by the distance of the curve from the line $AB'C$, positive when the curve is above the line and negative when it is below.

The moment at the center of l will be

$$\frac{3}{8}wl\cdot\frac{l}{2} - \frac{wl}{2}\cdot\frac{l}{4} = \frac{wl^2}{16}. \tag{2}$$

The point where the moment is zero is found from the equation

$$\frac{3}{8}wl\cdot x_0 - \frac{wx_0^2}{2} = 0$$

from which

$$x_0 = \tfrac{3}{4}l \tag{3}$$

The moment at any point distant x from A is

$$M = \frac{3}{8}wlx - \frac{wx^2}{2}. \tag{4}$$

This is a maximum where $\frac{dM}{dx} = 0$, or $x = \frac{3}{8}l$, and max. $M = \frac{9}{128}wl^2$. (5)

If the uniform load covers only one span, $M_2 = -\frac{wl^2}{16}$, and the end reaction of the unloaded span is $-\frac{wl}{16}$. At the end of the loaded span the reaction is found from the equation

$$-\frac{wl^2}{16} = Rl - \frac{wl^2}{2}$$

whence $$R = \frac{7}{16}wl.$$

From these values, the center reaction R_2 is

$$R_2 = wl + \frac{wl}{16} - \frac{7}{16}wl = \frac{5}{8}wl.$$

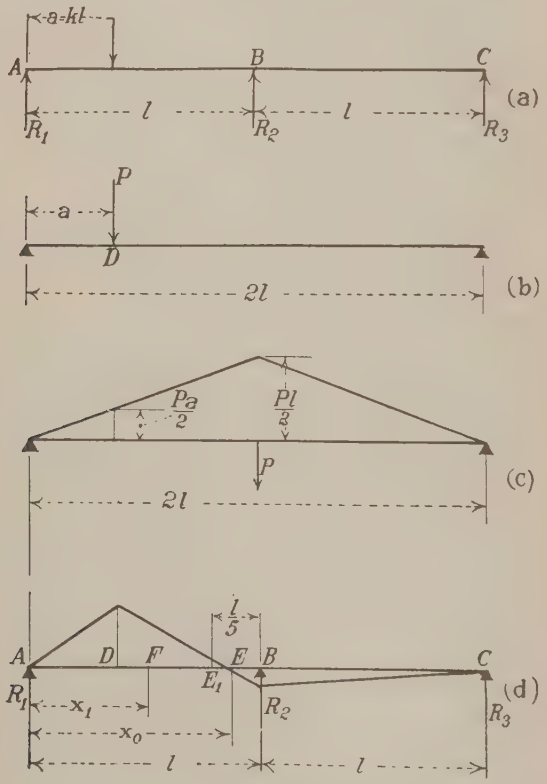


Fig. 312.

4. Two Equal Spans with Single Concentrated Load (Fig. 312*a*).—The deflection at the center of the simple span $2l$ due to the load P as in Fig. 312*b* is, by Eq. (36) of Chap. XII,

$$\frac{Pa}{12EI}(3l^2 - a^2). \qquad (6)$$

This is easily found as follows by the area-moment method: Fig. 312*c* gives the moment diagram for the span $2l$ with load P at center. The deflection of the center below the horizontal is the moment (divided by EI) of one-half the moment area, about one end, or

$$\frac{1}{EI} \cdot \frac{Pl}{2} \cdot \frac{l}{2} \cdot \frac{2}{3} l = \frac{Pl^3}{6EI}.$$

The deflection of D from the tangent at the center is the moment about D of the moment area between the center and D (divided by EI) or

$$\frac{Pa}{2EI}(l-a)\left(\frac{l-a}{2}\right) + \frac{(l-a)}{2EI} \cdot \frac{P(l-a)}{2} \cdot \frac{2}{3}(l-a).$$

The deflection of D below the horizontal is the difference of these or $\frac{Pa}{12EI}(3l^2 - a^2)$, as stated above.

The upward deflection at the center due to R_2 alone is $\frac{R_2 l^3}{6EI}$. Equating these two values

$$\frac{R_2 l^3}{6EI} = \frac{Pa}{12EI}(3l^2 - a^2)$$

$$R_2 = \frac{Pa(3l^2 - a^2)}{2l^3} = \frac{Pk(3 - k^2)}{2}. \tag{7}$$

The statical equations then give

$$R_1 = P - \frac{Pk(5 - k^2)}{4} \tag{8}$$

$$R_3 = -\frac{Pk(1 - k^2)}{4} \tag{9}$$

$$M_2 = -\frac{Pkl(1 - k^2)}{4}. \tag{10}$$

For such a single load in the left span, M_2 is negative, producing tension in the top fiber or chord; R_3 is therefore obviously downward, and R_1 and R_2 upward. It can easily be *seen*, without any equations, that such must be the case; for the load on the left span would deflect it and the elastic line at B would slope upward to the right, and this, since there is no load on the right span, would lift the beam from the right support unless it were held down. The moment diagram is shown in Fig. 312*d*. The point of inflexion, E, is found by placing

$$M = R_1 x_0 - P(x_0 - kl) = 0,$$

which gives

$$x_0 = \frac{4l}{5 - k^2}. \tag{11}$$

This value will be least when k is least, and when $k = 0$ it becomes $\frac{4}{5}l$. It follows that the point of inflexion for any single load, and hence for any combination of loads, is always in the fifth part of the span nearest to R_2, or to the right of the point E_1.

To determine now much of the structure to load with a uniform load to produce max. positive shear at any given point F in the left span, distant x_1 from A it is obvious that BC must be unloaded, since any load there causes a downward R_1; hence FB alone should be loaded. For max negative shear, load AF and BC.

The max. positive shear may be found from Eq. (8), by making $P = wdx$, $k = \frac{x}{l}$, and integrating between the limits x_1 and l;

$$\text{Max.} + S = \frac{w}{16l^3}(7l^4 - 16l^3x_1 + 10l^2x_1^2 - x_1^4) \tag{12}$$

$$\text{Max.} - S = \frac{w}{16l^3}(l^4 + 10l^2x_1^2 - x_1^4) \text{ numerically.} \tag{13}$$

For maximum positive moment at any point F in the left span, BC must be entirely unloaded; but it does not follow that AB must be fully loaded, because, as just shown, a single load in AB produces negative moments near B. It has been shown above that, for any point in the distance $AE_1 = \frac{4}{5}l$, a single load in AB produces a positive M, while a single load in BC produces negative M anywhere in AB. Hence, for any point in AE_1,

$$\text{for max.} + M_1 \text{ load } AB$$
$$\text{for max.} - M_1 \text{ load } BC.$$

It will easily be found that if x_1 is the distance of the point taken from A

$$\text{Max.} + M = \frac{7}{16}wlx_1 - \frac{wx_1^2}{2} = \frac{wx_1}{16}(7l - 8x_1) \tag{14}$$

$$\text{Max.} - M \text{ (numerically)} = \frac{wlx_1}{16}. \tag{15}$$

For any point in AB distant $x_1 > \frac{4}{5}l$ from A, it is necessary first to find the point ($a = k_1l$) where a single load must lie in order to produce zero moment at x_1. This is found from Eq. (11), in which x_0 is now x_1; or

$$k_1 = \sqrt{5 - \frac{4l}{x_1}}. \tag{16}$$

A single load at any point for which k_1 is less than this value produces negative moment at the point x_1, and at any point for which k_1 is greater than this value produces positive moment at x_1.

Hence, for max. $+M$ at any point for which $x_1 > \frac{4}{5}l$, load from B to a point distant $k_1l = l\sqrt{5 - \frac{4l}{x_1}}$ from A; and for max. $-M$, load the remainder of AB, and also BC. It will be found that

$$\text{Max.} + M = \frac{w}{2x_1}(2l^3 + 4lx_1^2 - 5l^2x_1 - x_1^3) \tag{17}$$

$$\text{Max.} - M \text{ (numerically)} = \frac{wl}{8x_1}(13x_1^2 + 8l^2 - 20lx_1). \tag{18}$$

Note how these values all check for known values of x_1, such as $x_1 = 0$ or $x_1 = l$; also that the results given by Eqs. (15) and (18) and by Eqs. (14) and (17) agree for $x_1 = \frac{4}{5}l$.

5. The Theorem of Three Moments.—The method just explained, by which the necessary elastic equation is obtained by the condition that the intermediate deflections are zero, is simple enough for two spans, but becomes very complicated for more than two spans, especially if they are of unequal lengths.

Fortunately, a relation may be obtained between the moments existing for any given loading at any three consecutive supports. This is known as the *Three*

Moment Equation. Applying 3 to every set of three consecutive supports gives $n - 2$ equations if there are n supports, and these supplied exactly the required number of conditions.

There are several methods of determining the moments. The usual one has been to write the equation of the elastic line, and finding by the calculus the slope over a support by considering first the span on the left and then the span on the right. These two values being identical, the moment is obtained, but the method involves several pages of equations. The following is the simplest method known to the writer, and involves only results already obtained.

Figure 313 represents two adjoining spans. Considering the left span, we see that the slope at the support n depends upon four elements, viz.: (1) the moment M_{n-1}; (2) the moment M_n; (3) the load P_{n-1}; (4) the difference of level of

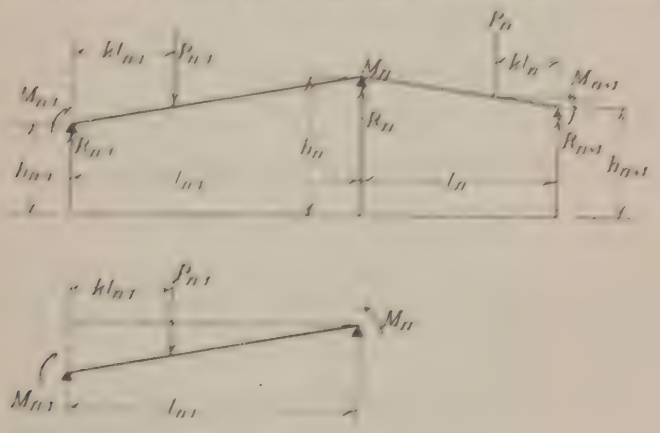


FIG. 313.

the supports. These elements will be considered separately and the results added. The slope at support n, *downward to the left*, will be;

Due to M_{n-1}: $$\frac{M_{n-1}l_{n-1}}{6EI} \tag{1}$$

Due to M_n: $$\frac{M_n l_{n-1}}{3EI} \tag{2}$$

Due to P_{n-1}: $$\sum \frac{P_{n-1}kl^2_{n-1}(1 - k^2)}{6EI}. \tag{3}$$

Note that k is not a constant quantity, but that each load has its own particular value of k.

Due to elevation: $$\frac{h_n - h_{n-1}}{l_{n-1}}. \tag{4}$$

The total slope is the sum of these, or

$$\alpha_n = \frac{1}{6EI}\left[M_{n-1}l_{n-1} + 2M_n l_{n-1} + \Sigma P_{n-1}kl^2_{n-1}(1 - k^2)\right] + \frac{h_n - h_{n-1}}{l_{n-1}}.$$

Similarly, for the right span the slope at the nth support, *downward to the right*, will be

$$\alpha_n' = \frac{1}{6EI}\left[M_{n+1}l_n + 2M_n l_n + \Sigma P_n kl^2_n(1 - k^2)\right] + \frac{h_n - h_{n+1}}{l_n}.$$

But $\alpha_n = -\alpha_n'$, hence

$$M_{n-1}l_{n-1} + 2M_n(l_{n-1} + l_n) + M_{n+1}l_n = -\Sigma P_{n-1}kl^2{}_{n-1}(1 - k^2) - \Sigma P_n kl_n{}^2(1 - k^2) - 6EI\left(\frac{h_n - h_{n+1}}{l_n} + \frac{h_n - h_{n-1}}{l_{n-1}}\right). \quad (19)$$

This is the general form of the Theorem of Three Moments, for the case of constant I and homogeneous material, which is the case usually assumed, even if not strictly true. The moments at the supports are plus when causing tension in bottom fibers. If all supports are on the same level, the last term disappears; or if the unloaded beam fits the supports; if such is not the case, and the position of the supports with reference to the unloaded beam is known, the stresses caused by making the beam fit the supports are computed once for all, by using only the last term of the second member, and in considering loads this term is omitted, the resulting stresses being combined.

For uniform load, P becomes wdx, k becomes $\frac{x}{l}$, so that the term with P_{n-1} becomes $-\frac{w}{l_{n-1}}\int_o^{l_{n-1}} (l^2 - x^2)xdx = -\frac{wl^3{}_{n-1}}{4}$ and similarly for the term with P_n; hence the equation becomes for uniform load covering both spans,

$$M_{n-1}l_{n-1} + 2M_n(l_{n-1} + l_n) + M_{n+1}l_n = -\frac{w}{4}(l^3{}_{n-1} + l_n{}^3). \quad (20)$$

The Theorem of Three Moments is a beautiful application of structural mechanics, and shows what simple relations may exist in apparently complex cases. It was first demonstrated by the French engineer Clapeyron, in 1857, and is universally used for treating beams over several supports.

This theorem gives as many equations, involving the unknown moments at the supports and the known loads, as there are intermediate supports. Hence it is possible to find from them the moment at each support. The reactions may then be found by writing the equation for the known moment at a support in terms of the known loads and the unknown reactions to the left (or right) of that support. Beginning with the second support, R_1 will be the only unknown quantity in the equation; then the equation for moment at the third support will have R_2 as the only unknown quantity, since R_1 is known

6. Examples.

(*a*) *Two Equal Spans.*—The equation of three moments, for uniform load over both spans, becomes

$$4M_2l = -\frac{wl^3}{2}; \; M_2 = -\frac{wl^2}{8}$$

as already found in Eq. (1).

Also

$$M_2 = -\frac{wl^2}{8} = R_1l - \frac{wl^2}{2}; \; R_1 = \frac{3}{8}wl$$

as already found.

(*b*) *Three Equal Spans* (Fig. 314).

$$4M_2l + M_3l = -\frac{wl^3}{2}$$

$$M_2l + 4M_3l = -\frac{wl^3}{2}.$$

Whence $$15M_3l = -\frac{3}{2}wl^3;\ M_3 = -\frac{wl^2}{10}.$$

By symmetry $M_2 = M_3$.

The reactions are thus easily found.

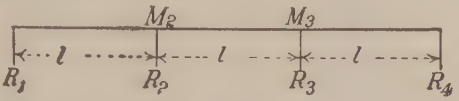


FIG. 314.

Problems

(1) For two equal spans loaded with a uniform load, show that when one span only is loaded, the max. neg. M is $-\frac{wl^2}{16}$, and the max. pos. M is $\frac{49}{512}wl^2$.

(2) In question 1, find where the max. + and − moments occur.

(3) Treat the cases where both spans are loaded and where one only is loaded by supposing first that there are two simple spans, and then finding the moment M_2. Check this with previous results.

(4) In the problem of Art. 4, if the left span is l_1 and the right span l_2, and $a = k_1l_1$, show that

$$M_2 = -\frac{pk_1l_1^2(1 - k_1^2)}{2(l_1 + l_2)}.$$

(5) In the problem of Art. 4, find, by integration, the reactions, and the moment M_2, when a uniform load covers one span, and check with Art. 3.

(6) Work out the proof of Eqs. (14), (15), (16), (17), and (18) of Art. 4.

(7) Look up the usual demonstration of the theorem of three moments, and compare with the one given.

(8) Find the reactions for a continuous girder of three spans:

(*a*) Loaded uniformly over all spans.

(*b*) Loaded uniformly over one end span.

(*c*) Loaded uniformly over the central span.

(*d*) With a single load at center of one end span.

(*e*) With a single load at the center of the center span.

CANTILEVER BEAMS

7. A cantilever is a beam which projects beyond a support. A beam fixed at one end is a cantilever; and a beam on two supports may project beyond either or both supports. Indeed, we have seen that a beam fixed at one end may be considered to have two points of support at the fixed end, so that it projects beyond one support which is at or near the face of the structure to which it is fixed.

We have seen that a beam continuous over more than one span has negative moments at the intermediate supports, positive moments near the centers of the spans, and points of inflexion where the moment is zero; generally (for full loading) two such points of inflexion in the intermediate spans and one in each end span. See the diagram for two spans in Fig. 311.

We have also seen that such a continuous beam, over n supports, requires $n - 2$ conditions in addition to the statical conditions, in order to render it determinate. Such a condition is furnished if the construction is such that the point

of inflexion is fixed. In the continuous beam the point of inflexion moves according to the loading. It may be fixed, and the moment there made zero, if the beam

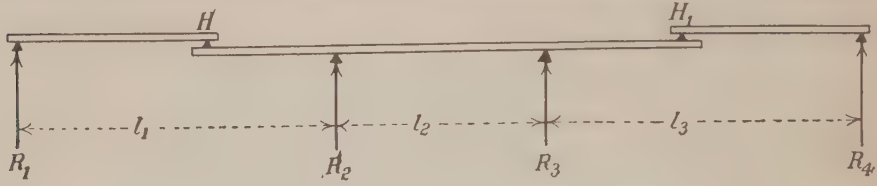


FIG. 315.

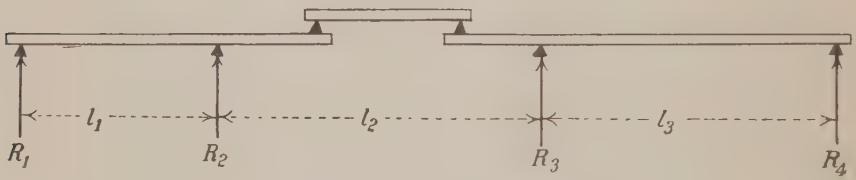


FIG. 316.

FIGS. 315 and 316.—Cantilever bridges.

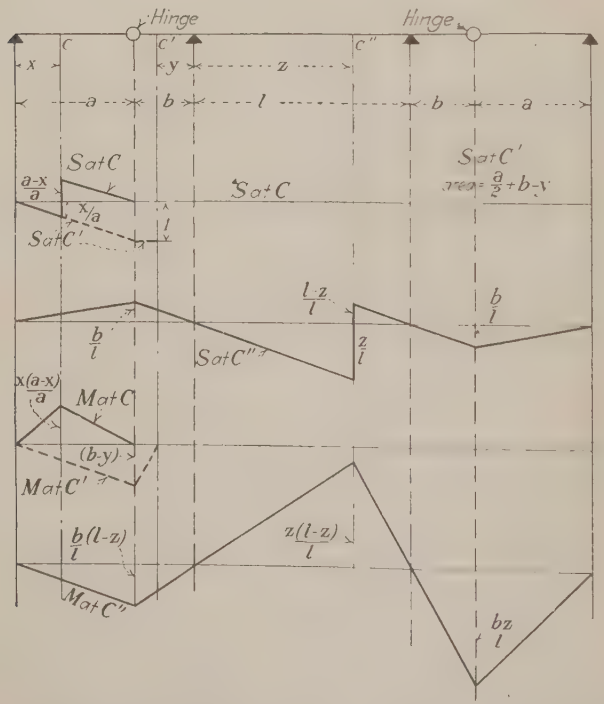


FIG. 317.—Influence lines for shear and moment on cantilever bridges.

is broken at that point and so connected that, while shear or load may be carried past the breaks, any tendency for a moment to exist there would cause rotation about the point; in other words, by introducing a hinge in the beam at that point,

or by supporting the part of the beam on one side upon the part on the other, as shown in Fig. 315. In that figure there are three spans, l_1, l_2, l_3; but span l_2 projects at each end beyond its supports and carries on each end the reaction from a part of the end spans. Span l_1 consists of a simple beam resting at one end on the support R_1 and at the other, at H, upon the projecting or cantilever end of span l_2. The point H is a hinge in span l_1, or the point where the moment is always zero—the point of inflexion. In this case there are four supports, $n = 4$, and the number of conditions necessary is $n - 2 = 2$, which are furnished by the two hinges H and H_1. There is one cantilever, HH_1, with two cantilever arms, and two simple beams, one in each end span.

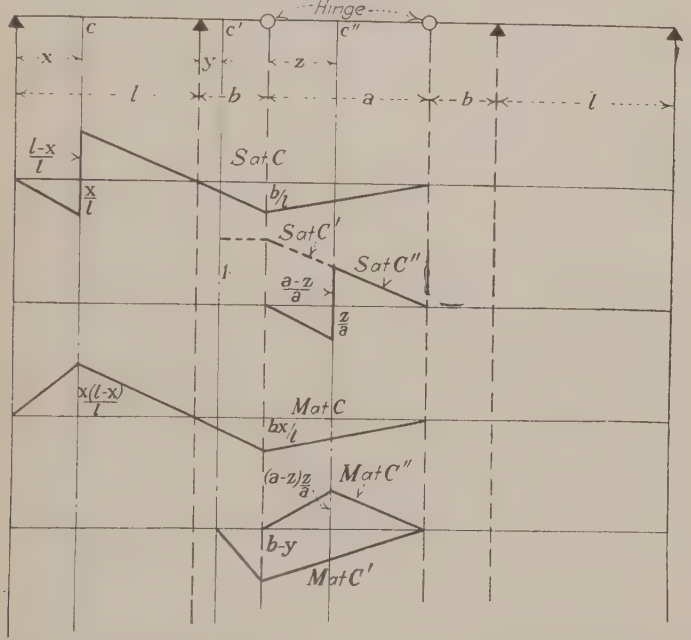


FIG. 318.—Influence lines for shear and moment on cantilever bridges.

Another arrangement for four supports, with two hinges, is shown in Fig. 316, which has two single cantilevers, each projecting into the central span, and one simple girder resting on the ends of the cantilevers.

If the reader understands what has preceded, he should have no difficulty with cantilever beams, since they involve no new principles.

The reactions, shears and moments may be found by influence lines. Figures 317 and 318 show influence lines for shear and moment for each of the arrangements above, and should be carefully studied. The points of inflexion or hinges are shown by small circles instead of showing knife-edge supports as in the previous figures.

CHAPTER XV

REINFORCED CONCRETE BEAMS

1. In previous chapters, mention has been made of the treatment of beams of different materials. In the present chapter, even with some repetition, the main principles of the strength of reinforced concrete beams will be given.

Reinforced concrete is concrete in which steel rods are placed in positions where the concrete is exposed to tension. The object is to let the steel carry the tension and the concrete the compression.

Concrete is well fitted to carry compression, being strong, durable, easily formed into any desired shape, and gaining strength with time; but it is not suited for carrying tension. Not only is its tensile strength small, but cracks are likely to exist, due to shrinkage or stress, which destroy all power of resisting tension. On the other hand, the most economical use of steel is in tension, and to resist tension it may be used in the form of plain rolled rods, either round or square, or rods whose surface is indented or "deformed" in various ways, in order to give better bond or adhesion to the concrete. Such rods could not be exposed to compression unless imbedded in a material like concrete, which would prevent them from bending. The combination of the two materials has the advantage of using each where it is best applicable, and the steel is protected from rust and fire by being imbedded in concrete.

In beams or other members of reinforced concrete, therefore, it would be desirable to have no tension on the concrete. This, however, is not possible in beams. If the deformation *tends* to produce tension on any plane, tension will be produced to the extent that the material can sustain it, subject to all static and elastic conditions. It is thus desirable to make the tension as small as possible, or to design so that if tension occurs and cracks are formed, no harm will be done. The assumption is therefore made that there is no tension on the concrete, on any cross-section of the beam.

Reinforced concrete is used in flexure, and in compression (columns); rarely, if ever, in tension members. In flexure, it is used as floor slabs, and in beams and girders supporting floors. In compression it is used in columns, and in arches, dams, retaining walls, foundations, etc., many of which structures carry flexure as well as compression.

2. If a steel rod is imbedded in concrete and the combination is exposed to loads which produce stress in the rod, and if the rod does not slip, it is clear that the unit elongation or contraction of the rod must equal that in the adjacent concrete. Hence the unit stresses must be proportional to the moduli of elasticity, which are generally assumed as about 15 to 1 ($E_c = 2{,}000{,}000$; $E_s = 30{,}000{,}000$). Hence the unit stress in the steel will be about 15 times that in the concrete. If the steel is in tension, and the unit stress in it is more than 15 times the ultimate tensile strength of the concrete, cracks will form in the latter. This assumes perfect bond, that is, equal deformation of the two materials.

It would be fatal to the integrity and proper action of the structure, and equivalent to a failure, for the steel to slip in the concrete. In that case, it would be almost like running a steel rod through a hole in the concrete with no bond between the two; the stress in the rod would be constant and this stress would be brought into the rod at the ends only. In a beam, the moment varies in general from point to point along the beam, and the tension in the steel must vary correspondingly; and the only way for stress to get into or out of the steel is by the bond between the steel and concrete, which is therefore essential.

A simple method of considering the subject is to imagine that the steel is replaced by 15 times its area of concrete if it is acting in tension; and if it and the concrete are acting in compression to use the full area of steel and concrete plus a concentrated area of concrete, at the steel, of 14 times the area of the steel, which is the same thing as replacing the steel by 15 times its area of concrete. In this way an imaginary "equivalent section" is taken instead of the actual section. This can be treated like any other beam section, subject to the condition that on the tension side of the neutral axis there is no stress on the actual concrete area, but only on the equivalent concrete area which replaces the steel, since no dependence is placed upon the concrete in tension, on a cross-sectional area.

There will, however, be shear on a cross-section, acting on the part on both sides of the neutral axis, and therefore, as we shall see, there will be diagonal tensile stresses, unless these, too, are taken up by diagonal steel rods.

3. The notation will be that of the Joint Committee on Concrete and Reinforced Concrete, *viz.:*

(*a*) *Rectangular Beams:*

f_s = tensile unit stress in steel;
f_c = compressive unit stress in concrete;
E_s = modulus of elasticity of steel;
E_c = modulus of elasticity of concrete;
$n = \frac{E_s}{E_c}$;
M = moment of resistance, or bending moment in general;
A_s = steel area;
b = breadth of beam;
d = depth of beam to center of steel;
k = ratio of depth of neutral axis to depth, d;
z = depth below top to resultant of the compressive stresses;
j = ratio of lever arm of resisting couple to depth, d;
$jd = d - z$ = arm of resisting couple;
p = steel ratio $= \frac{A}{bd}$.

(*b*) *T-beams:*

b = width of flange;
b' = width of stem;
t = thickness of flange.

(*c*) *Beams Reinforced for Compression:*

A' = area of compressive steel;
p' = steel ratio for compressive steel;
f_s' = compressive unit stress in steel;
C = total compressive stress in concrete;
C' = total compressive stress in steel;
d' = depth of center of compressive steel;
z = depth to resultant of C and C'.

(*d*) *Shear, Bond and Web Reinforcement:*

V = total shear;
V' = total shear producing stress in reinforcement;
v = shearing unit stress;
u = bond stress per unit area of bar;
o = circumference or perimeter of bar;
Σ_o = sum of the perimeters of all bars;
T = total stress in single reinforcing member;
s = horizontal spacing of reinforcing members.

(*e*) *Columns:*

A = total net area;
A_s = area of longitudinal steel;
A_c = area of concrete;
P = total safe load.

4. Rectangular Beams in Pure Flexure.—The general principles of pure flexure apply to the shaded section in Fig. 319, the wings representing concrete area equivalent to the area of steel, or nA_s. The intensity of stress varies as the distance from the neutral axis NN, and if the intensity on the wings is found, being on concrete, the intensity actually on the steel will be n times as great. The total

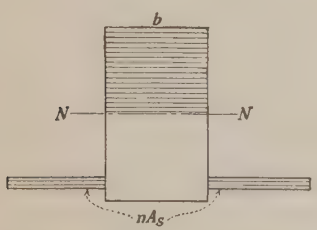


Fig. 319.

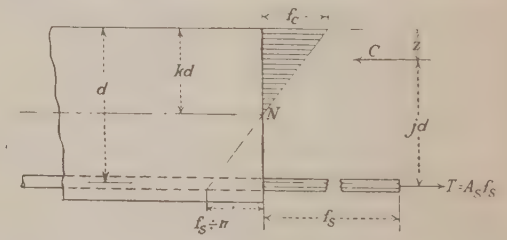


Fig. 320.

tension and the total compression are equal, and the neutral axis passes through the center of gravity of the section. However, since the concrete on the tension side of the neutral axis does not act, the section is not known till the neutral axis is known. But since the neutral axis goes through the center of gravity of the section, the moments, about it, of areas on each side must be equal, or, from Fig. 320.

$$\frac{k^2d^2 \cdot b}{2} = nA_s(d - kd) = nA_sd(1 - k) \tag{1}$$

from which (see notation) $$\frac{k^2}{2pn} = 1 - k \tag{1a}$$

$$k = \sqrt{2pn + p^2n^2} - pn. \tag{2}$$

The value of T may be found from the known external moment, M, by the condition

$$T = \frac{M}{jd}; f_s = \frac{M}{A_s jd} = \frac{M}{pjbd^2}. \tag{3}$$

Since $T = C$;

$$f_c = \frac{2C}{kdb} = \frac{2M}{kjd^2b} = \frac{2f_s p}{k} \tag{4}$$

$$z = \tfrac{1}{3}kd; j = 1 - \tfrac{1}{3}k. \tag{5}$$

If $f_s = 15{,}000$, $\frac{f_s}{n} = 1{,}000$; $f_c = 600$, from similar triangles

$$1{,}600 : d :: 600 : kd$$
$$k = \tfrac{3}{8}; z = \tfrac{1}{8}; j = \tfrac{7}{8}. \tag{6}$$

Usual values for the allowed stresses are,

$f_c = 500$ to 800 pounds per square inch
$f_s = 15{,}000$ to 16,000 pounds per square inch
$\frac{f_s}{n} = 1{,}000$ to 1,067 pounds per square inch.

From similar triangles in Fig. 320 it is clear that j will be greatest when f_c is smallest and f_s is greatest; j will be least when f_c is greatest and f_s is smallest.

For minimum j take $f_c = 800$; $\frac{f_s}{n} = 1{,}000$; then

$$1{,}800 : d :: 800 : kd$$
$$k = 0.44$$
$$j = 1 - \tfrac{1}{3}k = 0.85.$$

For maximum j take $f_c = 500$; $\frac{f_s}{n} = 1{,}067$; then

$$1{,}567 : d :: 500 : kd$$
$$k = 0.32; j = 0.89.$$

Hence, ordinarily j varies between 0.85 and 0.89. It will clearly be very close to $\frac{7}{8}$ for usual values of f_s and f_c. Taking this value, the computation of a beam is very simple: divide the moment M by $\frac{7}{8}d$ and we obtain $T = C$; f_c is then $2C$ divided by $\frac{3}{8}db$.

The position of the neutral axis depends upon p, the percentage of steel. The percentage required to secure desired fiber stresses f_c and f_s may be found from Eq. (1a), in the form

$$\frac{1}{p} = \frac{k}{p}\left(\frac{k}{2pn} + 1\right)$$

whence, from (4)

$$p = \frac{1}{\frac{k}{p}\left(\frac{k}{2pn} + 1\right)} = \frac{1}{2}\frac{f_s}{f_c}\frac{1}{\left(\frac{f_s}{nf_c} + 1\right)}. \tag{7}$$

5. T-beams in Pure Flexure (Figs. 321 and 322).—It is impossible to say at the start whether the neutral axis will lie in the flange or in the stem. If it is in the flange, the case is the same as a rectangular beam of width b, since there is no tension in the concrete below the neutral axis.

Consequently, let us assume the neutral axis in the stem, and find out if our assumption is correct. If not, we already have the formulae; if it is in the stem, the formulae are to be developed. It is customary, and allowable, to neglect the compression on the stem above the neutral axis, as it is very small provided the flange is large compared with the stem. The position of the neutral axis is

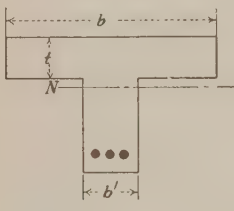


FIG. 321.

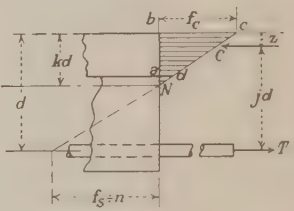


FIG. 322.

found from the condition that the moment of the flange about it equals that of the tensile area.

$$nA_s d(1 - k) = tb\left(kd - \frac{t}{2}\right)$$

$$kd = \frac{2ndA_s + bt^2}{2nA_s + 2bt}. \tag{8}$$

The next step is to find z and jd. This is done by finding the total compression and its point of application. If f_c is the intensity at the top of the flange, the intensity at the bottom is

$$f_c \frac{kd - t}{kd} = f_c\left(1 - \frac{t}{kd}\right).$$

The total compression is b times the area $abcd$, or

$$C = f_c bt\left(1 - \frac{t}{2kd}\right).$$

The moment of this compression about the top is found by dividing $abcd$ into a rectangle with base ab and height ad, and a triangle. The moment is

$$\frac{f_c bt^2}{2}\left(1 - \frac{2t}{3kd}\right).$$

Dividing the moment by C, the value of z is found,

$$z = \frac{t}{3} \cdot \frac{3kd - 2t}{2kd - t}. \tag{9}$$

Therefore

$$jd = d - z. \tag{10}$$

As in the rectangular beam, the total tension T is

$$T = f_s A_s = \frac{M}{jd} \tag{11}$$

and

$$f_s = \frac{M}{A_s jd}. \tag{12}$$

The fiber stress in compression is found from

$$C = f_c bt\left(1 - \frac{t}{2kd}\right) = T = \frac{M}{jd} = f_s A_s$$

$$f_c = \frac{Mkd}{jbdt\left(kd - \frac{t}{2}\right)} = \frac{f_s}{n} \cdot \frac{k}{1-k}. \tag{13}$$

If the flange is large, the neutral axis will be in the flange or but slightly below it, and for approximate results the formulae for a rectangular beam of width b may be used.

6. If the flange is small compared with the stem, and it is not desired to neglect the compression on the stem, the following formulae are easily derived. Taking moments about the neutral axis:

$$nA_s d(1-k) = tb\left(kd - \frac{t}{2}\right) + \frac{b'(kd-t)^2}{2}$$

from which

$$kd = \sqrt{\frac{2ndA_s + (b-b')t^2}{b'} + \left(\frac{nA_s + (b-b')t}{b'}\right)^2} - \frac{nA_s + (b-b')t}{b'}.$$

The value of C in the previous article becomes

$$C = f_c bt\left(1 - \frac{t}{2kd}\right) + \frac{f_c b'(kd-t)^2}{2kd}.$$

The moment of the compression about the top becomes

$$\frac{f_c bt^2}{2}\left(1 - \frac{2t}{3kd}\right) + \frac{f_c b'(kd-t)^2}{6kd}(kd + 2t).$$

Dividing the moment by C

$$z = \frac{bt^2(3kd - 2t) + b'(kd-t)^2(kd+2t)}{3bt(2kd-t) + 3b'(kd-t)^2}. \tag{9a}$$

The arm of the resisting couple formed by C and T is

$$jd = d - z. \tag{10}$$

Equations (11) and (12) hold here, as in the previous article. The fiber stress in pression is found from

$$C = f_c bt\left(1 - \frac{t}{2kd}\right) + \frac{f_c b'(kd-t)^2}{2kd} = T = \frac{M}{jd}$$

$$f_c = \frac{2Mkd}{jd[bt(2kd-t) + b'(kd-t)^2]}. \tag{13a}$$

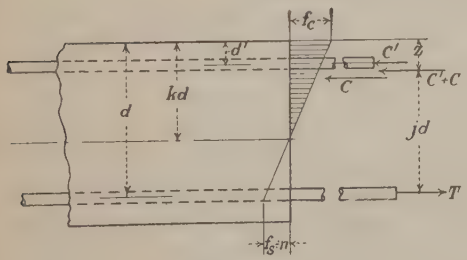


Fig. 323.

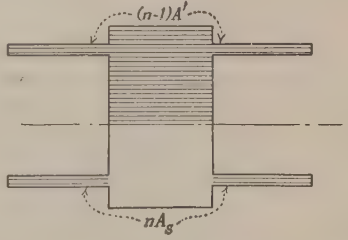


Fig. 324.

7. Beams in Pure Flexure Reinforced at Top and Bottom.—The preceding formulae assume tension reinforcement only. If there is reinforcement on the compression side also, as in Fig. 323, the equivalent concrete area is shown in Fig. 324, assuming A, to be the steel area on the tension side and A′ that on the compression side.

Taking moments about the neutral axis,

$$nA_s d(1 - k) = \frac{k^2 d^2 b}{2} + (n - 1)A'(kd - d') \tag{14}$$

$$k = \sqrt{2np + 2(n - 1)p'\frac{d'}{d} + (np + (n - 1)p')^2} - (np + (n - 1)p'). \tag{15}$$

(The formula of the Joint Committee for k is not quite correct, in that it takes np' instead of $(n - 1)p'$.)

The unit compression at the compression rods is

$$f_c \frac{kd - d'}{kd} \text{(on the concrete)}. \tag{16}$$

The total compression is

$$C = \frac{f_c bkd}{2} + \frac{f_c(kd - d')}{kd}(n - 1)A'. \tag{17}$$

To find z, take moments about the top; that moment is

$$\frac{f_c bk^2 d^2}{6} + \frac{f_c(kd - d')(n - 1)A'd'}{kd}. \tag{18}$$

Dividing this by C,

$$z = \frac{k^3 d^2 + 6(n - 1)p'd'(kd - d')}{3k^2 d + 6(n - 1)p'(kd - d')}. \tag{19}$$

The arm of the couple C and T is

$$jd = d - z. \tag{10}$$

The fiber stresses are

$$f_s = \frac{M}{pjbd^2} = nf_c \frac{1 - k}{k} \tag{20}$$

$$f_s' = nf_c \frac{kd - d'}{kd} \tag{21}$$

$$f_c = \frac{Mk}{pnjbd^2(1 - k)}. \tag{22}$$

These equations are not exactly those of the Joint Committee because we have assumed the equivalent compressive area of rods as $(n - 1)A'$, while the Committee has incorrectly used nA', while still allowing the undiminished section of the concrete.

8. If a beam is not in pure flexure, but has a normal component on a section, the above formulae do not hold. Here, as in any case of flexure, the resultant normal force on a section is known, and also its point of application. The neutral axis does not pass through the center of gravity of the section determined by it. The same principles may be used that have been explained in the chapter on flexure, but the solution is complicated. The expression for k is found from an equation of the third degree.

The total compressive stress on the section is not equal to the total tensile stress.

This case will not be treated here, but will be reserved for a later volume of this work.

All cases, including those treated above, may be solved by trial if the equations are forgotten or are not available. This has been indicated in Chap. X.

9. Shear and Bond in Reinforced Concrete Beams.—If there is a shear V at any section of a rectangular reinforced concrete beam, the increase of moment in a distance dx is Vdx, and the increase of stress in the tension rods, per unit length along the beam, is $\frac{V}{jd}$. This must be the horizontal shearing stress on a strip whose area is b; hence the unit horizontal shearing stress is

$$v = \frac{V}{jdb}. \tag{23}$$

For T-beams $v = \frac{V}{jdb'}$.

There must be an equal shearing stress on a transverse (vertical) plane; and, if there are no normal stresses on vertical or horizontal planes, there must be a tension of the same intensity on a plane at 45° to the horizontal, and an equal compressive stress on a plane at right angles to the latter.

Also, there must be a bond stress between the bar and the concrete, in a unit of length, equal to $\frac{V}{jd}$, hence the unit bond stress is (for rectangular or T-beams)

$$u = \frac{V}{jd\Sigma o}. \tag{24}$$

There is no escape from these conclusions. Really, however, the facts are not as they have been assumed. There is really some tension on a vertical plane, unless there is a crack there; it is assumed, for safety, that there is no tension on that plane, but this leads necessarily to the conclusion that there is tension on some other plane. There also may be some compression on a horizontal plane, and this is left out of account.

In the report of the Committee "it is recommended that two-thirds of the external vertical shear (total shear) at any section be taken as the amount of total shear producing stress in the web reinforcement" (whatever this may mean). Hence taking $V' = \frac{2}{3}V$, if vertical rods are placed at distances s apart, it is assumed that the stress in one of these vertical rods is

$$T = \frac{V's}{jd} = \frac{2}{3}\frac{Vs}{jd}. \tag{25}$$

If there are diagonal tension bars bent up at 45°, or at angles between 20 and 45°, the Committee gives the formula

$$T = \frac{3}{4}\frac{V's}{jd} = \frac{1}{2}\frac{Vs}{jd}. \tag{26}$$

It is obvious that if the beam is to be designed on the assumption that there can be no tension whatever in the concrete, there must be diagonal tension bars at every section where there is shear, and the sum of the vertical components of the stresses in these bars cutting any vertical section must equal the shear on the section.

This part of the report is not very clear, and the theory and facts of the subject are obscure and uncertain, as has been indicated. There is much difference of opinion.

This subject is a part of the subject of reinforced concrete design, and will not be further pursued here, but will be treated in a later volume of this work.

10. Columns.—Using the same method of imagining each area of steel replaced by an area n times as large of concrete, it is obvious that if P is the total axial load

$$P = f_c(A_c + nA_s) = f_cA(1 + (n - 1)p) \tag{27}$$

$$f_c = \frac{P}{A[1 + (n - 1)p]} \tag{28}$$

$$f_s = nf_c. \tag{29}$$

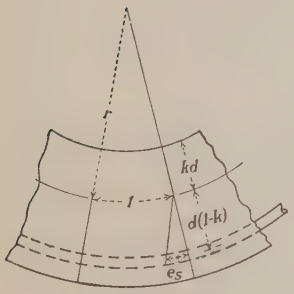


Fig. 325.

11. Deflection of Reinforced Concrete Beams.—Draw radii to the elastic line through points distant a unit apart *along the axis* (Fig 325). Then if e_S is the stretch of the rods in a unit distance,

$$1 : r :: e_S : d(1 - k)$$

and $$e_S = \frac{f_S}{E_s} = \frac{M}{jdA_SE_S}$$

whence, $$\frac{1}{r} = \frac{d^2y}{dx^2} = \frac{M}{jd^2A_SE_S(1 - k)}. \tag{30}$$

Hence in the usual deflection formulae, the value of I, the moment of inertia of the section, should be replaced by $jd^2A_S(1 - k)$, and E by E_S.

CHAPTER XVI

RIVETED CONNECTIONS

1. Rivets and Pins.—When plates or shapes of steel or other metals are to be connected together, the connection (if not welded) may be made by a pin, or by rivets. Thus, if the bar a (Fig. 326) is to be connected to the bars b, a single pin or bolt, with a nut on each end, or a nut on one end and a head on the other, may be put through the bars as shown. The hole weakens each bar, and in

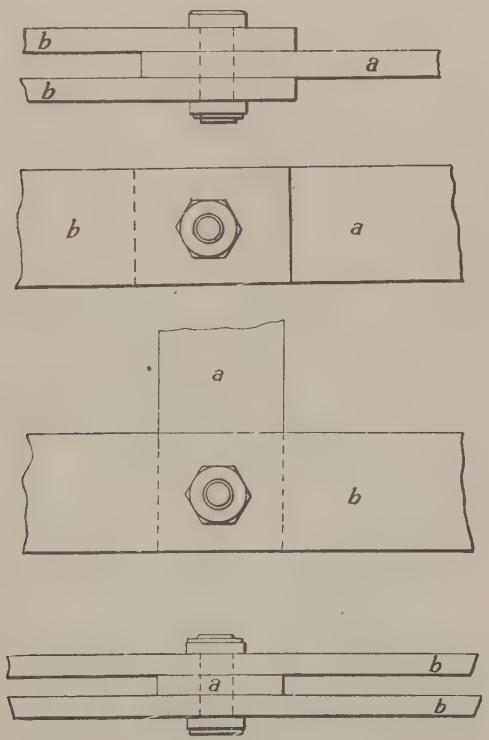


FIG. 326.—Pin connections.

order to avoid this, eye bars (Fig. 327) are used as explained in Chap. VI. Each bar, if disconnected at the other end, would be free to turn around the pin. Instead of a single pin, rivets may be used, as in Fig. 328.

A rivet is a bolt or rod of steel (or other metal) with a head at each end, forming one piece with the body (Fig. 329). One head is formed with the body, in a rivet machine, by pressing the metal, red hot, in a suitable die. The body is of a

suitable length, depending upon the number of plates to be connected, and may taper slightly toward the end. It is heated red hot, put through the hole, and the other head is then formed, either by ordinary hammers, by continuous pressure exerted by a hydraulic or compressed air machine, or by repeated blows of

FIG. 327.—Eye bar pin connection.

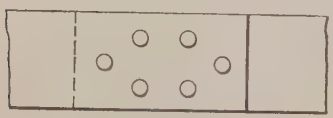


FIG. 328.—Riveted connection.

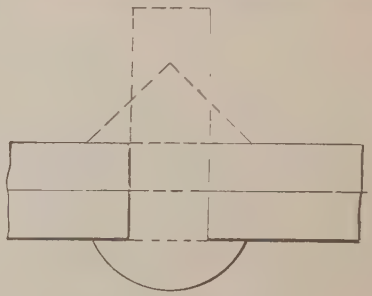


FIG. 329.

FIG. 330.—Hydraulic riveter. (*Southwark Foundry and Machine Co.*)

the hammer of a riveting machine operated by compressed air. In stationary machines the head already made is held in a die and another die is forced against the projecting body with great force, by steam pressure, hydraulic pressure,

or compressed air, pressing the plates together, squeezing the red hot metal into the hole till it completely fills it, and forming the other head. This is called *shop riveting* in distinction from *field riveting*, which is done where the structure is being erected, by hand or by portable machines. Formerly field riveting was done by hand, called *hand riveting;* the head is held against the plates by a die on the end of a long steel lever acting through a fulcrum provided by chains around the plates, or otherwise, while the head is formed by hammering down the projecting end, either into a conical head (Fig. 329) by hammers alone, or into any desired form of head by means of a die. Field riveting is now generally done, on large work, by a pneumatic riveter, the head being held up as in hand riveting, and the

FIG. 331.—Riveting angles to plates. (*Southwark Foundry and Machine Co.*)

new head being formed by a die which receives the rapidly repeated blows of a pneumatic hammer.

When the rivet is put in, it must necessarily be tapering or slightly smaller than the hole, and if the holes in the two plates are not exactly opposite, or not quite cylindrical, there will be spaces around the rivet. These are supposed to be filled by the squeezing in of the red hot plastic metal under the operation. Hand riveting is obviously less effective in doing this than shop or pneumatic field riveting, since in the latter the pressures are larger and more continuous. The plates must be held tightly together by bolts while rivets are being driven, in order to prevent the formation of collars or fins on the rivet where it would be forced into spaces between the plates.

For some time after the introduction of steel in structural work, it was thought that rivets should be of wrought iron, partly because of its softness, and partly because of the danger of injuring the quality of steel rivets by the reheating before driving, with regard to which some troubles had occurred. Now, however, rivets are made of a special soft grade of steel, known as rivet steel.

A pin connection may be taken apart by removing the nut and driving out the pin, but a riveted connection is permanent and can only be taken apart by chipping off one head of each rivet and driving the rivets out, or else by drilling out the rivets.

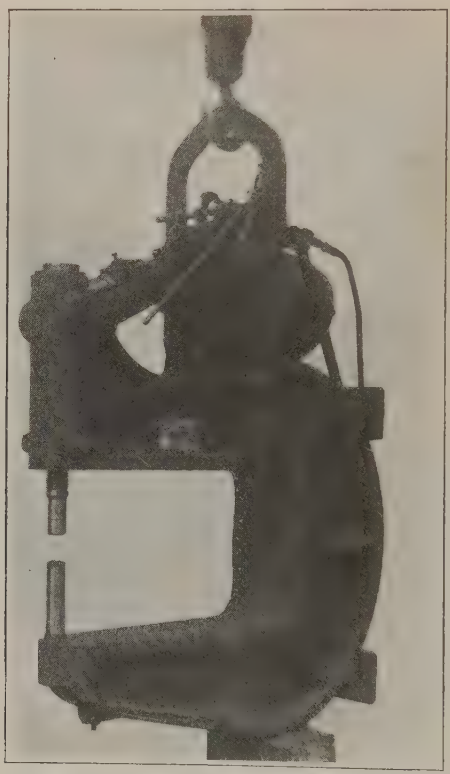


FIG. 332.—Shop pneumatic portable riveter. (*Southwark Foundry and Machine Co.*)

2. Riveting Machines.—Figures 330, 331 and 332 show types of shop riveting machines, and Fig. 333 a portable compressed air riveter. Everyone must have heard the rapid rat-tat of a portable riveter used in the erection of steel frame buildings.

3. Punched, Reamed and Drilled Holes.—The holes for rivets are commonly punched. This leaves a rough hole, and one that is conical, being larger at the side opposite the punch. Even where the punch works into a hole in a die-block beneath, this hole must be larger than the punch (Fig. 334). It is practically impossible to punch a smooth cylindrical hole in steel. Frequently there is a roughness or fin at the edge. The pressure of the punch is very great, and this

causes a lateral flow of the metal which may result in small cracks around the hole, especially in thick plates and hard metal. Also the spacing of holes may not be exact and two holes will not be precisely opposite. It was formerly the custom, after plates had been assembled together before riveting, to drive tapered

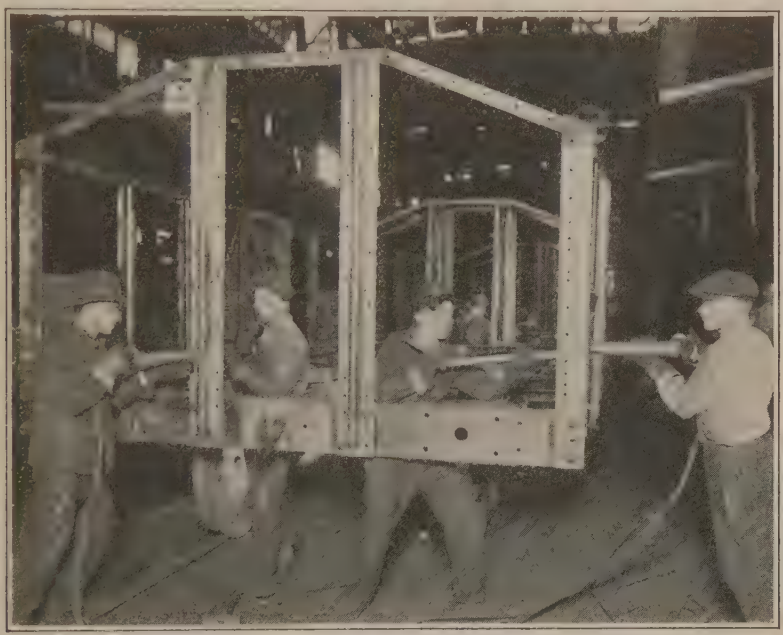


FIG. 333.—Portable pneumatic riveters. (*Chicago Pneumatic Tool Co.*)

pins, called *drift pins*, into the hole, so that a rivet of the proper size could be put in; but this process evidently merely forcibly distorts the metal.

There are three remedies for these evils: annealing, reaming and drilling. If the metal is *annealed* after punching, the injury to the quality done by the cold punching (and the metal is practically always punched cold) is removed, but the cracks and injured material are not removed, nor inaccuracy in spacing corrected. By *reaming* out the punched hole any injured metal is removed, and the hole is made almost cylindrical, but there is often a burr left at the edge of the hole. If reaming is done, the hole must of course be punched with a smaller diameter than the driven rivet is to be. As a matter of fact, in reamed work in order to be sure to get the rivet in, holes are punched a little smaller than the cold rivet, and reamed till they are larger, so that *the actual diameter of a rivet after driving is larger than the nominal diameter.* The metal may be reamed before assembling, but this of course does not correct

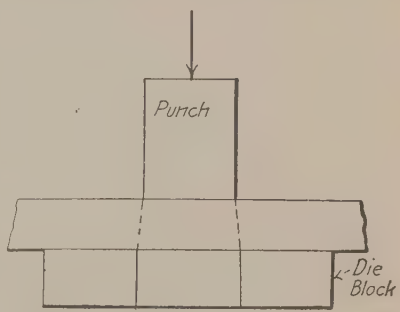


FIG. 334.

inaccuracy in spacing, and it is better to ream after assembling, thus making the entire hole, through all the parts to be connected, nearly cylindrical.

Any burr on the edge of a hole should be removed by a slight countersinking. (Fig. 335), which also makes the head stronger.

The most accurate method is obviously to drill all holes, preferably (if practicable) after assembling, thus producing a cylindrical hole through all the parts. This is not always practicable, and sufficiently good work can generally be obtained by punching and reaming, and often by punching alone. Probably the great majority of existing structures have had rivet holes merely punched. If the material is ductile, and the plates not too thick, punching may be done with

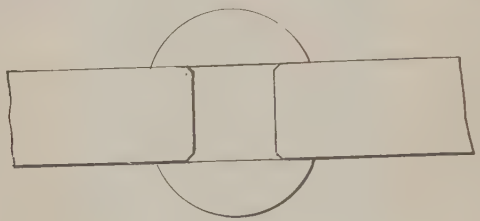


FIG. 335.

practically no injury to the metal. Certain important riveted connections, however, should be drilled after assembling; in the field if necessary. The general practice now is to require reaming or drilling in metal over a certain thickness, and to reject material in which the holes are not reasonably opposite. Thus the specifications of the Massachusetts Public Service Commission contain the following requirements:

"All riveted work, except in the case of steel over $\frac{3}{4}$ of an inch thick, shall be punched accurately with holes $\frac{1}{16}$ of an inch larger than the size of the rivet. The pieces forming one built member must be so accurately punched that when they are put together the holes will be truly opposite; no drifting to distort the metal will be allowed; if the hole must be enlarged to admit the rivet it must be reamed.

"In steel over $\frac{3}{4}$ of an inch thick all holes shall be drilled from the solid or else they shall be punched $\frac{1}{16}$ inch smaller than the rivet, and drilled or reamed to a diameter of $\frac{1}{8}$ of an inch larger than the punched holes, so as to remove all the sheared surface of the metal; and in pieces used in tension all sheared edges shall be planed.

"All holes for field rivets in floor beam and stringer connections, and splices in tension members, shall be accurately drilled to an iron templet, or reamed while the connecting parts are temporarily put together."

The 1920 specifications of the American Railway Engineering Association are (for railroad bridges):

"206. *Class of Work.*—The work shall be "Punched Work" or "Reamed Work" as stipulated.

"210. *Punched Work.*—In punched work, holes in material whose thickness is not greater than the diameter of the rivet plus $\frac{1}{8}$-inch, may be punched full size. Holes in material of greater thickness shall be drilled." (This practically allows punched work, if stipulated, for all usual material.)

"211. *Reamed Work.*—In reamed work, holes in material $\frac{7}{8}$-inch thick and less, used for lateral, longitudinal and sway bracing, lacing, stay plates, and diaphragms, may be punched full size.

"212. Holes in other material $\frac{3}{4}$-inch thick and less, shall be sub-punched and reamed.

"213. Holes in material more than $\frac{3}{4}$-inch thick shall be drilled.

"214. *Punched Holes* —Full size punched holes shall be $\frac{1}{16}$-inch larger than the nominal diameter of the rivets. The diameter of the die shall not exceed the diameter of the punch by more than $\frac{3}{32}$-inch. If any holes must be enlarged to admit the rivets, they shall be reamed. Holes must be clean cut, without torn or ragged edges. Poor matching of holes may be cause of rejection.

"215. *Sub-punched and Reamed Holes.*—In sub-punched and reamed work, the holes shall be punched $\frac{3}{16}$-inch smaller, and, after assembling, reamed $\frac{1}{16}$-inch larger than the nominal diameter of the rivet. The diameter of the punch used shall be $\frac{3}{16}$-inch smaller than the nominal diameter of the rivet and the diameter of the die not more than $\frac{3}{32}$-inch larger than the diameter of the punch. Outside burrs shall be removed with a tool making a $\frac{1}{16}$-inch fillet.

"216. *Accuracy of Punching in Reamed Work.*—In sub-punched and reamed work, the punching shall be so accurately done that, after assembling and before reaming, a cylindrical pin $\frac{1}{8}$-inch smaller in diameter than the nominal size of the punched hole may be entered, perpendicular to the face of the member, without drifting, in at least 75 of any group of 100 contiguous holes in the same plane. If this requirement is not fulfilled, the badly punched piece shall be rejected. If any hole will not pass a pin $\frac{3}{16}$-inch smaller in diameter than the nominal size of the punched hole, this shall be cause for rejection.

"217. *Reaming After Assembling.*—Reaming shall be done after the pieces forming a built member are assembled and so firmly bolted together that the surfaces are in close contact. Before riveting, they shall be taken apart, if necessary, and any shavings removed. When it is necessary to take the members apart for shipping or handling, the respective pieces reamed together shall be so marked that they may be reassembled in the same position in the final setting up. No interchange of reamed parts will be permitted.

"218. *Accuracy of Reaming and Drilling.*—When holes are reamed or drilled, 85 of any group of 100 contiguous holes in the same plane shall, after reaming or drilling, show no offset greater than $\frac{1}{32}$-inch between adjacent thicknesses of metal.

"219. *Reamed Holes.*—Reamed holes shall be cylindrical, perpendicular to the member, and not more than $\frac{3}{32}$-inch larger than the nominal diameter of the rivets. Reamers preferably shall not be directed by hand. Outside burrs shall be removed with a tool making a $\frac{1}{16}$-inch fillet.

"220. *Drilled Holes.*—Drilled holes shall be $\frac{1}{16}$-inch larger than the nominal size of the rivet. Burrs on the outside surfaces shall be removed.

"221. *Assembling for Drilling.*—Connecting parts requiring drilled holes shall be assembled and securely held together while being drilled."

The reader should also consult Arts. 222 to 234 of the specifications.

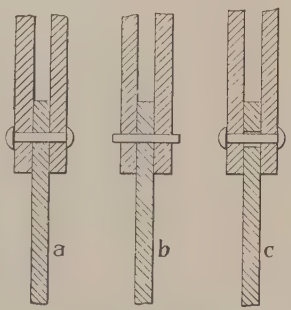


FIG. 336.

4. A rivet being red hot when placed in the hole, it shrinks in cooling, and produces a tension on the rivet and a friction between the plates, which may amount to as much as 10,000 pounds per square inch of rivet area. This friction, however, is always neglected, because it is uncertain, variable, and likely to be destroyed by vibration and other causes. It may be ascertained by testing connections with a single rivet (Fig. 336) first with the heads made in the usual man-

ner (*a*), and then with no heads (*b*), or better by having an elongated hole in the central plate (*c*). It was at one time customary in Germany to take account of friction; and in France it was long ago the practice of some builders to depend entirely on friction, neglecting the shearing strength, because when the hot rivet shrinks, it must leave some space, however small, around it, unless circumferential initial compression has been produced in riveting, which the shrinkage merely

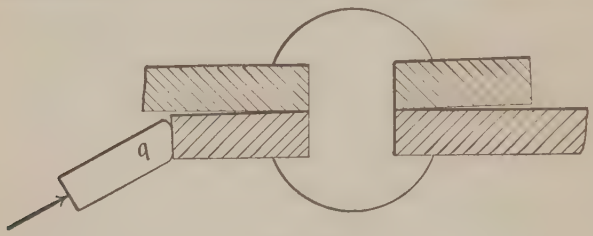


FIG. 337.

relieves. A well-riveted joint, if cut through the rivets, shows no visible space around the rivets, which appear to fill the holes completely.

A riveted joint exposed to water or steam pressure may leak, because of the deformation, or because of rust or magnetic oxide on the plates, or because the dimensions are unsuitable. After riveting, the plates are sometimes found to be sprung or opened as shown in Fig. 337. They may be made tight by "caulking" with a caulking tool *b*, but this must be carefully done with a tool rounded at the edge, or a groove may be cut in the plate. The caulking edges of the plates are beveled as shown to an angle not sharper than 70°, and as near thereto as practicable. Rust and scale should be removed by scraping, or by washing with a solution of sal-ammoniac. Caulking is now done with a pneumatic machine that is run along the edge of the plate.

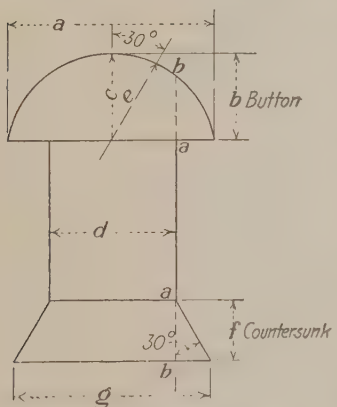


FIG. 338.—American Bridge Co. Standard rivet.

Rivets sometimes become loosened, especially when subject to repeated stresses in opposite directions, or to vibration; and when loose they must be cut or drilled out and replaced. Looseness may be detected by striking one head with a hammer, at the same time placing the finger on the other head if practicable.

5. Shape of Head.—When made by hand, without a cup, with a flat-faced hammer (a poor method) rivet heads are conical. Structural rivets almost always have button heads. Figure 338 shows the standard of the American Bridge Co. Figure 339 is from the Boiler Code of the A.S.M.E. Sometimes the head is made with a single radius. Since the rivet is often in tension, due to cooling or to stress produced by the loads, the cylinder with height *ab* (Fig. 338), should have a shearing strength equal to or greater than the tensile

resistance of the rivet shank. It will be found that in the standard head this area is 20 to 40 per cent greater than the area of the rivet, varying with the size.

Countersunk rivets have the heads flush with the surface of the metal, and are required when another piece must be flush with it. The hole is cut conical by a

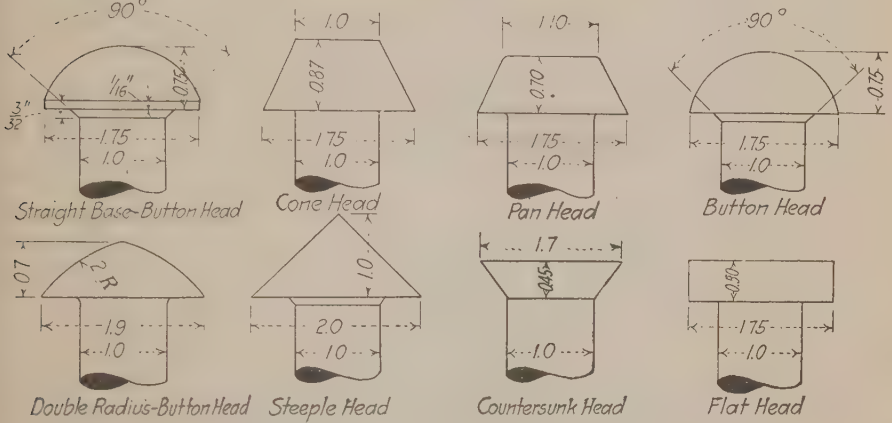


FIG. 339.—Acceptable forms of rivet heads. (*Boiler Code of Am. Soc. M. E.*)

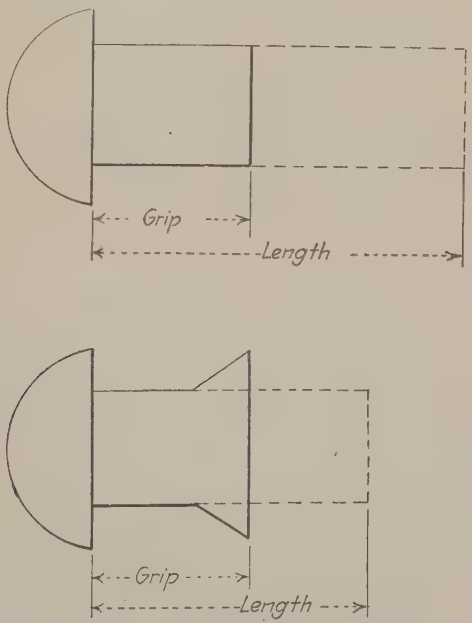


FIG. 340.

countersinking tool. In a countersunk rivet the area of the cylinder *ab* is twice that of the shank. Sometimes the heads are flattened to some given thickness and partly countersunk, if another member does not allow space for a full head. Countersunk rivets obviously weaken the plate more than others.

6. Length of Rivet.—For rivets with ordinary heads the length under heads is called the *grip:* for countersunk rivets the grip is the length to outside of head (Fig. 340). The length of the cold rivet equals the grip plus the length necessary to form one head. The "Carnegie Pocket Companion" gives the required length for various grips and diameters, for plain and countersunk heads. The length must also allow for some extra material to be forced into the rivet hole in order to fill it completely, so that the length will exceed the grip, for the same diameter, by an amount which increases with the grip.

Long rivets are objectionable on account of the difficulty of forcing the rivet in so that it will completely fill the hole (called *upsetting* it), and the grip is generally limited to some fixed number of times the diameter (d). It is desirable

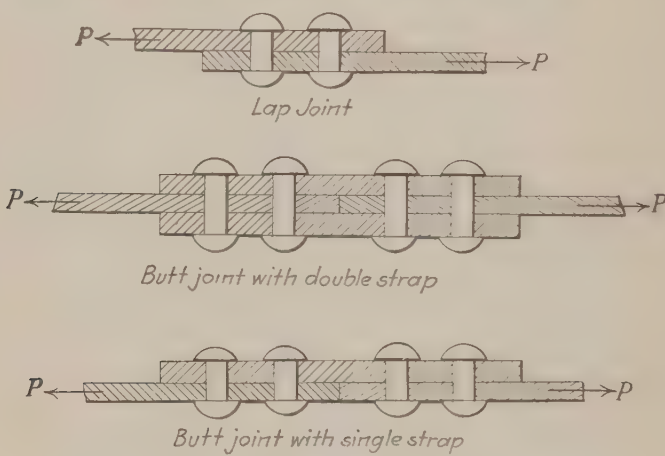


FIG. 341.

that $\frac{grip}{d}$ should not exceed 4 or 4.5; and, where long rivets must be used, the diameter may be increased to the proper ratio. The A.R.E.A. specifications state:

"Rivets which carry calculated stress and whose grip exceeds four and one-half diameters shall be increased in number at least 1 per cent for each additional $\frac{1}{16}$-inch of grip. If the grip exceeds six times the diameter of the rivet, specially designed rivets shall be used."

7. Size of Rivets.—The diameter of rivets varies, according to the sizes of parts to be connected, from $\frac{3}{8}$ to $1\frac{1}{8}$ or even $1\frac{1}{4}$ inch. The most common sizes for ordinary structural work are $\frac{3}{4}$ and $\frac{7}{8}$ inch diameter, the area being respectively 0.44 and 0.6 square inch. These should be remembered. In heavy structural work or where long rivets are required, the size may be 1 or $1\frac{1}{8}$ inch. Larger rivets than $1\frac{1}{4}$ inch are seldom used. If the length exceeds 6 inches it is generally better to use bolts, but the design can generally be so made that this is unnecessary; and unless turned bolts are used, which exactly fit the holes, a bolt is not as good as a rivet which may be upset so as to completely fill the hole.

8. Definitions.—The distance between centers of rivets in the same row is called the *pitch*. When one plate laps over another, and the two are riveted together, the joint is called a *lap joint:* where two plates are in the same plane and

are connected by a cover plate (or *butt strap*) on one or both sides, the joint is called a *butt joint* (Fig. 341). The distance from the center of a rivet to the edge of a plate is called the lap or *edge distance* or in England the *margin*.

In a lap joint, since the forces in the plates are not in the same line, there is a tendency to bend the plate as shown in Fig. 342, thus causing a tension on the rivets and a tendency to pull off the heads. A butt joint with double strap has

FIG. 342.

not this objection, but a butt joint with single strap has; it is merely two lap joints.

When two plates are connected by a lap or butt joint having one row of rivets in the lap, or one on each side of the butt joint, it is called *single riveting;* if there are two rows in the lap, or two on each side of the butt joint, it is *double riveting;* treble and quadruple riveting may also be used.

A lap joint may also be made with a cover plate over the lap (Fig. 343), in which case it is called a *lap joint with cover*, or a *combined lap and butt joint.*

FIG. 343.

A double riveted joint may have the rivets in the two rows opposite each other, in which case it is called *chain riveting;* or spaced alternately, when they are said to be *staggered* (Fig. 344). The distance, *b*, between the two rows, is the *back pitch; c* is the *diagonal pitch.*

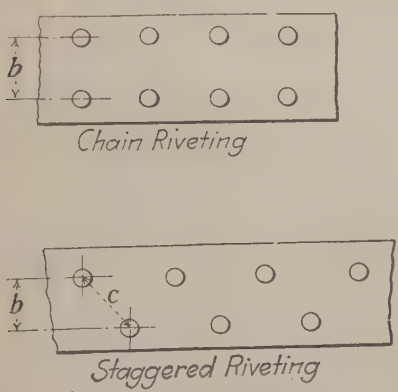


FIG. 344.

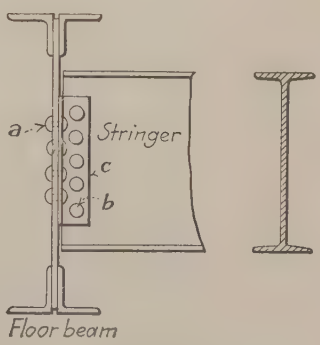


FIG. 345.

9. Stresses on a Rivet.—The primary and in fact only purpose of a rivet, so far as stress is concerned, is to resist a force tending to shear it on a cross-section. A rivet should not be designed primarily to resist tension, though riveted connections are often exposed to forces which bring tension on a rivet, and in this case the

tension should be allowed for. Thus, if a stringer is fastened to a floor beam by a connection as shown in Fig. 345, the connection will be more or less rigid, depending upon the yielding of the floor beam, and there will be a bending moment on the end of the stringer, which will expose the upper rivets in row *a* to tension. Nevertheless the primary function of rivets *a* is to support the stringer, the vertical reaction from which tends simply to shear them off at the floor beam web. Rivets in row *b* are exposed to the same vertical force, which tends to shear them off vertically at two sections, one on each side of the stringer web; the moment acts also on these rivets, but tends to shear them off *horizontally* at the stringer web, the upper rivets in one direction and the lower ones in the opposite direction, so that these rivets are not exposed to tension, but to two shears at right angles to each other. The lower rivets in row *a* are not exposed to compression, for the compressive force which goes to make up the end moment is given by the direct pressure of the connecting angles *c* (*hitch angles*) against the floor beam web.

Rivets may therefore be exposed to shearing and to tension; or to two shears at right angles, the resultant of which is the real shear acting.

Otherwise stated, a group of rivets may be subjected to:

(1) A force acting at the center of gravity of a group of rivets at right angles to the rivets in which case the tendency is merely to shear the rivets off in the direction of the force:

(2) A force as in (1) together with a moment in a plane perpendicular to the rivets, in which case the tendency is merely to shear the rivets off, but the resultant force having different directions on the different rivets, depending on the relative shears produced by the single force and by the moment:

(3) A moment alone as in (2), in which case the resultant force on the group is zero, but some rivets are subject to shear in one direction and some in the opposite direction:

(4) A force perpendicular to the rivets, together with a moment in a plane parallel to the axis of the rivets (as in rivets *a*, Fig. 345) in which case the rivets are exposed to shear and, in addition, some (but not all) of them are exposed to tension.

Specifications sometimes require that rivets shall not be used in tension. As a matter of fact, they are often used where they must be in tension, though the tension may be disregarded, as in hitch angles connecting stringers to floor beams. Even the A.R.E.A. specifications require the rivets in hitch angles to carry the entire load transmitted (even if there is a shelf below the stringer for convenience in erection), and they reduce the allowable shearing stress on these rivets by 25 per cent. The writer has seen many connections by hitch angles where the rivet heads near the top had been pulled off, and he has seen many cases where old structures have been strengthened in a way to put rivets purposely in tension, and to depend upon it. The reason why it is usual not to depend on rivets in tension is the danger that in heating and driving they may be injured (see Chap. XXVII).

10. Stresses on a Rivet Resisting Shear.—Let a rivet which passes through three plates (Fig. 346) be exposed to the forces shown. This rivet transmits or transfers a force P from the single plate to the two plates, and is in shear on two planes, *a* and *b*. Such a rivet is in *double shear*, because the force it transmits acts

on two sections, *one-half on each*. If the rivet is as shown in Fig. 347, the case is *not* one of double shear, because the rivet transmits the force P from one plate to another and although it is in shear on plane a and on plane b, the force P is not divided into two parts, one-half on each plane, but the force on each plane is P.

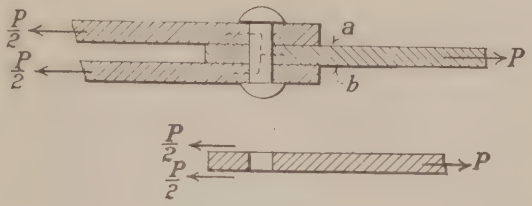


Fig. 346.

In Fig. 346, if the two left-hand plates were not equal but one double the thickness of the other, the force P would be divided, two-thirds going into one, and one-third into the other; it would still be technically double shear, but not in the

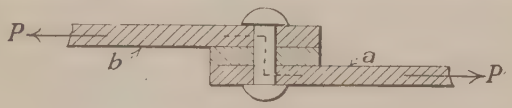


Fig. 347.

usual sense, which is generally understood to be where the stress is the same on each section, namely $\frac{P}{2}$.

Double shear may always be distinguished from single shear by this criterion: in single shear, the whole force transmitted by the rivet goes through each section which is in shear; in double shear, the whole force transmitted by the rivet is divided into two parts, not necessarily equal but generally understood so, one part going through one (or more) sections, the other part going through other sections.

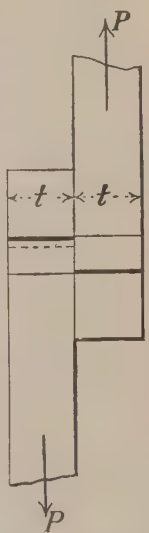


Fig. 348.

When a rivet transmits shear, as in Fig. 348, the tendency is to shear it across its section and the surfaces indicated by heavy lines will be in compression. The force will not be uniformly distributed across the thickness t, but will tend to be more concentrated at the surfaces of contact, and instead of the left-hand plate moving with reference to the other so that the rivet edges are parallel to their former positions there will be a bending for some unknown distance on each side of the surface of contact; in other words there will not be pure shear. Nevertheless it is always assumed that pure shear exists, and that the force P is uniformly distributed over the area of the rivet; hence if d is the diameter, and f_s the intensity in shear, the force transmitted by the rivet is always taken as $f_s \frac{\pi d^2}{4}$, or

$$f_s = \frac{4P}{\pi d^2}. \qquad (1)$$

Similarly, the force P is distributed over the cylindrical area acb, (Fig. 349) and it is not uniformly distributed, being clearly greatest at c and least at a and b, where it is really zero. Nevertheless, the maximum compressive, or as it is always called, *bearing stress intensity*, is always considered to be the same as if P were uniformly distributed over a diametrical section ab; hence, if t is the thickness of the plate and f_b the intensity of bearing stress, the force transmitted to the plate by the rivet is always taken as $f_b td$,

or
$$f_b = \frac{P}{td} \tag{2}$$

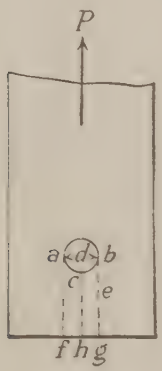


FIG. 349.

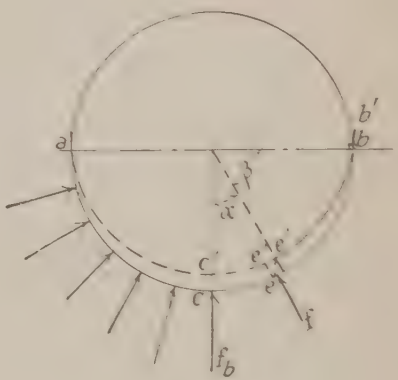


FIG. 350.

A more accurate expression for f_b may be obtained by the aid of a few assumptions which seem reasonable. Thus (Fig. 350) it is clear that the surface of the rivet acb will be compressed into some form $ac'b$. Assume that the pressure on the cylindrical surface is really *normal* everywhere, and also that each point moves the same distance in the direction of the force P, c to c', e to e', b to b', with $cc' = ee' = bb'$. Then remembering that the deformations are very small, and calling f the intensity of normal bearing at any point and f_b its maximum value, at c, it is clear that ee'' will be a constant k times f, k depending upon the modulus of elasticity in bearing, and since $ee' = \frac{ee''}{\cos \alpha}$,

$$cc' = kf_b = ee' = \frac{kf}{\cos \alpha}; \quad \therefore f = f_b \cos \alpha.$$

Since the components of the forces fds in the direction cc' must equal P,

$$P = 2t \int_0^{\frac{\pi}{2}} fds \cos \alpha = 2rf_b t \int_0^{\frac{\pi}{2}} \cos^2 \alpha d\alpha = \frac{rf_b t}{2}\pi.$$

Hence
$$f_b = \frac{P}{\frac{\pi r}{2} t} \tag{3}$$

This means that the max. bearing pressure is the same as if the force P were uniformly distributed over the surface of a *quadrant* $0.78dt$ *instead of over a diametrical section* dt as universally assumed.

The result in Eq. (3) is based on the assumption that f varies as $\cos \alpha$ or as $\sin \beta$. If it is assumed that f is normal, the usual practice requires the assumption that it is constant around the semi-circumference, or $f = f_b$, for then

$$P = 2tf_b r \int_0^{\frac{\pi}{2}} \cos \alpha d\alpha = f_b t d.$$

This last assumption, of a constant f, is clearly impossible, and (3) is no doubt more accurate than the usual practice.

Since there is *some* bending in a rivet, even if but little, so that the shear is not exactly uniformly distributed over a section, it follows that the actual shearing stress and the actual bearing stress, if the above is correct, are somewhat larger than generally assumed. The maximum stresses, however, being strictly local and confined to small areas, may safely be taken as larger than values given by tests showing average shearing strength.

In addition to shearing on the rivet and bearing on rivet and plate, a riveted connection involves other stresses. There is a tendency to shear out the plate along the lines af and bg in Fig. 349. Let e be these distances; then

$$2f_s et \geq f_s \frac{\pi d^2}{4}$$

$$e \geq \frac{\pi d^2}{8t} > 0.39 \frac{d^2}{t}.$$

There is also a tendency to split the plate back of the rivet along ch.

Commonly e is made not less than $1.5d$ (see Art. 12), which is often if not generally larger than the above value. In some cases, as in some of Tetmajer's tests, $1.5d$ has been found too small, and it is better to make it $2d$ if practicable.

In designing or studying riveted joints, clearly d should be the diameter of the driven rivet or of the hole. This is specified by the "Boiler Code." It should always be done in studying the cause of a failure unless it is found that the rivets do not fill the holes. Frequently, however, it is not done; and while, in deducting rivet holes to get the net section, the hole is taken as $\frac{1}{8}$ inch larger than the nominal diameter of the rivet, yet in computing the strength of the rivet the nominal diameter is used. This is on the safe side, but not logical.

11. Efficiency of a Riveted Joint.—When a plate is connected by rivets, it is weakened by the rivet holes, and cannot carry as great a force as the unbroken plate could. The ratio of the force which the joint will carry to that which the unbroken plate could carry is the *efficiency of the joint*, and is always less than unity. The larger it is, the better the joint is designed. In the study of riveted joints

f_s = the ultimate (or the allowable) stress intensity in shearing
f_t = the ultimate (or the allowable) stress intensity in tension
f_b = the ultimate (or the allowable) stress intensity in bearing
f_c = the ultimate (or the allowable stress intensity in compression
t = thickness of a plate or shape connected by the rivets.

f_c and f_b are not the same, the latter being greater than the former, because it is a local pressure not extending over the entire area of a piece.

In the standard "Boiler Code" of the A.S.M.E. the *ultimate values* for steel are taken as

$$f_t = 55{,}000$$
$$f_s = 44{,}000$$
$$f_b = 95{,}000.$$

These values will be used in what follows, unless otherwise mentioned. *The code specifies a factor of safety of five.*

12. Limiting Spacing of Rivets.—If rivet holes are punched too near together, or too near the edge of the piece, there is danger of distorting or even cracking the material if it lacks ductility, either in punching or in driving the rivet. While this minimum clear distance, in ductile material, might be as low as the diameter of the rivet (which would make the minimum distance between centers of holes $2d$ and between center of hole and edge of plate $1.5d$), these distances are too small. *The minimum pitch should be three diameters of rivet.* The A.R.E.A. specifications state that it "preferably shall be not less than 3½ inches for 1-inch rivets, 3 inches for ⅞-inch rivets, and 2½ inches for ¾-inch rivets." The standard of the American Bridge Company is three diameters, but preferably the same as the A.R.E.A. using also 2 inches for ⅝-inch rivets and 1¾ inches for ½-inch rivets. The "Pocket Companion" of the Carnegie Steel Co., gives the following minimum spacing:

Size of rivet, inches	¼	⅜	½	⅝	¾	⅞	1	1⅛
Minimum pitch, inches	1	1¼	1¾	2	2¼	2⅝	3	3⅜

Where rivets are staggered, the minimum diagonal distance should be as given in the table (*i.e.*, 2¼ inches for ¾-inch rivets). The minimum distance from the center of a rivet hole to a sheared edge should be: 1¾ inches for 1-inch rivets, 1½ inches for ⅞-inch rivets, 1¼-inches for ¾-inch rivets; to a rolled edge, which is less liable to crack, 1½, 1¼, and 1⅛ inches respectively (A.R.E.A.). (See Art. 10.)

There is also a maximum allowable pitch. In boilers or similar vessels, this must be such as will ensure tightness; in structures not liable to leak, it must be such as to hold the parts well together and in compression pieces to prevent wrinkling of plates.

13. Stresses on a Group of Rivets Resisting a Moment.—We have seen that a group of rivets may be exposed to a moment. If the moment acts in a plane perpendicular to the axis of the rivets, it is similar to torsion and causes shearing; double shearing if the piece transmitting the moment is enclosed between two others (Fig. 351*a*), single shearing if it is not so enclosed (Fig. 351*b*). If the moment acts in a plane parallel to the axis of the rivets, it causes tension on some rivets and compression between the pieces that the rivets connect (Fig. 351 rivets in lines *d*). Figure 351*b* is not a good design.

Considering the first case, the stress on any rivet should be considered as proportional to the distance of that rivet from the center of gravity of the group. If that distance is d, the stress on any rivet will be kd, where k is a constant, equal to the stress on the extreme rivet divided by its distance. The moment of all the stresses about the center of gravity must equal the applied moment M about the same point, or

$$M = \Sigma\, kd^2. \tag{4}$$

The stress S_1 on the extreme or most stressed rivet will be

$$S_1 = \frac{Md_1}{\Sigma d^2}. \qquad (5)$$

Example (Fig. 352).—Suppose a force of 20,000 pounds to be carried by an angle, attached to a plate *a*, which is riveted by three rivets to the bottom of a plate girder as shown. Let the center of gravity line of the angle, along which P acts, meet the line of rivets in the girder 4 inches above the upper rivet. Then at this point there are a horizontal force of 20,000 sin 30° = 10,000, and a vertical force (in the figure) of 20,000 cos 30° = 17,320. The moment M about the center rivet is 10,000 × 7 = 70,000 inch-pounds. The center rivet, at the center of gravity of the group, carries no stress due to the moment; hence the stress in each outer rivet, due to the moment,

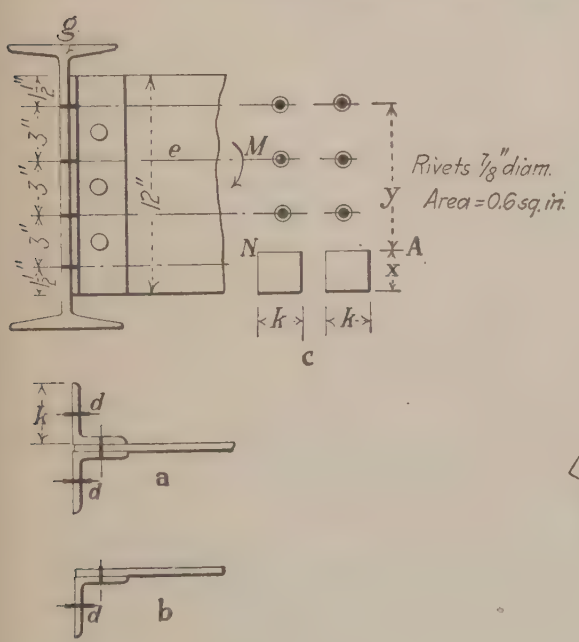


Fig. 351.

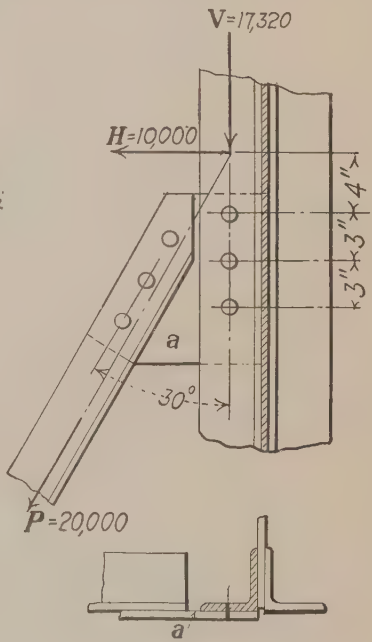


Fig 352.

is simply $\frac{70,000}{6}$ = 11,667 pounds. The same result is given by the formula, for d_1 = 3 inches and Σd^2 = 18. The stress in the outer rivet, 11,667 pounds, is a horizontal shear. Each rivet also carries a horizontal shear of $\frac{10,000}{3}$ = 3,333, assuming equal distribution of H; and a vertical shear of $\frac{17,320}{3}$ = 5,773 pounds. Hence the outer rivets have a horizontal shear of 15,000 pounds and a vertical shear of 5,773; or a resultant shear of close to 16,000 pounds.

If there were six rivets in the row instead of three, the stress on the extreme rivet due to M would be $\frac{10,000 \times 11.5 \times 7.5}{2 \times 78.75}$ = 10,950 pounds, and the resultant stress on the extreme rivet 12,940 pounds. Here increasing the number of rivets does not much reduce the stress due to M, since the lever arm of P increases also. This is a

very bad design; the line of action of P should be made to pass through the center rivet, so that there would be no M.

14. Moment Carried by a Line of Rivets (Fig. 353).—Equation (5) gives the stress on the extreme rivet in this case, and the dotted line indicates planar distribution over the rivets. If the center of gravity comes on a rivet, and p is the pitch, the problem may also be solved as follows: if S is the stress on the first rivet, the others will be as shown, and

$$M = S \times 2p + 2S \times 4p + 3S \times 6p + 4S \times 8p = 60Sp$$

$$S = \frac{M}{60p}; S_1 = 4S = \frac{M}{15p}.$$

This is exactly the same as Eq. (5), for $d = 4p$, $\Sigma d^2 = 60p^2$.

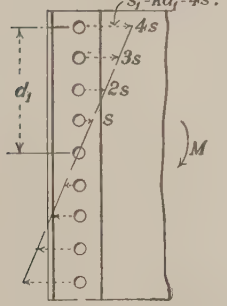


Fig. 353.

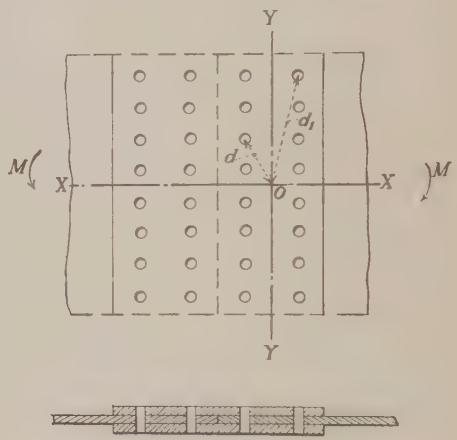


Fig. 354.

15. Moment Carried by Two Lines of Rivets (Fig. 354).—In this case the center of gravity of the group is midway between the two lines. The moment couple acts parallel to the plane of the paper, and tends to twist the group about an axis perpendicular to the paper through the center of gravity. The case is one of torsion. If the diagonal distance from any rivet to the center of gravity of the group is d, the stress on that rivet due to the moment M acts perpendicular to that diagonal distance, and is kd, if k is the stress on a rivet at a unit distance. or, if S_1 is the stress on the rivet farthest from the center of gravity, at a distance d_1 from it, the stress on any other rivet at a distance d will be $S_1 \frac{d}{d_1}$ and its moment about the center of gravity will be $S_1 \frac{d^2}{d_1}$, and the sum of these must equal M, or

$$M = \Sigma S_1 \frac{d^2}{d_1} = \frac{S_1}{d_1} \Sigma d^2$$

whence $$S_1 = \frac{Md_1}{\Sigma d^2}, \text{ as in Eq. (5).}$$

16. Moment Causing Tension in Rivets.—In Fig. 351 the rivets in lines d would be in tension for a certain distance from the top down, below which there

would be compression between the two angles and the web of the beam. The end of the plate e should not be considered as in contact with the web of the beam.

If the case is one of pure flexure, that is, if there is no resultant direct compression or tension between e (which may be a beam or girder) and the beam g, the cross-section, on the face of the web of g, which resists the moment, has the form shown in Fig. 351*c*, and the position of the neutral axis NA is determined by the condition that it passes through the center of gravity of the section. This can be found by trial; or, assuming it to be between certain rivets, its position can be found algebraically. Thus, assuming it to be between the first and second rivets from the bottom, as shown, calling x the height of the rectangular part, and omitting to deduct the area of the two rivet holes in the rectangular portion, which carry no compression, we have

$$2kx \cdot \frac{x}{2} = 2 \times 0.6\,[(4.5 - x) + (7.5 - x) + (10.5 - x)] = 1.2(22.5 - 3x)$$

from which,

$$x = -\frac{1.8}{k} + \sqrt{\frac{27}{k} + \frac{3.24}{k^2}}.$$

If this value of x lies between 1.5 and 4.5, the assumption made was correct, and the neutral axis is found. If not, the result will indicate another assumption to be made. The rivet holes in the rectangular portion may be deducted; the expression then will be

$$2kx \cdot \frac{x}{2} - 2 \times 0.6\,[(x - 1.5)] = 1.2(22.5 - 3x)$$

$$x = -\frac{1.2}{k} + \sqrt{\frac{25.2}{k} + \frac{1.44}{k^2}}.$$

Having found the N.A., the value of I about that axis must be calculated, and then

$$\text{Max. total stress on a rivet} = \frac{0.6My}{I} \text{ (for } \frac{7''}{8} \text{ rivets)}.$$

If the case is not one of pure flexure, that is, if there is a resultant compression between the hitch angles and the web of the floor beam, it is more difficult, and the solution will not be given here.

17. Three Methods of Considering Riveted Connections.—There are three ways of looking at riveted connections, all correct so far as they go, but from different points of view. From some of these points of view certain things can be seen that cannot be seen from the others. Each has advantages, all should be understood by the student, and one seems practically unknown. They are:

(*a*) By considering possible cases of failure. (This is most used.)

(*b*) By considering transmission of stress.

(*c*) By considering strands carried by rivets.

The second method is very useful. A rivet always acts to transmit stress from one piece, A, to another piece B. It must be clearly seen just what the rivets do and how they do it; that is, the stress they carry must be seen, their mode of action, and the pieces from which and to which they carry the stress. In every case some such statement as the following should be mentally made: "these rivets carry—stress (in pounds) from piece A to piece B, acting in (single or double) shear, and in bearing against either (one thickness or another in single

shear, or one thickness and the sum of two others in double shear) whichever is the smaller."

We shall first consider continuous riveted joints, as in boilers and tanks, where two plates are riveted together along the entire length of adjoining edges; and then the joints by which a tension or compression piece is connected to other parts.

CONTINUOUS JOINTS

18. In such a joint, consisting of one or several lines of rivets across the entire width of a plate, the first thing to do is to find a "repeating section" of the plate; that is a strip with the rivets in it, which is merely repeated across the entire width. Then this strip or section may be alone considered, and not the entire width.

19. Single Riveted Lap Joint (Fig. 355). —In this, the most simple case, the repeating section is clearly a strip ab, whose width is p, carrying a load P. The

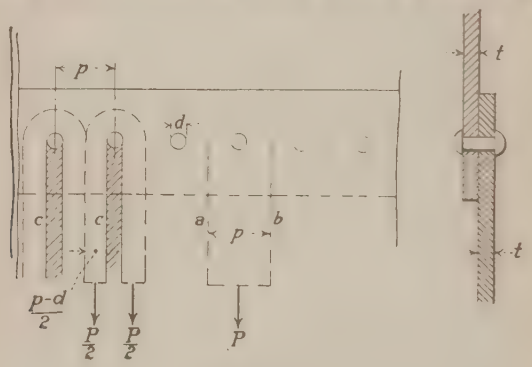


FIG. 355.

two plates are lapped one over the other and connected by a single row of rivets, with pitch p. Each rivet carries the stress P in a width of plate p. The strength of the unbroken plate is $f_t pt$.

(*a*) *Method by Cases of Failure.* —The strip may fail by tearing the plate through the rivet, for which its strength is $f_t(p - d)t$, (1); or by shearing the rivet, for which its strength is $f_s \frac{\pi d^2}{4}$, (2); or by bearing on rivet and plate, for which its strength is $f_b dt$, (3). The other possible methods, as by shearing out the plate behind the rivet, will be assumed to be made stronger than the weakest of the three just named. The smallest of these three is the strength of the joint, and the smaller of the last two may be made equal to the first.

The last two may be made equal, giving $\frac{d}{t} = \frac{4f_b}{\pi f_s} = 2.75$, using the ratios of Art. 11. Making now the third and first equal, we find $p = \frac{f_b + f_t}{f_t} d = 2.73\, d$. These dimensions will give the maximum efficiency possible in this form of joint: or since (3) = (1); max. efficiency $= \frac{p - d}{p} = \frac{f_b}{f_b + f_t} = 0.633$. In any given

case, the efficiency is the smallest of the three strengths named above divided by $f_t pt$. If the rivets are weaker in shearing than in bearing, which will be when $t > 0.36d$, and if the tearing strength is made equal to the shearing strength,

$$p = d + \frac{f_s \pi d^2}{4 f_t t} = d\left(1 + 0.63\,\frac{d}{t}\right) \tag{6}$$

which increases with d; and the efficiency

$$\frac{p - d}{p} = \frac{1}{1 + \dfrac{4 f_t t}{f_s \pi d}}$$

which also increases with d.

If the rivets are weaker in bearing than in shearing ($t < 0.36d$), and if the tearing and bearing strengths are made equal,

$$p = d\left(1 + \frac{f_b}{f_t}\right) = 2.73d \tag{6a}$$

which increases with d; and the efficiency

$$\frac{p - d}{p} = \frac{f_b}{f_b + f_t} \text{ which is consatnt.}$$

Therefore as long as the rivets will fail by shearing, with the pitch as computed, the efficiency is increased by increasing the diameter of rivet; but when the rivets will fail by bearing there is no increase in efficiency by such increase.

If exactness were possible, the following procedure might be employed; generally the stress in the plate *per unit width* is known, and calling this s:

$$ps = t(p - d)f_t = f_s \frac{\pi d^2}{4} = f_b t d.$$

From these three equations, knowing f_t, f_s and f_b, we find, for the assumed values,

$$t = s \cdot \frac{f_b + f_t}{f_b f_t} = 0.0000287s$$

$$p = 4s\,\frac{(f_b + f_t)^2}{\pi f_s f_t^2} = 0.000215s$$

$$d = 4s\,\frac{f_b + f_t}{\pi f_s f_t} = 0.0000789s.$$

Proportions necessary for maximum efficiency cannot often be used. The necessary pitch will sometimes conflict with the rules of Art. 12. Good practice requires the use of as few different rivet sizes as practicable on one job. Generally the size of rivet is assumed, the value of a rivet in shearing and in bearing found, the smaller used; and then, knowing the stress in the plate per unit width, the pitch is made such that the stress in a width p equals the strength of a rivet.

(*b*) *Method of Transmission of Stress.*—The tension P in the width p is carried by one rivet and by the strength of the net section $t(p - d)$, and the same results are reached as by the first method.

(*c*) *Strand Method* (Fig. 355).—The force in the plate is considered acting on strands, each passing around a rivet which carries the stress in the two ends of the strand. Each strand in this joint has a width $\frac{p - d}{2}$ and carries a stress $\frac{P}{2}$, so that each rivet carries P. The distance of the end rivet from the edge of the plate must be at least $\frac{p - d}{2}$, but as already stated it should not, for practical

reasons, be less than a specified distance. The shaded areas in front of the rivets are practically lost so far as concerns strength. Of course the plate will not really act exactly as if it were composed of strands; nor must it be inferred that the necessary lap back of a rivet need be only the width of a strand. The strand method gives no special information in this joint.

20. Double Riveted Lap Joint (Fig. 356).—The riveting may be chain or staggered. In either case the width of the repeating section is p, but the stress P on this section is now carried by two rivets. By making equal the tensile strength of the plate $(p - d)tf_t$; the bearing strength of two rivets, $2f_b dt$; and the shearing strength of two rivets, $\frac{1}{2}\pi d^2 f_s$, the relations found are: with the assumed stresses, $d = 2.75t$; $p = d\dfrac{2f_b + f_t}{f_t} = d\left(1 + \dfrac{2f_b}{f_t}\right) = 4.46d$, and maximum efficiency $= \dfrac{(p-d)}{p} = \dfrac{2f_b}{2f_b + f_t} = 0.775$. This however, is again impractical,

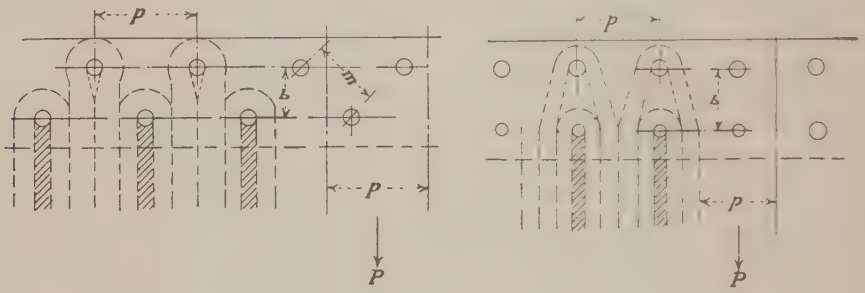


Fig. 356.

for the three strengths will never be equal. The actual efficiency, as before, will be the smallest strength divided by tpf_t.

The strands are shown in the figures. Staggered riveting is better than chain riveting, as the strands are less curved. In general, if there are n rivets in a repeating section, the width of a strand is $\dfrac{p-d}{2n}$.

The "Boiler Code" of the A.S.M.E. provides that the back pitch b shall be, for staggered rivets

if $\dfrac{p}{d} \leq 4$; minimum $b = 2d$

if $\dfrac{p}{d} > 4$; minimum $b = 2d + 0.1\,(p - 4d)$;

the minimum diagonal distance is given in Art. 12; it must obviously be greater than $d + 2\dfrac{p-d}{4} = \dfrac{p+d}{2}$, as is shown by the strands.

21. Other Lap Joints.—Sometimes a lap joint is riveted as in Fig. 357. The stress on a width p is carried by four rivets. The cases of failure are now:

(a) By tearing along the outer row; strength $= (p - d)tf_t$.

(b) By tearing along the main row and shearing or bearing on one outer rivet;

$$\text{strength} = (p - 2d)tf_t + \frac{\pi d^2}{4}f_s; \text{ or } (p - 2d)tf_t + f_b dt;$$

(*c*) By shearing four rivets; strength $= \pi d^2 f_s$.

(*d*) By bearing of rivets; strength $= 4dtf_b$.

The first row will be weaker than the second in tearing if the tensile strength dtf_t is less than the shearing (or bearing) strength of one rivet, or if $d > 1.59t$.

Assuming the first row weaker than the second, and making its strength equal to the smaller of the shearing and bearing strengths, the efficiency will be $\frac{p-d}{p}$. If the shearing and bearing strengths can be made equal, we shall find

$$d = 2.75t;\ p = d\left(1 + \frac{4f_b}{f_t}\right) = 7.91d$$

max. efficiency $= 0.86$.

Lap joints are sometimes treble riveted with rivets in middle row staggered (Fig. 358).

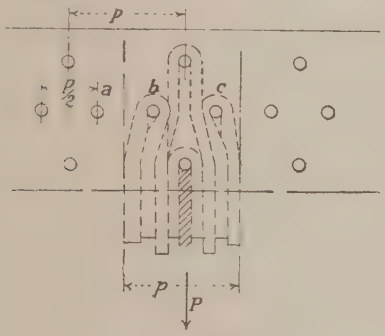


FIG. 357.

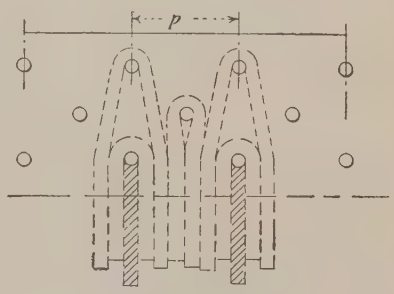


FIG. 358.

22. The strand method often indicates or suggests what might otherwise be difficult to see. Thus in Fig. 357 the strands are more crowded together between rivets *b* and *c* where four strands must pass, than between rivets *a* and *b* where only two pass. In looking at joints by the method of possible failure, the stress is always assumed equally distributed over a net section or over a group of rivets. This is generally untrue. In this case it is probable that along the middle row of rivets the stress averages more between *b* and *c* than between *a* and *b*; and it seems to the writer probable that the strength of the joint would be increased if the distance between *a* and *b* were less than that between *b* and *c*. This, however, is never done.

23. Example of a Double Riveted Staggered Lap Joint (Fig. 356).—Suppose a tank to have a hoop tension of 6,420 pounds per inch of height (see Chap. XVIII). Let the plates be ½ inch, rivet holes ⅞ inch, $p = 2\frac{1}{16}$ inches. Then stress in unbroken plate $= f = 6{,}420 \times 2 = 12{,}840$ pound per squares inch.

Net strength of plate $= 1\frac{3}{16} \times \frac{1}{2} \times 55{,}000 = 49{,}850$ pounds

Gross strength of plate $= 2\frac{1}{16} \times \frac{1}{2} \times 55{,}000 = 73{,}900$ pounds

Strength of two rivets, shearing $= 0.6 \times 2 \times 44{,}000 = 52{,}800$ pounds

Strength of two rivets, bearing $= 2 \times \frac{7}{8} \times \frac{1}{2} \times 95{,}000 = 83{,}100$

Joint is weakest in tearing: efficiency $= \frac{49{,}850}{73{,}900} = 0.674$.

$$\text{Max. stress on net section} = \frac{12{,}840}{0.674} = 19{,}100 \text{ pounds per square inch}$$

$$\text{Max. stress on rivet, shearing} = \frac{6{,}420 \times 2^{11}/_{16}}{1.2} = 14{,}400 \text{ pounds per square inch}$$

$$\text{Max. stress on rivet, bearing} = \frac{6{,}420 \times 2^{11}/_{16}}{2 \times {}^{7}/_{8} \times {}^{1}/_{2}} = 19{,}710 \text{ pounds per square inch.}$$

24. Example of a Treble Riveted Lap Joint (Fig. 358).—Let the hoop tension be 7,920 pounds per inch width; plates $^{1}1/_{16}$ inch; rivets $1^{1}/_{8}$ inches, pitch 4 inches.

Stress in gross section = $7{,}920 \div {}^{1}1/_{16} = 11{,}550$ pounds per square inch

Net strength of plate = $2^{7}/_{8} \times {}^{1}1/_{16} \times 55{,}000 = 108{,}700$ pounds

Gross strength of plate = $4 \times {}^{1}1/_{16} \times 55{,}000 = 151{,}250$ pounds

Strength of three rivets, shearing = $3 \times 0.994 \times 44{,}000 = 131{,}208$ pounds

Strength of three rivets, bearing = $3 \times {}^{9}/_{8} \times {}^{1}1/_{16} \times 95{,}000 = 220{,}400$ pounds

Strength of one rivet, shearing = 43,736 pounds

Tensile strength of area $dt = 1^{1}/_{8} \times {}^{1}1/_{16} \times 55{,}000 = 42{,}540$ pounds.

Hence middle row in tearing is stronger than first row, and joint is weakest in tearing on first row, for which efficiency is $\frac{108{,}700}{151{,}250} = 0.719$

$$\text{Max. stress on net section} = \frac{11{,}550}{0.719} = 16{,}020 \text{ pounds per square inch}$$

$$\text{Max. shearing on rivets} = \frac{7{,}920 \times 4}{3 \times 0.994} = 10{,}600 \text{ pounds per square inch}$$

$$\text{Max. bearing on rivets} = \frac{7{,}920 \times 4}{3 \times {}^{9}/_{8} \times {}^{11}/_{16}} = 13{,}700 \text{ pounds per square inch.}$$

For the joint shown in Fig. 357, the "Boiler Code" of the A.S.M.E. gives the back pitch as stated in Art. 20, but p in the formula there given is "pitch in the outer row less pitch in inner row," (or one-half the pitch in the outer row).

25. Butt Joints.—In these joints the main plates are butted together, and covered by two straps, one on each side, either of the same or of different widths. In the lap joint, the stress in one plate is transmitted through the rivets, acting in single shear and bearing, directly to the other plate. In the butt joint the stress in one plate is transmitted through the rivets, acting in double shear and in bearing against the main plate or the sum of the two covers to the cover plates, and from them in the same way to the other plate. Hence the cover plates must together be as strong as the main plate. It follows that the two covers must together have a minimum net section at least as great as that of the main plate; or the net section of the main plate at the row of rivets farthest from the joint must not exceed the net section of the two covers at the row of rivets nearest the joint, assuming that these are the weakest sections of main plate and covers respectively. Each cover must therefore be at least half as thick as the main plate, or a little more, since the first row of rivets in the main plate often has fewer rivets than the row nearest the joint. The covers are always given quite a little excess above this minimum.

Thus the "Boiler Code" of the A.S.M.E. has the following requirements for minimum thickness of *double butt straps:*

Thickness of shell	¼	$^{9}/_{32}$	$^{5}/_{16}$	$^{11}/_{32}$	⅜	$^{13}/_{32}$	$^{7}/_{16}$	$^{15}/_{32}$	½	$^{17}/_{32}$	$^{9}/_{16}$	⅝	¾	⅞	1	1⅛	1¼
Thickness of strap	¼	¼	¼	¼	$^{5}/_{16}$	$^{5}/_{16}$	⅜	⅜	$^{7}/_{16}$	$^{7}/_{16}$	$^{7}/_{16}$	½	½	⅝	$^{11}/_{16}$	¾	⅞

and "for plate thicknesses exceeding 1¼ inches, the thickness of butt straps shall be not less than two-thirds of the thickness of the plate."

Occasionally one cover is thicker than the other, in which case they must be assumed to carry stress in proportion to their thicknesses, and the rivets, while in double shear, do not carry equally on the two shearing sections.

26. Single Riveted Butt Joint (Fig. 359).—The resistance to tearing is $f_t(p - d)t$, to shearing $f_s\frac{\pi d^2}{2}$, to bearing $f_b dt$. Equating the last two, $\frac{d}{t} = \frac{2f_b}{\pi f_s}$, or half that for a lap joint. Equating the first and last, $p = d \cdot \frac{f_b + f_t}{f_t}$, or the same as for a lap joint, and the maximum efficiency $\frac{p - d}{p}$ is also the same, or 0.633. The maximum efficiency, being $\frac{f_b}{f_b + f_t}$, is independent of d, but in the butt joint it requires smaller rivets than in the lap joint.

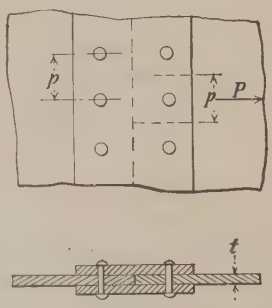


FIG. 359.

27. Other Forms of Joint.—Butt joints are made double or triple or even quadruple riveted, with covers of equal or unequal width, as in Fig. 360. For more complicated joints, see the "Boiler Code" of the A.S.M.E.

Considering Fig. 360*b*, the example given in the Code is as follows: let t = ⅜ inch; t_1 = 5/16 inch; p = 6.5 inches; d = 1 3/16 inch; area rivet = 0.5185 square

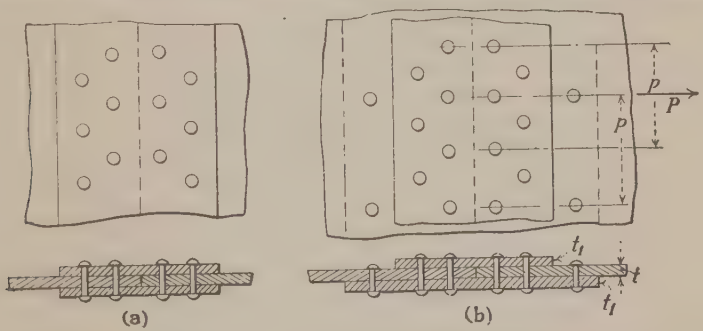


FIG. 360.

inch; the symmetrical repeating section will go through the centers of two inside rivets; then there are four rivets in double and one in single shear;

Strength of solid plate = $6.5 \times 0.375 \times 55{,}000$ = 134,062 pounds

Net strength of main plate = $(6.5 - 0.8125) \times 0.375 \times 55{,}000$ = 117,304 pounds

Shearing strength of rivets = $0.5185 \times 44{,}000 \times 9$ = 205,326 pounds

Crushing strength of rivets = $95{,}000 \times (4 \times 0.8125 \times 0.375 + 1 \times 0.8125 \times 0.3125)$ = 139,902 pounds

Crushing strength of four rivets + single shearing strength of one rivet

$$= 95{,}000 \times 4 \times 0.8125 \times 0.375 + 1 \times 44{,}000 \times \times 0.5185 = 138{,}595 \text{ pounds}$$

Strength of plate at second row of rivets plus shearing strength of one rivet $= (6.5 - 2 \times 0.8125) \times 0.375 \times 55{,}000 + 1 \times 44{,}000 \times 0.5185 = 123{,}360$ inches
Strength of plate at second row plus bearing strength of one rivet $= (6.5 - 2 \times 0.8125) \times 0.375 \times 55{,}000 + 1 \times 95{,}000 \times 0.8125 \times 0.3125 = 124{,}667$ inches.

The smallest of these is 117,304, and hence

$$\text{Efficiency} = \frac{117{,}304}{134{,}062} = 0.875.$$

The butt joint with covers of unequal width (Fig. 360*b*) is in the opinion of the writer bad design and should not be used. If the two covers are of equal thickness, they can take equal stress, but require the same number of rivets to take the stress in and out, and hence should be of equal width. If they do not carry equal stress there is a tendency for the joint to bend like a lap joint. If a butt joint is to be used the covers should have the same thickness and width.

28. The preceding paragraphs illustrate the relations between pitch, rivet diameter, etc. for the various forms of joints. In a tank or boiler, the most suitable size of rivet may be chosen and used throughout. Generally the diameter is greater than the thickness of the plate. In Germany and France a common formula has been d (in inches) $= 1.5t + 0.16$.

In England, Unwin gives as common practice, $d = 1.2\sqrt{t}$ to $1.4\sqrt{t}$. We have seen that the larger the diameter, the greater, generally, the efficiency of the joint. Also, larger rivets mean larger heads and greater tightness. Further details are given in books on boiler design.

29. Lap vs. Butt Joints.—On account of the fact that the pulls are not in the same line, a lap joint, if it does not straighten out, exposes each plate to an apparent bend-

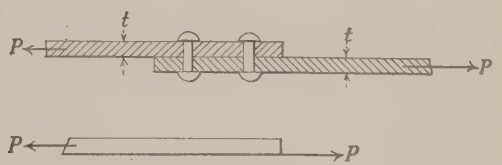


Fig. 361.

ing moment $\frac{Pt}{2}$ in addition to the direct stress (Fig. 361). This moment would produce a maximum fiber stress $\frac{Pt}{2} \cdot \frac{t}{2I} = \frac{3P}{bt}$, making the total maximum stress four times the load P divided by the area. The bending moment causes the plates to bend, thus reducing the lever arm and the stress, and if the total stress does not exceed the elastic limit the joint will straighten again when the load is removed; if above the elastic limit, a permanent bend will be produced. This bending stress, or the alternate bending and relieving of the plates is by some engineers considered injurious and they strongly condemn such joints. Cracks along one of the rivet lines have sometimes been attributed to this cause. The "Boiler Code" only allows lap joints in boilers 36 inches in diameter or less, and for such

diameters only for pressures of 100 pounds or less per square inch. But such a boiler would have a hoop tension of 1,800 pounds, and would require the minimum allowed thickness of ¼ inch, which would give 7,200 pounds per square inch on the gross section (and more on the net), and allowing for the full eccentricity of the pulls P would allow a maximum stress of 28,800 pounds per square inch instead of the supposedly allowed 11,000.

In a butt joint the plates and pulls are in line and there is no bending on them; but there must be bending on the covers, as indicated in Fig. 362, which would tend to make them bend away from the main plates at the ends. This tendency is resisted by the rivet heads and by tension on the outer rivets, especially if there are several rows, and is a good reason for making the covers thicker than $\frac{t}{2}$ and for using joints that are more than single riveted. There is a similar restraining action in lap joints, which makes the above computed stresses erroneously high.

Altogether, the writer believes that if the material is ductile the lap joint is not at all as bad as many consider it; though he has no question that properly

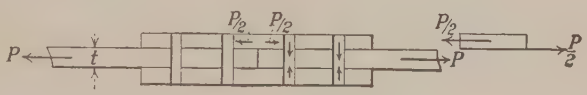


Fig. 362.

designed butt joints are better. *Tests of riveted joints* (*see Art.* 32) *do not indicate any special difference in strength between the two forms.* The main thing in a lap joint is to have good ductile material, which would bend flat on itself without cracking.

The presence of initial stress (see Chap. XIX) also enters into the question. If the plates of a lap joint are bent cold so as to bring the pulls in line, not only is the moment reduced, but the cold bending beyond the elastic limit leaves initial stresses of tension at the edge where the moment produces compression, and compression at the edge where the moment produces tension, thus partly or perhaps wholly counteracting the additional stress due to the moment. In large tanks for oil, gas, and water, it is common practice to lap all the joints in a ring for right-handed caulking, that is, lapping the left-hand plate outside the right-hand plate. It can be seen by drawing a figure that it is impossible to make a circular ring in this way without making a very excessive bend at one joint, sufficient to take up the combined thickness of all the plates in the ring, or else to bend each joint so as to bring the pulls in line. With alternate right-hand and left-hand caulking the straight lap could be made.

30. Beveling Edges.—A tank is made of rings with the joints in one ring opposite solid plates in the rings above and below, which lap over or under the first ring, and are riveted by circumferential rivets to form the circumferential joint. In such case one of the plates forming the lap must be forged down to a thin edge as shown in Fig. 363.

With butt joints, the covers need not lap over the plates above or below and there need be no beveling.

WATERTOWN ARSENAL TESTS OF METALS (See Art. 32)

Description of joint	Tension on net section at time of failure		Sum of ratios stress on net section / Tens. strength of plate		References	
	Above tensile strength of plate	Below tensile strength of plate	Plus	Minus	1882	1891
Lap:						
Single riveted ¼-inch iron plate	4	7	4.295	6.265	p125–126	
Single riveted ¼-inch steel plate	23	0	29.245	0.000	p127–128	p564–568
Double riveted ¼-inch iron plate	2	1	2.095	0.823	p128–129	
Double riveted ¼-inch steel plate	10	0	11.200	0.000	p128–129	p564
Double riveted 5/16-inch steel plate	7	1	7.793	0.973	.....	p572
Single riveted ⅜-inch iron plate	2	3	2.545	2.525	p131	
Single riveted ⅜-inch steel plate	8	0	9.660	0.000	p132–133	
Double riveted ⅜-inch iron plate	3	2	3.165	1.955	p133–134	
Double riveted ⅜-inch steel plate	8	0	9.390	0.000	p133–134	p578–580
Single riveted 7/16-inch steel plate	7	0	8.390	0.000	.....	p582–588
Double riveted 7/16-inch steel plate	6	0	7.055	0.000	.....	p584
Triple riveted 7/16-inch steel plate	4	0	4.625	0.000	.....	p586
Single riveted ½-inch iron plate	0	3	0.000	2.315	p136–137	
Single riveted ½-inch steel plate	1	1	1.055	0.865	p136–137	
Double riveted ½-inch steel plate	2	0	2.358	0.000	p137	
Single riveted ⅝-inch iron plate	1	2	1.050	1.548	p138–139	
Double riveted ⅝-inch iron plate	3	1	3.635	0.975	p138–139	
	91	21	107.556	18.244		
			18.244			
			112)125.800(1.123			

Butt:						
Single riveted ¼-inch iron plate	2	0	2.180	0.000	*p*130	
Single riveted ¼-inch steel plate	6	0	6.570	0.000	.....	*p*560–562
Double riveted ¼-inch steel plate	15	1	16.485	0.990	.....	*p*560–564
Single riveted 5/16-inch steel plate	2	0	2.280	0.000	.....	*p*568–570
Double riveted 5/16-inch steel plate	6	0	6.625	0.000	.....	*p*568–572
Double riveted ⅜-inch steel plate	12	2	13.200	1.955	.....	*p*574–578
Single riveted 7/16-inch steel plate	5	2	5.495	1.627	.....	*p*580–582
Double riveted 7/16-inch steel plate	2	0	2.155	0.000	.....	*p*580–582
	50	5	54.990	4.572		
			4.572			

55)59.562(1.083
55
456
440
162

31. Grooved Specimens.—It has been shown that in computing riveted joints the stresses are considered uniformly distributed over a tearing section and over the rivets. In Chap. VI the effect of a groove in a tension piece was discussed. As a matter of fact, a riveted joint consists of a number of strips with semi-circular grooves on each side (Fig. 364) placed close together. The case is also similar to that of a plate with a hole in it, discussed in Chap. VI. There is undoubtedly in such cases an excess of stress at the edges of the holes, but on the other hand the metal there has greater strength than elsewhere, owing to its inability to stretch. Theoretical investigations and Professor Coker's experiments, by which it is claimed that the stress at the edge of the hole is three times the average stress in the gross section have caused apprehension in the minds of some engineers; but if this stress really existed and were not counteracted by greater strength, many existing boilers and structures would have failed long ago. Tests of riveted joints show in fact that the net section through a line of rivets has an ultimate strength some 10 or 12 per cent greater than an equal section of a prismatic bar.

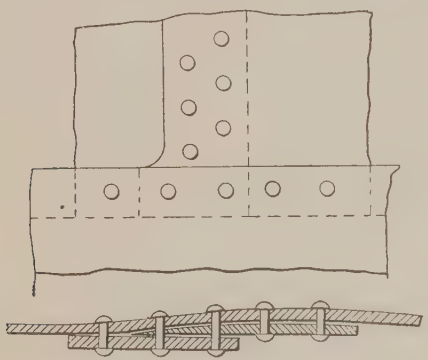


Fig. 363.

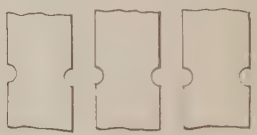


Fig. 364.

32. Tests of Riveted Joints.—The best and most extensive series of tests of riveted joints known to the writer were made at the Watertown Arsenal, and are reported in "Tests of Metals" for 1882 and 1891. They are quite fully described in Lanza's "Applied Mechanics," but the original reports should be studied. In all, 112 lap joints were broken, and in 91 of them the average stress on the net section at time of fracture exceeded the tensile strength of the material as found from tests on standard pieces. This is shown in the table on pages 404 and 405:

For each test the ratio of the tensile stress on the net section, at failure (which did not in all cases occur by tension), to the tensile strength of the plate, as determined by a special test, has been computed. Thus, in the first set of 11 tests, there were seven in which, when the failure was by tension on the net section, the stress on that section was less than the tensile strength of the plate; and four in which, at failure (whether by tension on the net section or otherwise), the stress on the net section exceeded the tensile strength of the plate. The sum of the ratios for the seven cases (each ratio less than unity) was 6.265; and the sum of the ratios for the four cases (each greater than unity) was 4.295. Summing up the ratios, subtracting the minus from the plus, and dividing by the total number of tests, it appears that, on the average, the lap joints showed, at failure, a greater strength on the net section than the strength of the unbroken plate, by 12.3 per cent.

Doing the same for the butt joints listed, there is an excess strength of 8.3 per cent. This is rather surprising, and it certainly indicates no inferiority of the lap joints, but rather the reverse.

These tests support the writer's view that the most important thing, if lap joints are used, is to see that ductile material is used; and that if such material is used, lap joints are not as faulty as some engineers think.

JOINTS IN STRUCTURES

33. As distinguished from continuous joints such as occur in boilers and tanks, where wide plates are connected along one edge, joints in structures are for the purpose of connecting comparatively narrow members, at their ends, to other parts. The principles of design of such joints will be fully treated in another volume, and only a few will be referred to here.

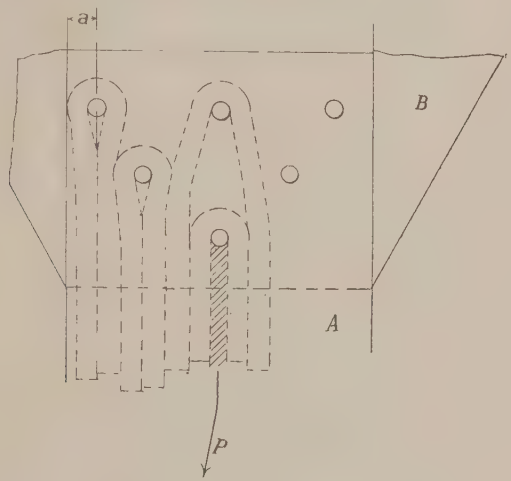


FIG. 365.

Suppose a plate A (Fig. 365) to be connected to the gusset plate B. The total stress in A must be transmitted to B through the rivets, and the problem is to determine the number and arrangement of rivets. The number will be the total stress divided by what a rivet will carry, assuming equal distribution. The figure shows six.

In order to weaken the main plate as little as possible, only one rivet is placed in the first row. If the tensile strength of a strip of plate with area dt is less than the stress that goes out of A through the first rivet, the second row is stronger than the first.

The figure shows the strands. If there were another row of three rivets, the distance a should be large enough for two strands unless the strands are to be very much bent out of line. The stress will certainly be most uniformly distributed when the strands are least bent.

34. A row of rivets in a line parallel to the stress do not carry stress equally; the end rivets carry most, and the stress decreases toward the center rivet, which carries least. Let two plates be connected by a lap joint with three rivets in a line

(Fig. 366). Let R_1 = stress on first rivet; R_2 = that on second rivet; $R_3 = R_1$ = stress on third rivet; E = modulus of elasticity in tension; E_S = modulus of elasticity in shear. Now assuming that all rivets carry equally, each carrying $\frac{P}{3}$, then each rivet will distort equally, and p_1' will equal p_1''; but the stress in portion a of the upper plate will be $\frac{2}{3}P$, and that in portion b of lower plate will be $\frac{1}{3}P$. The distance p_1' is the elongated distance which was originally p, and so is p_1'', and since these lengths have unequal stresses they cannot elongate equally, and

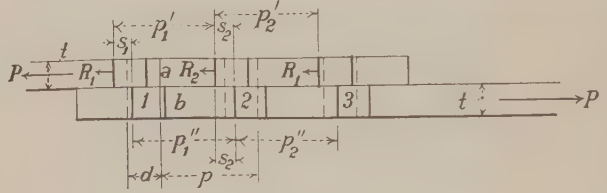


Fig. 366.

p_1' cannot equal p_1'', as must be the case if the rivets carry equally; hence they cannot carry equally.

The distribution can be found if it is assumed that the stress is uniformly distributed over the section between two rivets (which is not true). Thus, if b is the breadth of the plates,

$$p_1' = p + (P - R_1)\frac{p}{btE} = p + k(P - R_1)$$

$$p_1'' = p + \frac{R_1 p}{btE} = p + kR_1$$

calling $k = \frac{p}{btE}$. Also, if A = area of rivet section,

$$s_1 = \frac{R_1 d}{AE_S} = cR_1;\ s_2 = \frac{R_2 d}{AE_S} = cR_2$$

$$p_1' + s_2 = p_1'' + s_1;\ \text{or}\ p_1' - p_1'' = s_1 - s_2;\ \text{or}$$
$$p + k(P - R_1) - p - kR_1 = c(R_1 - R_2);\ \text{or}$$
$$k(P - 2R_1) = c(R_1 - R_2). \tag{1}$$

Similarly; $k[P - 2(R_1 + R_2)] = c(R_2 - R_3) = c(R_2 - R_1)$. (2)

Solving these two equations for R_1 and R_2, we find

$$R_1 = P\frac{k + c}{3c + 2k} \tag{3}$$

$$R_2 = P\frac{c}{3c + 2k}. \tag{4}$$

Obviously $R_1 > R_2$ in the ratio $\frac{R_1}{R_2} = 1 + \frac{k}{c} = 1 + \frac{pAE_S}{btdE}$. (5)

Example.—Let two plates $3 \times \frac{1}{2}$ inches, with pitch = 3-inch and $\frac{7}{8}$-inch rivets, be connected as shown; then $A = 0.6$, and $\frac{E_S}{E} = 0.4$ (see Chap. IV); and (using nominal rivet areas)

$$\frac{R_1}{R_2} = 1.55.$$

Hence $R_1 = R_3 = 0.378P$; $R_2 = 0.244P$.

If it is not considered proper to assume the stress in portions a and b as uniformly distributed over b, but that it should be distributed over the net section, say 2 inches wide,

$$R_1 = R_3 = 0.3925P;\ R_2 = 0.215P.$$

35. The inequality of distribution will increase with the number of rivets in a line parallel to the stress. With but two rivets, they will carry equally; with three, the results have just been shown. The case of six rivets gives results as follows (Fig. 367); the equations will be

$$k(P - 2R_1) = c(R_1 - R_2)$$
$$k[P - 2(R_1 + R_2)] = c(R_2 - R_3)$$
$$k[P - 2(R_1 + R_2 + R_3)] = c(R_3 - R_4) = 0.$$

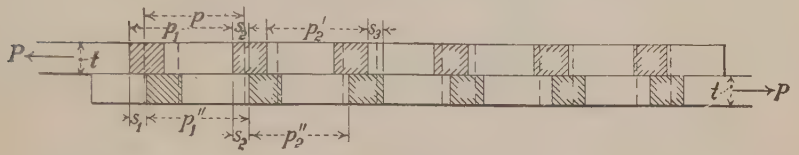


FIG. 367.

The solution of these equations gives

$$R_1 = P\frac{c^2 + 6ck + 4k^2}{2(3c + 2k)(c + 2k)}$$

$$R_2 = P\frac{c}{2(3c + 2k)}$$

$$R_3 = P \cdot \frac{c^2}{2(3c + 2k)(c + 2k)}$$

$$R_1 + R_2 + R_3 = \frac{P}{2},\ \text{as it should.}$$

Example.—Assuming plates $4\frac{1}{2} \times \frac{3}{4}$ inches; $p = 3$ inches; $\frac{7}{8}$-inch rivets; $\frac{E_s}{E} = 0.4$.

$$\frac{R_2}{R_3} = 1 + 2\frac{k}{c} = 1 + \frac{2pAE_S}{bdtE} = 1.488$$
$$R_1 = 0.26P;\ R_2 = 0.143P;\ R_3 = 0.096P$$

$R_1 + R_2 + R_3 = \frac{P}{2}$, as it should (within errors of approximation). With equal distribution, each rivet should carry $0.1667P$; hence R_1 carries 56 per cent more than the average, and R_3 only about 0.58 of the average.

These percentages are not to be taken too seriously. What it is desired to show is that the stress is not carried equally by a row of rivets and that the error in the usual assumption of equal distribution may be considerable. This point is generally entirely overlooked in designing.

36. Indirect Transmission of Stress.—The rivets thus far considered transfer stress directly from one piece to another which is in contact with it, the rivet passing through both. Sometimes, however, there is an intermediate piece or *filler*, as in Fig. 368. This is indirect transmission. The question thus arises, can a rivet transfer stress just as effectively in this way as where there is no filler? Clearly not. Each rivet will be in shear in the same direction on the planes above

and below the filler, and even if it fills the hole there will be more bending of the rivet than if there were no filler, because the filler can slip between the two main plates if the rivets yield. Such a filler is called a *loose filler*, and it is clear that more rivets will be required than if there were no filler. It may be made a *tight filler* by extending it along either plate, as shown by the dotted lines, and riveting it to that plate. Rivets A then serve to transmit a part of the force P from plate 1 to the filler, and rivets B to carry it from the filler to plate 2. There is no doubt, however, that a rivet may transmit force through a loose filler, though less effectively than through a tight filler or with none at all.

If the filler is made tight, and given a net area equal to that of the main piece, as, for instance, when it has the same width and thickness of the plate to be connected in the figure, there will be twice as many rivets necessary as for a loose

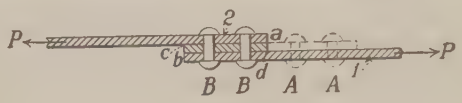


Fig. 368.

filler; one-half of these rivets (A) to be considered as carrying the stress to the filler and one-half carrying it from the filler to plate 2. This assumes that rivets B carry no stress from plate 1. Often, however, fillers are thin, and the case is more complicated. Tetmajer's experiments, with plates of equal thickness, led him to the conclusion that for indirect splices with one intermediate plate, at least twice as many rivets are necessary as for a direct splice.

The case is further complicated if there is more than one intermediate plate, in which case still more rivets are necessary. The question obviously is, whether, and to what extent, a rivet can carry stress out of a plate, and through an intermediate plate or plates, to another plate; or whether a rivet can only carry stress from a plate to another in contact with it. To assume the latter is safest, but no doubt too safe.

CHAPTER XVII

COLUMNS

1. Definitions.—A *column*, *post*, or *strut* is a body with a straight axis, acted upon by two equal and opposite compressive forces (P) at the ends. The cross-section of the column is usually the same at all points, though sometimes it varies. If the forces P act at the center of gravity of the section at the ends, and if the column is short, it will not bend laterally, and the resultant force on any section will act at its center of gravity. In such case the stress on any section is uniformly distributed over it, and the maximum stress intensity equals the average or $f = \frac{P}{A}$. Such a piece is called a short strut.

The *slenderness ratio* of a column is the ratio of the length to the least radius of gyration of a cross-section. If this ratio is great enough, the column may bend laterally, and if it does, though the stress on the ends is uniformly distributed, as assumed, the stress on any other section will not be, the resultant force on any section except the ends will not act at the center of gravity, there will be a bending moment, the stress at any point of the section will be the sum of that due to the direct compression and that due to the bending moment, and the maximum intensity will be greater than the average $\frac{P}{A}$. Such a column is sometimes called a long column, but there is no sharp line of demarcation between long and short columns.

If f is the maximum allowable compressive stress intensity, the formula $f = \frac{P}{A}$ is often used up to some arbitrary limit of $\frac{l}{r}$, and above this limit a formula is used which is designed to allow for bending. A theoretically correct formula should apply to all cases, and for $\frac{l}{r} = 0$ should give $f = \frac{P}{A}$. A theoretically correct formula, however, has not yet been found.

2. Pin-ended Columns.—A column in which the force at each end acts at the center of gravity of the section has no bending moment at the end. If the column tends to deflect, the ends tend to rotate, and there is no restraint to prevent such rotation. This condition is generally assumed to exist when the force is applied through a pin passing through the center of gravity of the section, and such a column is called a *pin-ended column* (Fig. 369). In an actual column, however, the condition is different, and can never be accurately known. If the pin fits the hole, there will be friction, and this will to some extent restrain the rotation, and will constitute a bending moment on the end of the column, thus making the resultant force on the column act at one side of the center of gravity, even if the pin is placed there. The friction will depend upon the load, and upon

the state of lubrication of the bearing surface, and its moment will depend upon the radius of the pin.

Practically a pin never fits the hole exactly. In this case the pressure between pin and column is distributed over less than a semi-circumference. The column, in tending to deflect, will exert friction as before, or it may roll on the pin; but in either case there is restraint against deflection, as will be seen.

By the term "pin-ended column," therefore, is meant—not a column with pin ends—but an ideal or imaginary column, in which there is absolutely no bending moment at either end.

Columns differ greatly, as we shall see, with regard to the end conditions, but all long prismatic or approximately prismatic bodies exposed to a compressive force at each end are called columns, the term being qualified in each case by some adjective such as "pin-ended" or "fixed-ended."

It is possible to conceive a pin-ended column, no matter how long or slender, which would remain perfectly straight, under the action of compressive forces, and for which, therefore, the formula would be $P = fA$. But such a column could not practically exist. In the first place, no long column could be *absolutely straight*, and any deviation from straightness, with loads centrally applied at the ends, would result in bending moments, and therefore in bending, which bending would increase the lever arms, the bending moments, and therefore the bending, until a state of equilibrium would establish itself. In the second place, even if absolutely straight, there would be differences of homogeneity of the material, which would result in bending. In the third place, even if absolutely straight and homogeneous, any small lateral force would produce some lateral deflection, which would introduce bending moments due to the compressive forces, and so increase the bending caused by the transverse load. In the fourth place, if not exactly vertical, its own weight would bend it.

Fig. 369.

As showing the effect of a lack of homogeneity, suppose a column were made of two channels, as in Fig. 383, and suppose the right-hand channel to have a slightly lower modulus of elasticity than the other, for a short distance at the center of the length. Then the former would be compressed more than the latter, and this would produce a bend, and a small deflection, which would at once become the arm of the compressive force, which would immediately increase the moment and the deflection until equilibrium were attained, or failure resulted.

It must therefore be assumed that a long, slender column will bend laterally, even under the action of compressive forces applied centrally at the ends.

Let Fig. 369 represent an ideal pin-ended column, which bends laterally in the plane of the paper. The problem then is, to find the maximum unit stress.

3. Moment of Friction on a Pin.—If a column has a pin at the end through which the load is applied (Fig. 370), and if the pin fits the hole closely, the force P is distributed over the area of the semi-circle, *abc*. It may be assumed that at each point there is a normal pressure of intensity n, which varies from zero at a and c to a maximum n_1 at b. This will cause a compression of the pin to some curve like *aec*, and of the column to *adc*, and these two will coincide as the resulting surface of contact. The shape of this surface is unknown. It seems reasonable, however, to assume that

every point, as g, moves the same distance $gf = cd$, parallel to P. The normal compression, gh, will be some constant times n, or cn, and gf will then equal de, or cn_1.

Hence
$$\frac{gh}{gf} = \frac{cn}{cn_1} = \frac{n}{n_1} = \cos\theta$$
$$n = n_1 \cos\theta.$$

Then, if t is the thickness of the bearing surface,

$$P = \int_0^{\frac{\pi}{2}} 2t \cdot rd\theta \cdot n\cos\theta$$
$$= 2rn_1 t\int_0^{\frac{\pi}{2}} \cos^2\theta \cdot d\theta = \frac{n_1 \pi rt}{2}$$
$$n_1 = \frac{2P}{\pi rt}. \qquad (1)$$

Thus, the maximum intensity of pressure equals P divided by the area of a quadrant $\left(\frac{\pi rt}{2}\right)$, as was shown in the previous chapter for rivets.

Now if the column tends to bend, and does rotate on the pin, there is developed, on each area $rtd\theta$ of the surface, a tangential friction equal to $rtd\theta.nk$, if k is the coefficient of friction. The total frictional force, acting tangent to the pin, will be

$$F = 2\int_0^{\frac{\pi}{2}} rtnkd\theta = 2rtk \cdot \frac{2P}{\pi rt}\int_0^{\frac{\pi}{2}} \cos\theta d\theta$$
$$= \frac{4Pk}{\pi} = 1.27Pk. \qquad (1a)$$

The moment of this force about the center of the pin will be

$$M = Fr = 1.27Pkr. \qquad (2)$$

This will be the moment at the end of the column which resists the tendency to rotate.

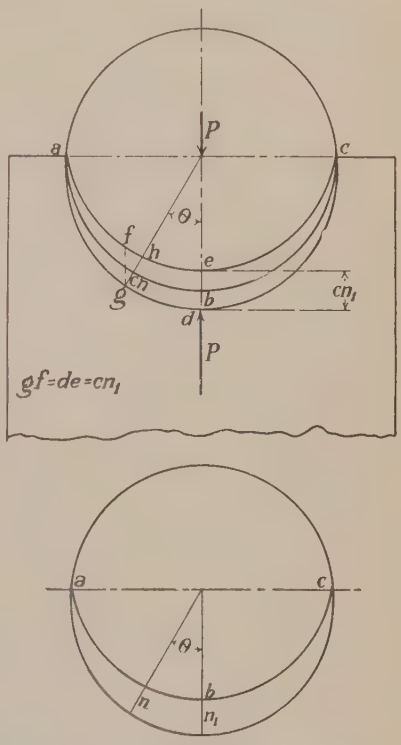


FIG. 370.

The above only applies if the pin closely fits the hole. Equation (1) applies to the distribution of stress over a well-driven rivet; but common practice is to assume $n_1 = \frac{P}{di}$, or smaller than given by Eq. (1). A pin does not fit the hole closely. If it did, it would be difficult to insert it, owing to inevitable inaccuracies in workmanship; the holes in the different bars connected by the pin are often not exactly opposite. Therefore a certain play is specified, generally $\frac{1}{50}$ inch for pins up to 5 inches in diameter, and $\frac{1}{32}$ inch for larger pins.[1] In this case, if the load is applied to the pin along the axis of the straight column, one of two things will happen when the column end tends to rotate: either there will be a tendency to slipping of the column on the pin, with friction resisting or perhaps preventing it, or there will be rolling of the column on the pin.

If there is slipping (Fig. 371) the point b of the axis will move to b', and there will be a tangential force F, practically horizontal, exerted by the column on the pin since

[1] Specifications of the A.R.E.A., 1920.

bb' is very small. Applying two equal and opposite forces at C, the center of the pin, the resultant of P and one of these is P'(or practically P), while a right-handed couple remains which must be balanced by another (left-handed) couple $M = Fr$ exerted upon the pin elsewhere, to keep it in equilibrium. The end of the column axis is now b'. The force exerted by the pin on the column is the resultant of P' and M, or a force P' acting to the left of P' and to the left of b', and this combined with a force equal and opposite to F. Hence the final resultant is a force parallel to P but acting to the left of b', and hence tending to oppose the leftward deflection of the column. The pressure on the pin is distributed over a surface to the right and left of b', which is at the axis of the column. The resultant pressure, however, does not pass through b', but to the left of it, because of the rotation.

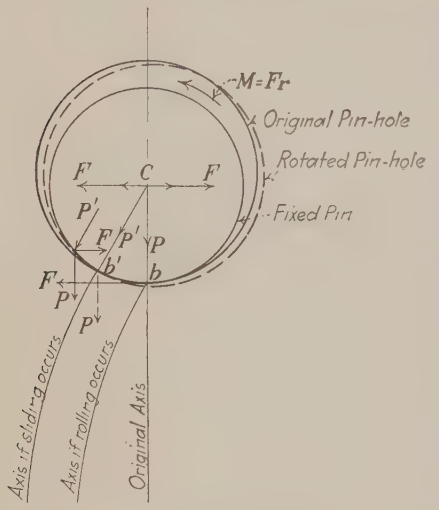


FIG. 371.

If the column *rolls* on the pin, then the axis goes through b, and b' is the central point of contact with the pin, through which P acts (as shown by the dotted force P), and this, being to the left of the end b of the axis, tends to oppose the deflection, as before.

However the pin acts, therefore, the tendency of the column to deflect is opposed.

If the pin does not fit the hole, as is always the case, then if the material were rigid there would be contact between pin and column only at the point of tangency; but the pressure is actually distributed, though over less than a semi-circumference of the pin. The effect of this is to make the frictional moment, if there is slipping or tendency to slip, greater than if the pressure were distributed over half the circumference. For if the pressure P were distributed over a small arc $2\theta_1$, or θ_1 to the left and θ_1 to the right of b, n would, as before, equal $n_1 \cos\theta$, and $P = 2\int_0^{\theta_1} trd\theta \cdot n \cos\theta = 2\,trn_1\int_0^{\theta_1} \cos^2\theta d\theta = trn_1\left(\theta_1 + \frac{\sin 2\theta_1}{2}\right)$. This becomes Eq. (1) when $\theta_1 = 90° = \frac{\pi}{2}$, but otherwise

$$n_1 = \frac{2P}{tr}\cdot\frac{1}{2\theta_1 + \sin 2\theta_1}$$

$$F = 2\int_0^{\theta_1} nk \cdot trd\theta = 2trkn_1\int_0^{\theta_1}\cos\theta d\theta = 4Pk\frac{\sin\theta_1}{2\theta_1 + \sin 2\theta_1}$$

which is always less than the value given by Eq. (1a). Hence the play of the pin in the hole decreases the resisting frictional moment.

4. Columns differ (1) in regard to the condition of the ends with reference to restraint of motion, (2) in the fact that the loads may be central or eccentric, and (3) in the fact that the section may or may not be constant.

Most columns are of practically the same section from one end to the other, except at the very ends, where, for a short distance, there may be a variation for purposes of connecting to other parts.

Eccentricity will be shortly considered. It has very important effects.

5. End Conditions.—For classifying, several ideal conditions may be named. These are called ideal, because they never exist exactly as imagined, but occur always in more or less modified form, as will be explained. These are

(*a*) Both ends pivoted (pin-ended).

(*b*) Both ends fixed.

(*c*) One end fixed and one pivoted.

(*d*) Both ends rounded.

A flat-ended column is a modification of (*b*). There are several modifications of (*c*), as will be seen.

6. The Ideal Pivoted or Pin-ended Column (Fig. 372*a*, *a'*).—The characteristics of this ideal column are:

(1) It is originally absolutely straight;

(2) It is originally absolutely homogeneous (E the same everywhere);

(3) It is of constant section;

(4) The loads are applied centrally (along the axis) at the ends;

(5) The ends are absolutely free to rotate;

(6) The ends are not allowed to move laterally;

(7) There are no transverse loads; *i.e.*, the column is vertical, and its own weight is neglected.

This column may remain straight under the axial load, in which case the stress will be uniform on each section and equal to $\frac{P}{A}$.

If it bends, it may bend as in (*a*) or (*a'*), or it may have more than one node between the ends. The bending moment at the ends, and at the nodes, is zero. At any point distant x from one end, the moment is Py.

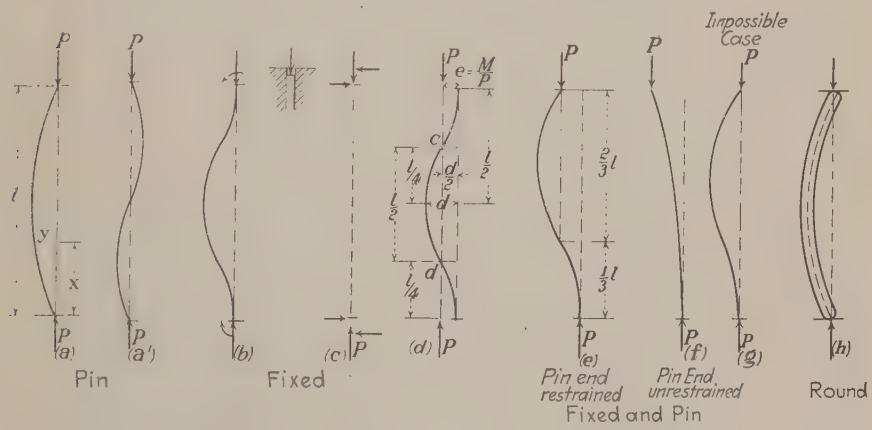


FIG. 372.

7. The Ideal Fixed-ended Column (Fig. 372*b*, *c*, *d*).—The characteristics of this column are:

(1) (2) (3) are the same as in the pivoted column;

(4) The ends are rigidly maintained in their original direction, and are not allowed to move laterally.

This requires that the loads shall be applied along the axis at the ends, and that, in addition, there shall be a moment at each end if the ends tend to rotate, as in *b*. This restraining moment may be produced if the ends were fitted into close-fitting holes, as indicated, which prevented rotation or lateral movement, but the restraint must be complete or, if the column tends to bend, and the restraint is exerted, the resultant force acting on the end is the resultant of P and M, or it is equal to the force P acting at a distance $\frac{M}{P}$ to one side of the axis at the end, as indicated in *d*.

The axis of this column at the ends is parallel to the axis at the middle of the length, since the two halves must be symmetrical. If it bends as indicated, the bending alone causes tension at the right at the ends, and at the left at the middle. The moment at the ends is Pe, and there is no moment at *c* or *d*. From symmetry it follows that the moment at the middle equals that at the ends, numerically, that the eccentricity e equals $\frac{d}{2}$, and that $cd = \frac{l}{2}$. Hence, if d is the deflection at the center, the end moment is

$$M = P\frac{d}{2} = Pe.$$

The center half of this column, or *cd*, is an ideal pin-ended column, hence *the ideal fixed-ended column has the same strength as the ideal pin-ended column having half its length*, with the same section and material. This result is necessary, and independent of the formula used. If the proper formula for either form of column is known, the formula for the other may be deduced from it.

As in the pin-ended colu mn, the fixed-ended column is supposed to have no lateral forces, except those that may be exerted at the ends; it is supposed vertical, and its weight is neglected. Columns are frequently tested in a horizontal position, but if this is done, the weight should be counterbalanced or supported at the center and perhaps at other points.

A *flat-ended column* is one in which the ends are flat, parallel and at right angles to the axis and abut against flat surfaces. If these surfaces prevent lateral movement and maintain the axis of the column at the ends in exactly its original line and direction, such a column is a true fixed-ended column. Frequently, however, the tendency of the ends to rotate causes them to open at the edge opposite to the direction of deflection so that they do not remain everywhere in contact with the bearing surfaces. To prevent this would require tension to be exerted over part of the ends, and there is no way of supplying this tension. This action will not occur unless the deflection d is so great that the load P acts outside the kernel of the section, that is, outside the middle third for a rectangular section;[1] but it may occur when a column is tested to destruction. Unfortunately, this point has not always been observed and recorded in tests on flat-ended columns, and it is impossible to tell whether they really were

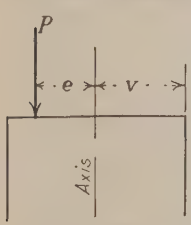


FIG. 373.

[1] Referring to Fig. 373 there will be no tension at the right edge if $\frac{P}{A} - \frac{Pev}{Ar^2} \geq 0$, where r is the radius of gyration, that is, if $e \geq \frac{r^2}{v}$ $\left(\geq \frac{b}{6}\right.$ for a rectangular section$\left.\right)$. Hence the greatest lateral deflection at the center of the length consistent with no opening at the edge would be $d = 2e = \frac{2r^2}{v}$.

fixed-ended (see remarks in Arts. 22 and 29). Flat-ended columns frequently occur in structures. In testing, flat ends are generally used, and are assumed to be fixed ends, or else pin ends are used.

Observe that an ideal fixed-ended column must have the direction of the axis at the ends not merely parallel, but in the same line. A column like that in Fig. 374, though the ends are fixed in direction, is not what we have called the ideal fixed-ended column, but is weaker than the latter. In this case the loads P will be equal and in opposite directions, but the moments at the ends will not be equal. In a riveted framed structure, when it deforms under loads, the ends of a strut are moved laterally with reference to each other, and moments are introduced at the ends which cause so-called secondary stresses that may be very different from those merely necessary to fix the direction of the ends of a fixed-ended strut, and which may greatly modify the stresses in the strut.

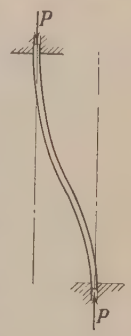


FIG. 374.

8. The Ideal Column with One End Fixed and the Other Pivoted.—Such a column may be the same as the lower three-quarters of a fixed-ended column, as in Fig. 372*e*, or it may be the same as the lower quarter, as in Fig. 372*f*, these being two radically different cases. Both involve a lateral movement of the pin end; but in the first case this movement is restrained and is very small, while in the second case the pin end is unrestrained and free to move laterally to any extent.

Obviously, in the first case, *the ideal column with one fixed end and one pin end (the lateral movement of the pin end being restrained) has the same strength as an ideal pin-ended column with the same section and two-thirds its length.*

In the second case, *the ideal column with one fixed end and one pin end, the pin end being unrestrained, has the same strength as the ideal pin-ended column with the same section and double its length.*

These results are necessary, and independent of the formula used. The constants for one column may be found from those that are correct for the other.

Of all these forms of columns, Fig. 372*f* is the weakest (for a given length and section), the pin-ended column next, then Fig. 372*e*, and the fixed-ended is the strongest.

The column fixed at the bottom and pin-ended at the top is frequently represented with the pin directly in line with the axis at the bottom, as in Fig. 372*g*, instead of laterally deflected as in Fig. 372*e* or 372*f*. This is clearly impossible, for if P is in line with the bottom and is the only force at the top there will be no moment at the bottom; and yet, if the column is bent, there must be a moment at the bottom. If there were a horizontal force at the top, causing this moment, the column would have to deflect.

9. The Column with Round Ends (Fig. 372*h*).—This has rounded ends, free to *roll* on the parallel end surfaces. If the radius of the rounded ends is r, this column is the same as the ideal pivoted column having a length $l - 2r$, and hence is slightly stronger than the ideal pivoted column of length l.

The round-ended column never exists in practice, but is sometimes used in testing, being the nearest approach that is possible to the ideal pin-ended or pivoted column. If the radius of the end is zero, it becomes a pivoted column.

THEORY AND FORMULAE FOR THE IDEAL COLUMN

10. Gordon's Formula.—Lewis Gordon, an English engineer, proposed the formula

$$\frac{P}{A} = \frac{f}{1 + a\frac{l^2}{d^2}}$$

where P is the load, A the area, l the length, and d the least diameter of the section. He gave the values for *fixed-ended columns:*

	f	a
Wrought iron, solid rectangular section	36,000	$\frac{1}{3{,}000}$
Cast iron, solid cylinder	80,000	$\frac{1}{400}$
Cast iron, hollow cylinder	80,000	$\frac{1}{800}$
Stone and brick	.....	$\frac{1}{600}$

This formula was modified by Rankine, as will be shown in the next article, where the theoretical basis of the formula will be explained. Gordon's formula is seldom used now except for wooden columns.

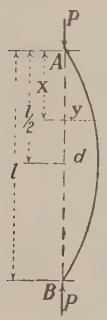


Fig. 375.

11. Rankine's Formula.—At any section there is a compression P, and hence an average intensity of $\frac{P}{A}$ over the entire area.

At any section there is a moment Py (Fig. 375), whose max. value occurs at the center of the length and is Pd; to which corresponds a max. stress intensity, if v is the distance from the center of gravity to the extreme fiber, in the plane of the paper, of $\frac{Pdv}{I}$. Hence the max. stress intensity, which equals the allowed value, is

$$f = \frac{P}{A}\left(1 + \frac{dv}{r^2}\right). \tag{3}$$

Here r is the radius of gyration of the section about an axis of the section perpendicular to the paper and through the center of gravity.

This formula is absolutely correct if the equation for beams $\left(M = \frac{fI}{v}\right)$ is correct, which it is rather late to dispute, though it holds, of course, only within the elastic limit.

In order to use this equation, d must be found in terms of given quantities. It cannot be used in the form given above. The quantity $\frac{dv}{r^2}$ is a number, with no dimensions. It is not a distance, nor a force, but a number, otherwise it could not be added to unity. If dv is to be replaced by some function of l, the simplest function would be cl^2, which would have the same dimensions as dv, if c is a number. This would mean the assumption that

$$dv = cl^2 \tag{4}$$

that is, for a column of given section, d varies as l^2. This seems reasonable. Equation (4) is of the same dimension on both sides, but this would also be true if $dv^2 = cl^3$, but the latter would be more complicated, and probably no more correct.

Using, then, the assumption involved in Eq. (4), Eq. (3) becomes

$$\left.\begin{aligned} f &= \frac{P'}{A}\left(1 + c\frac{l^2}{r^2}\right) \\ \text{or} \quad \frac{P}{A} &= \frac{f}{1 + c\frac{l^2}{r^2}} \end{aligned}\right\} \tag{5}$$

which is Rankine's[1] formula.

The value of c must be determined by experiment. It will obviously depend upon the material and upon the shape of the section (whether round, square, I-shaped, etc.) and its general proportions.

If the column is free to bend in any direction, it will bend in the direction which requires the smallest P, *i.e.*, about the axis for which r is least. If it can only bend in a given direction, r is to be taken about the axis through the center of gravity perpendicular to this direction.

Furthermore, while for a given P the relation $dv = cl^2$ may be reasonable, it is obvious that d, and therefore c, must increase as P increases. Logically, then, c should be a variable number increasing as P increases; but it is always assumed as a constant for a given kind of column.

Comparing Rankine's formula with Gordon's, the former replaces d^2 by r^2, and since for a rectangular section $d^2 = 12r^2$. Rankine proposed, for *fixed-ended columns of wrought iron*

$$\frac{P}{A} = \frac{36{,}000}{1 + \frac{l^2}{36{,}000r^2}} \text{ the ultimate strength.}$$

According to the principles of Arts. 6, 7, 8, therefore for pivoted (pin-ended) columns;

$$\frac{P}{A} = \frac{36{,}000}{1 + \frac{l^2}{9{,}000r^2}}$$

for one end pin and the other fixed;

$$\frac{P}{A} = \frac{36{,}000}{1 + \frac{l^2}{20{,}250r^2}}$$ [2] with pin end restrained

with pin end unrestrained;

$$\frac{P}{A} = \frac{36{,}000}{1 + \frac{l^2}{2{,}250r^2}}.$$

The fact that bending does not occur, with a straight and homogeneous column, until the slenderness ratio exceeds a certain limit, has led some engineers to propose a formula in which no reduction is made until the ratio exceeds that limit. Thus Cooper[3] proposed the formulae

For square ends;

$$\frac{P}{A} = \frac{f}{1 \pm \frac{\left(\frac{l}{r} - 80\right)^2}{18{,}000}}$$

[1] Wm. J. Macquorn Rankine, a Scotch engineer of great eminence (*b.* 1820; *d.* 1872) long professor of engineering at the University of Glasgow.

[2] 18,000 is often used instead of 20,250.

[3] *Trans. Am. Soc. C. E.*, Vol. XI, 1882.

For pin ends;

$$\frac{P}{A} = \frac{f}{1 \pm \frac{\left(\frac{l}{r} - 33\right)^2}{18{,}000}}$$

The + sign is to be used when $\frac{l}{r}$ exceeds 80 or 33

The − sign is to be used when $\frac{l}{r}$ is less than 80 or 33

These formulae have not been much used.

12. Euler's Formula.—With the shape of the elastic line as indicated in Fig. 375, it would naturally occur to the mind to solve the problem by using the equation of the elastic line, which is based on the equation

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = -\frac{Py}{EI} = -m^2y. \tag{6}$$

The sign is minus because x is positive downward and y to the right, and M is the left-handed moment of the outer forces acting on the part above the section. The slope $\frac{dy}{dx}$ clearly *decreases numerically* from the top downward to the center.

The general integral of this equation is

$$y = A \cos mx + B \sin mx \tag{7}$$

in which A and B are constants, determined by the conditions that

When $x = 0$, or $x = l$; $y = 0$

When $x = \frac{l}{2}$; $y = d$, and $\frac{dy}{dx} = 0$.

For $x = 0$, Eq. (7) reduces to $0 = A$; hence the equation of the elastic line is

$$y = B \sin mx. \tag{7a}$$

For $x = l$, $y = 0$; hence

$$B \sin ml = 0.$$

The conditions would therefore obviously be fulfilled if $A = B = 0$; but this would of course mean that $y = 0$, or that the column did not bend at all, which, as above explained, is conceivable but not practically possible. The only other alternative is

$$\sin (ml) = 0; \text{ or } ml = n\pi$$

n being any whole number. Hence

$$m = \frac{n\pi}{l}; \; m^2 = \frac{P}{EI} = \frac{n^2\pi^2}{l^2}$$

$$P = \frac{n^2\pi^2 EI}{l^2}. \tag{8}$$

The smallest value of P would be for $n = 1$, and therefore we must have

$$P = \frac{\pi^2 EI}{l^2} = \frac{9.87EI}{l^2} = \frac{9.87EAr^2}{l^2}$$

or

$$\frac{P}{A} = \frac{\pi^2 E}{\frac{l^2}{r^2}} = \frac{9.87E}{\frac{l^2}{r^2}} \tag{9}$$

where r is the radius of gyration about the axis perpendicular to which bending occurs.

Differentiating Eq. (7*a*)

$$\frac{dy}{dx} = Bm \cos (mx). \tag{10}$$

For $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; hence either $B = 0$, which means no bending, or $\cos \frac{ml}{2} = 0$, which again leads to Eq. (8).

For $x = \frac{l}{2}$, $y = d$; hence from Eq. (7*a*)

$$d = B \sin \frac{ml}{2}$$

$$B = \frac{d}{\sin \frac{ml}{2}}$$

and the equation of the elastic line is

$$y = d \frac{\sin mx}{\sin \frac{ml}{2}}.$$

Since $\cos \frac{ml}{2} = 0$; it follows that $\sin \frac{ml}{2} = 1$, hence with $n = 1$, which is represented in Fig. 375, the equation of the elastic line is

$$y = d \sin mx \tag{7b}$$

or since, as shown above, $ml = n\pi$, and n must be taken as 1, $m = \frac{\pi}{l}$, hence

$$y = d \sin \frac{\pi x}{l}. \tag{7c}$$

Equations (8) and (9) are Euler's[1] formula for columns, and are correct if Eq. (6) is correct as applied to this case

If $n = 2$,

$$P = \frac{4\pi^2 EI}{l^2}$$

$$y = d \sin \frac{2\pi x}{l} \tag{11}$$

and y will be zero for $x = 0$, $\frac{l}{2}$, or l; hence this case corresponds to Fig. 376. If $n = 3$, y will be zero for $x = 0$, $\frac{l}{3}$, $\frac{2l}{3}$, or l, and there will be two nodes between the ends. Equation (9) will always apply if l is the distance between points of inflexion.

FIG. 376.

In Eq. (9), if $\frac{P}{A} = y$ and $\frac{l}{r} = x$, the equation of the line representing Euler's formula, Fig. 377 is

$$y = \frac{\pi^2 E}{x^2}.$$

Differentiating, $$\frac{dy}{dx} = -\frac{2\pi^2 E}{x^3} = -\frac{2y}{x}.$$

[1] LEONHARD EULER (*oi'-ler*): b. 1707 in Basel, Switzerland, d. 1783 in St. Petersburg: pupil of J. Bernoulli.

If the tangent is drawn at a point, x_1,y_1, its intercept on the Y axis will be

$$y_0 = y_1 - x_1 \frac{dy}{dx} = 3y_1$$

as used in Art. 13.

13. The Straight Line Formula.—If there is bending in a column, the average stress intensity $\frac{P}{A}$ must be less than the maximum stress intensity f. By Rankine's formula the average equals f divided by something greater than unity.

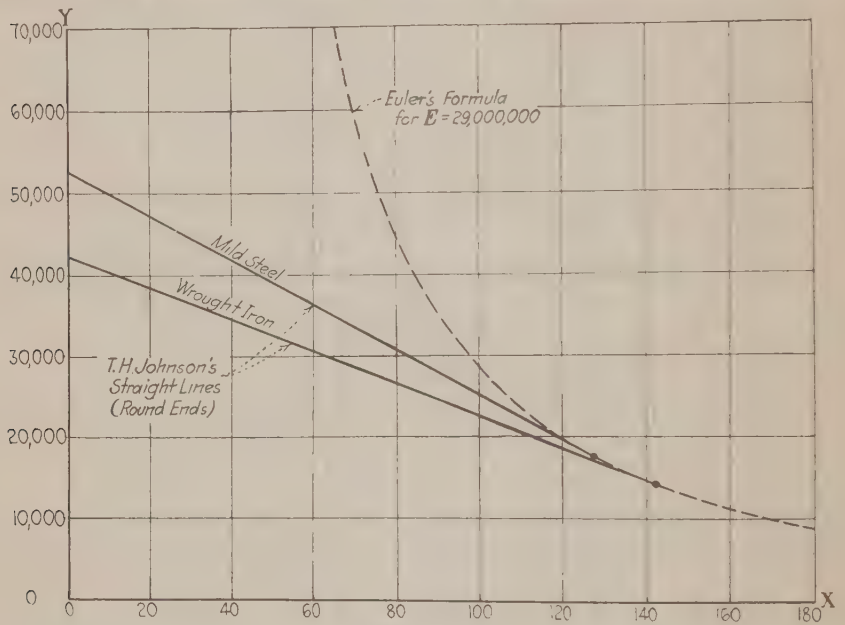


FIG. 377.

Instead of this, the average may be placed equal to f minus some positive quantity which increases with $\frac{l}{r}$. The simplest expression would be

$$\frac{P}{A} = f - k\frac{l}{r} \tag{12}$$

$$f = \frac{P}{A} + k\frac{l}{r} \tag{13}$$

in which k is some constant to be determined from experiment.[1]

This is called the *straight line formula*, because the equation of $\frac{P}{A}$ in terms of $\frac{l}{r}$ represents a straight line. In Rankine's formula, if values of $\frac{P}{A}$ are plotted as ordinates for different values of $\frac{l}{r}$ as abscissas, the result is a curved line—concave upward except for small values of $\frac{l}{r}$.

[1] Professor Merriman suggested the straight line formula in 1882, using the least diameter d instead of r. See *Trans. Am. Soc. C. E.*, 1882, p. 115.

The straight line formula combined with Euler's curve was proposed by Thos. H. Johnson,[1] who thought that the straight line should run from a point on the Y axis, representing the value of the allowable stress intensity for $\frac{l}{r} = 0$, tangent to the curve representing Euler's formula. In other words, he proposed that the value of $\frac{P}{A}$ for various values of $\frac{l}{r}$ should be represented by a straight line and Euler's curve. In Fig. 377, the curved line represents the value of $\frac{P}{A}$ given by Euler's formula, Eq. (9). As shown in Art. 12, a tangent to that curve at any point (x_1,y_1) intersects the Y axis at a point for which $y = 3y_1$; hence Johnson's straight line may be drawn easily by making $y_1 = \frac{y}{3}$ and finding the point of tangency. Johnson thought that the ordinate at the origin, or the value of f in his formula for columns, should be about $\frac{3}{4}$ the compressive strength as obtained from tests of short prisms, that is, it would be above the elastic limit. Johnson's proposed formulae contained the following values:

T. H. JOHNSON'S STRAIGHT LINE FORMULAE

Material	Wrought iron	Mild steel, carbon 0.12	Hard steel, carbon 0.36	Cast iron
Value of f	42,000	52,500	80,000	80,000
Value of k in Eq. (12), for				
flat ends	128	179	337	438
Hinged ends	157	220	414	537
Round ends	203	284	534	693

The constants given by Mr. Johnson for the different forms of columns do not at all fulfill the necessary relations which must exist, if the columns are the ideal ones (see Art. 21). Hence Johnson's formulae must be considered to be those which he recommends for practical use, and not those which are correct for ideal columns. This is also shown by the forms of Euler's formula which Johnson uses with his straight line formulae. For round-ended columns, which are nearly the ideal pin-ended column, he uses the equation $f = \frac{\pi^2 E}{\frac{l^2}{r^2}}$. For hinged ends he uses $f = \frac{5\frac{1}{3}\pi^2 E}{\frac{l^2}{r^2}}$, or quite different from the first.

For flat-ended columns, for which, if ideal, the formula would be $f = \frac{4\pi^2 E}{\frac{l^2}{r^2}}$, he uses $f = \frac{5\frac{1}{2}\pi^2 E}{\frac{l^2}{r^2}}$. (See Art. 20.)

[1] *Trans. Am. Soc. C. E.*, vol. 15, p. 517, 1886.

The value of k will depend upon the value of E in Euler's formula. Figure 377 does not agree exactly with values of k in the above table, which are based on a smaller value of E.

14. The Parabolic Formula.—The quantity subtracted from f in Eq. (12) may be such that, instead of varying with $\frac{l}{r}$, it varies with $\frac{l^2}{r^2}$, or any other proper function of $\frac{l}{r}$. By some it is thought that the so-called parabolic formula[1] is more correct, in which

$$\frac{P}{A} = f - k_1 \frac{l^2}{r^2}. \tag{14}$$

This formula may be made to represent a curve tangent to Euler's curve. Equation (14) may be written

$$y = f - k_1 x^2 \tag{14}$$

whence

$$\frac{dy}{dx} = -2k_1 x.$$

The slope of the tangent to Euler's curve at a point x, y, is (Art. 12)

$$\frac{dy}{dx} = -\frac{2y}{x} = -\frac{2\pi^2 E}{x^3}.$$

Placing these values equal, since the two curves are to be tangent to each other

$$2k_1 x = \frac{2y}{x}; \; x = \sqrt{\frac{y}{k_1}}.$$

Since the values of y must be the same for the two curves, at the point of tangency,

from (14); $$y = f - k_1 x^2 = f - y; \; \therefore y = \frac{f}{2}; \; x = \sqrt{\frac{f}{2k_1}}$$

from (9); $$y = \frac{\pi^2 E}{x^2} = \frac{b}{x^2} = \frac{2bk_1}{f} = \frac{f}{2}$$

$$\therefore f = \sqrt{4bk_1}; \; b = \frac{f^2}{4k_1}; \; k_1 = \frac{f^2}{4b}.$$

Hence the equations of the two curves are,

the parabola; $$y = f - \frac{f^2}{4b} \cdot \frac{l^2}{r^2} = f - \frac{f^2}{4\pi^2 E} \cdot \frac{l^2}{r^2} \tag{14a}$$

$$\text{from } \frac{l}{r} = 0 \text{ to } \frac{l}{r} = \sqrt{\frac{f}{2k_1}} = \sqrt{\frac{2b}{f}} = \sqrt{\frac{2\pi^2 E}{f}};$$

the Euler curve; $$y = \frac{\pi^2 E}{\frac{l^2}{r^2}}$$

$$\text{for } \frac{l}{r} > \sqrt{\frac{2\pi^2 E}{f}}.$$

[1] Proposed by the late Professor J. B. Johnson.

15. Euler's Formulae by Moment-area Method.—Euler's formula may be obtained by the moment-area method. Thus, if Fig. 378 represents the bent column, DC is the deflection at the center, d, and the moment there is Pd, hence this curve, to the proper scale, or with all ordinates multiplied by P, represents also the moment curve. The area $CDEF$ (multiplied by P), represents the slope at F with reference to AB; the moment of this area (multiplied by P) about EF represents the deflection FG; the area ADC represents the slope α at the end; the moment of ADC about A represents d. The equation of the elastic line is

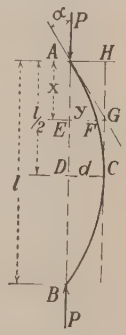


FIG. 378.

$$y = d \cdot \sin \frac{\pi x}{l}.$$

$$\text{The area } ADC = \frac{dl}{\pi}.$$

$$\text{The moment of } ADC \text{ about } A = \frac{dl^2}{\pi^2}.$$

Hence;

$$\alpha \cdot EI = \frac{Pdl}{\pi}; \alpha = \frac{Pdl}{\pi EI}$$

$$d = \frac{Pdl^2}{\pi^2 EI}; P = \frac{\pi^2 EI}{l^2} = \frac{9.87EI}{l^2}.$$

If the curve ACB is assumed to be a parabola

$$\text{Area } ADC = \frac{dl}{3}.$$

$$\text{Moment of } ADC \text{ about } A = \tfrac{5}{48} dl^2.$$

Hence

$$\alpha \cdot EI = \frac{Pdl}{3}; \alpha = \frac{Pdl}{3EI}$$

$$d = \frac{5Pdl^2}{48EI}; P = \frac{9.6EI}{l^2} \qquad (9a)$$

which agrees very closely with Euler's formula, and shows that the curve is very nearly a parabola.

16. Interpretation of Euler's Formula.—According to Euler's formula the load P does not depend upon the deflection d. P is the load which will just hold the column in equilibrium in the deflected shape shown in Fig. 375; but the same P would also hold it in equilibrium at any other deflection, within the elastic limit. It is therefore important to examine the assumptions on which the formula is based, in order to judge of their applicability. Those assumptions are,

(1) Stresses are within the elastic limit;

(2) There is planar distribution of stress;

(3) There is no effect of shear or shortening of the column;

(4) $\frac{dy}{dx} = 0$, or $ds = dx$, *i.e.*, the deflection is zero.

Of these, the fourth is the only one that may affect materially the validity of the formula (within the elastic limit). This assumption is made in all formulae for deflection of beams, and is universally and properly considered admissible in such cases; not only because the deflections are very small, but mainly because, in beams, the deflection is due to transverse loads. In both beams and columns, the deflection is due entirely to the bending moments; but in beams the bending moments are practically entirely independent of the deflection, and would be the same whether there were any deflection or not; whereas in pin-ended columns the

moments are due entirely to the deflection and to nothing else. If there were no deflection there would be no bending moment; and the deflection of a column (if originally straight) may therefore, paradoxically but accurately, be said to be caused by the deflection, so that it cannot be correct to assume the deflections non-existent and thereby obtain a formula which will hold when they do exist.

It is possible to use the exact value of the radius of curvature in the formula for deflection, not assuming $\frac{dy}{dx} = 0$. This, however, leads to elliptic functions. Schneider[1] finds that a closer value than that in Eq. (8) but still not exact, is the following, for $n = 1$,

$$P = \frac{\pi^2 EI}{l^2}\left(1 + \frac{\pi d^2}{8l^2}\right). \tag{15}$$

This difficulty is obviated if we conclude that Euler's formula gives the maximum value of P which will just hold the column *without bending at all* about a given axis (for which I is taken). If I is taken about the axis which gives I a minimum, the smallest of these values of P will be obtained. Any greater load than this must be assumed to bend the column suddenly and considerably, or at least to the elastic limit, and probably to fracture. *If Euler's formula is accepted, therefore, it must be considered to give the ultimate strength.*

It is a fact that when a column is tested in a testing machine it frequently happens that when it begins to bend it does so suddenly and to a considerable amount, but not to fracture. Often, however, it does not show this phenomenon, but bends gradually as the load is increased. This, however, may be due to the fact that it is not exactly straight or homogeneous at the start. Assuming it absolutely straight and homogeneous, according to the Euler theory it must be assumed to act as follows:

(1) So long as P is less than the value given by Eq. (9) it will not bend at all. If a bending moment is produced by a lateral load, the resulting deflection will be greater than would be produced by that lateral load alone without P; but when the lateral load is removed, the column will become straight again. The larger P is, the smaller the lateral load required to produce a given deflection.

(2) If P is exactly equal to the value given by Eq. (9), the column will still not bend; but it will be in a sort of state of unstable or, at least, indifferent equilibrium. The smallest possible lateral force will deflect it, and it will remain in equilibrium at any deflection so produced; in other words, it is ready to collapse if the lateral deflection is such that the stress anywhere exceeds the elastic limit, and such a deflection could be produced by a very small force.

(3) If P exceeds by even the smallest possible amount the value given by Eq. (9), while it is still possible to conceive of it as remaining straight, even the minutest lateral deflection will cause immediate collapse.

Equation (9) therefore gives the breaking load. Since this formula assumes Hooke's law to be true, it should not be used if it gives values above the elastic limit; if it does, the elastic limit is the breaking strength.

It should be observed that by Euler's formula the breaking load varies as E, but is independent of the ultimate crushing strength of the material. Hence a long ideal pin-ended column of the strongest steel will, by that formula, carry no more than one of wrought iron or soft steel, because E is practically the same

[1] *Zeitschr. d. Oester Ing. u. Arch. Vereins*, 1901, p. 649.

in both. No matter how great the crushing strength of the material is, if the load is sufficient to overcome the resistance to deflection, or what has been called the resilient strength of the column, the increasing deflection due to the load will at last cause the max. fiber stress to exceed the crushing strength. The higher this crushing strength, the greater the *deflection* that would have to occur before that crushing strength is reached; but the load given by Eq. (9) is, as we have seen, sufficient to cause *any* deflection.

The Euler theory thus assumes that the ideal column will not bend, as the load is increased, until the value given by Eq. (9) is reached, when it will suddenly bend and break. It is claimed that experiments prove this. On the other hand, Hodgkinson found that flexure began with very small loads, and increased with the loads, and such is the general observation. This may be due, however, to the fact that the columns tested were not exactly straight or homogeneous, as will be shown in Art. 23.

When $\frac{l}{r} = 0$, Euler's formula does not give $\frac{P}{A} = f$, but $\frac{P}{A} = \infty$. This means that no force, however great, could make a column of an infinitesimal length bend, which is obviously true.

For small values of $\frac{l}{r}$, Euler's formula gives values of $\frac{P}{A}$ far above what is generally considered the ultimate, $\left(\text{if } \frac{l}{r} = 50, \frac{P}{A} = 118{,}000 \text{ for steel}\right)$ and as the formula is only correct, if at all, below the elastic limit (because above it E is a variable), it should not be used for values of $\frac{l}{r}$ which will give $\frac{P}{A}$ equal to, or above, the elastic limit. This means that it should not be used at all for wooden columns.

If f' is the average stress intensity, Euler's formula may be written

$$P = f'A = \frac{\pi^2 EI}{l^2},$$

whence

$$\frac{f'l}{E} = \text{total compressive deformation} = \frac{\pi^2 r^2}{l}$$

which says that the elastic shortening of the column just before it begins to bend is independent of the material, and dependent only upon the dimensions, which seems singular and doubtful.

Notwithstanding these difficulties with Euler's formula, the late Professor Tetmajer, of Vienna, stated that "its applicability is incontrovertibly proved for central loading inside the elastic limit," and many continental engineers believe in it. On the other hand, many continental engineers reject it. Professor Vierendeel, of the University of Louvain, in his recent work on structures, points out the logical errors in it, in assuming a column bent by a force that would not bend it, and says that the formula "has always appeared lame to engineers, particularly to those who, making designs, must take the responsibility for their structures," and that for a long time it has not had the approval of engineers. He adds that those who believe in it seek to justify it by the presence of defects, which will cause bending, but remarks that they only commit another logical error in supposing that a cause not governed by law (the presence of defects) can be

governed by the law of Euler's formula. Professor Vierendeel also criticizes Rankine's formula, but in the end proposes one which is in form Rankine's. Euler's formula is generally accepted in this country for large values of $\frac{l}{r}$, but not for values generally occurring in practice.

17. Euler's formula may be used in another way, leading to a final expression similar to Rankine's. It is incontestable that the max. stress intensity on a section of a bent column is the sum of that due to direct stress and that due to bending, or

$$f = f_c + f_b = \frac{P}{A} + f_b.$$

The question is to find f_b. Equation (9) is deduced solely with reference to flexure, and has nothing to do with stress at all, but simply shows the greatest load that the column will carry without bending at all, or the load that will hold it in equilibrium if bent by a lateral force which is then removed. Placing $E = \frac{f_b}{\lambda}$, where λ is the compressive deformation per unit length of the outside fiber *due to the bending only*, at the point where the stress intensity due to bending is f_b, and the total stress is f, Eq. (9) becomes

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 f_b I}{\lambda l^2}$$

$$f_b = \frac{P\lambda l^2}{\pi^2 I}$$

whence $$f = \frac{P}{A} + \frac{P}{A}\cdot\frac{\lambda l^2}{\pi^2 r^2} = \frac{P}{A}\left(1 + \frac{\lambda}{\pi^2}\frac{l^2}{r^2}\right) = \frac{P}{A}\left(1 + c_1\frac{l^2}{r^2}\right)$$

where $$c_1 = \frac{\lambda}{\pi^2} = \frac{f_b}{\pi^2 E}$$

or
$$\frac{P}{A} = \frac{f}{1 + \frac{f_b}{\pi^2 E}\cdot\frac{l^2}{r^2}} \tag{16}$$

which has the same form as Rankine's formula, but in which the coefficient of $\frac{l^2}{r^2}$ is a variable, increasing with P, since f_b must increase with P. This agrees with what we have seen (Art. 11), and Eq. (16) would be a rational form of Euler's formula for bent columns, except that there is no way of finding f_b. This formula shows that the greater E is, the greater $\frac{P}{A}$.

18. In *Engineering News*, for July 14, 1894, Professor Merriman deduced a formula similar to Eq. (16). If values of $\frac{P}{A}$ be plotted as ordinates and values of $\frac{l}{r}$ as abscissas, then Rankine's formula has the form $y = \frac{f}{1 + ax^2}$, and Euler's $y = \frac{n\pi^2 E}{x^2}$. If these two curves are made tangent to each other at some common point (x, y), $\frac{dy}{dx}$ must be the same for each curve at that point.

For the Rankine curve; $$\frac{dy}{dx} = -\frac{2fax}{(1 + ax^2)^2}.$$

For the Euler curve; $$\frac{dy}{dx} = -\frac{2\pi^2 En}{x^3}.$$

Equating these and the values of y and solving for x and a, it is found that $x = \infty$, and $a = \frac{f}{n\pi^2 E}$; whence the Rankine curve can only be tangent to the Euler curve at infinity, and, if so tangent, its equation is

$$\frac{P}{A} = \frac{f}{1 + \frac{f}{n\pi^2 E} \cdot \frac{l^2}{r^2}}. \tag{16a}$$

For the ideal pivoted column, $n = 1$;
For the ideal fixed-ended column, $n = 4$;
For the ideal column with one end pivoted and one fixed, $n = 4\frac{1}{9}$;

19. Comparison of Formulae.—The main formulae for columns are:

Rankine's; $$\frac{P}{A} = \frac{f}{1 + c\frac{l^2}{r^2}} \tag{5}$$

Straight line; $$\frac{P}{A} = f - k\frac{l}{r} \tag{12}$$

Parabolic; $$\frac{P}{A} = f - k_1\frac{l^2}{r^2} \tag{14}$$

Euler's; $$\frac{P}{A} = \frac{9.87E}{\frac{l^2}{r^2}}. \tag{9}$$

Rankine's formula, like all based on the theory of flexure, assumes Hooke's law, and planar distribution. Aside from these, it involves only the assumption that $dv = cl^2$. We have seen that strictly speaking c must increase with P. It is difficult to introduce this condition, but is it well to remember that c must be a variable. In the usual form (Eq. (5)) with c a constant, or d independent of P, it involves the same error that was criticized in Euler's formula. This formula is only applicable within the elastic limit.

Euler's formula, as already shown, applies only within the elastic limit, and gives the ultimate load under which the column will remain straight. It does not apply to a bent column, says nothing about maximum stress, and so cannot be made use of in any investigation of cases of combined compression and bending. Rankine's formula, on the other hand, gives the max. stress intensity f as made up of two terms, one due to direct compression, the other to bending moment. When $\frac{l}{r} = 0$, $\frac{P}{A} = f$, as it should. It can be used in any theoretical investigations.

The straight line formula, like Rankine's, gives the maximum stress intensity in two terms, but the term representing bending, $k\frac{l}{r}$, is constant and does not depend upon P, which is obviously incorrect. No doubt a column may be safely designed by this formula, but on account of this defect the formula is not suited for applications involving a use of the stress due to bending. It may be claimed that the term $k\frac{l}{r}$ does vary with P, because, for any given length, the larger P is, the larger A and r; but this leads to the result that as P is *increased* $k\frac{l}{r}$ is *decreased*. This is an indirect relation, which exists in Rankine's formula as well, and merely expresses the fact that a column carrying a heavy load, and designed for it, is

less likely to bend than one designed for a light load. It would seem that there should also be some direct relation by which, as P increases, f increases also, as it does in Rankine's formula.

The same remarks apply to the parabolic formula.

Other difficulties will appear when we come to consider the values of the constants f, c, and k, which will be done after other forms of columns have been discussed. It is plain, however, that none of the four formulae are entirely satisfactory. All have been, and are, widely used; Euler's in continental Europe and somewhat in England, Rankine's in England and America, while of late years the preference in America has been for the straight line formula, which has been adopted by the American Railway Engineering Association. The parabolic formula has apparently been little used though it was tentatively recommended by the Committee of the A.R.E.A.

Probably the subject of columns is as uncertain as any in the strength of materials; and the writer believes that the weak points in structures are usually the columns, aside from weak points in details.

20. The Writer Prefers Rankine's Formula, for the Following Reasons. —

(1) It expresses separately the stress due to direct compression and that due to bending, both increasing with P.

(2) For $\frac{l}{r} = 0$, it gives $P = fA$. (The straight line and parabolic formulae do the same, but Euler's does not.)

(3) It is most convenient for use in applying the formula to practical, non-ideal cases, such as the flanges and webs of plate girders, and to combinations of direct compression and bending.

(4) It seems to be the most nearly rational formula, for reasons already discussed. Euler's formula says that the load is independent of the deflection. The straight line and parabolic formulae say that the term representing f_b is independent of P except through the indirect relation that the *larger* P is, the larger r will be, and hence the *smaller* $k\frac{l}{r}$ which represents f_b, which is clearly wrong. The only fault with Rankine's formula appears to be that c should increase with P.

In working with a committee to prepare specifications for structural steel for buildings, for the American Institute of Steel Construction, in 1923, the writer formulated as follows the reasons for adopting the Rankine formula:

"Any formula for columns must be obtained by finding the maximum stress and placing that equal to the allowable. A column bends, and the maximum stress intensity will be that due to direct compression plus that due to bending. If y is the maximum deflection from the line of action of the axial compression, and v is the distance of the extreme compression fiber from the axis of the section through its center of gravity, then the formula for columns must be

$$f = \text{allowable stress intensity} = \frac{P}{A} + \frac{Pyv}{I} = \frac{P}{A} + \frac{Pyv}{Ar^2}$$

or

$$\frac{P}{A} = \frac{f}{1 + \frac{yv}{r^2}}.$$

This formula must be correct, within the elastic limit, if the theory of flexure is sound. It cannot be used, because y is unknown.

"Rankine's formula assumes, if c is a constant,

$$yv = cl^2.$$

This expression is homogeneous, each side being the product of two lengths. It is reasonable, for as v increases, other things equal, y must decrease; and as l increases, y must increase in greater ratio. If the length of the column, and all its dimensions, were increased all in the same ratio, y would increase in the same ratio; this would be merely like looking at the deformed column through a magnifying glass. If the length is increased while the cross-section and v remain the same, clearly y would increase in greater ratio. The exponent of l may *not* be exactly 2, but with all the uncertainties involved it is folly to split hairs here.

"From this assumption, Rankine's formula follows:

$$\frac{P}{A} = \frac{f}{1 + c\frac{l^2}{r^2}}.$$

It contains one inherent inaccuracy, namely, that y should increase as P increases, that is, that c should really be a variable, increasing with P. It is not practicable to make c variable for practical designing, and furthermore, the law of its variation is unknown. It must, therefore, be assumed a constant for practical purposes.

The straight line formula is

$$\frac{P}{A} = f - c\frac{l}{r}$$

or

$$f = \frac{P}{A} + c\frac{l}{r}.$$

The term $c\frac{l}{r}$ must be the stress intensity due to bending, or

$$c\frac{l}{r} = \frac{Pyv}{Ar^2}.$$

This expression is not homogeneous: The first member is a constant times a ratio, or a constant; the second member is pounds per square inch. Hence the expression cannot be correct. Further, it gives

$$yv = \frac{clAr}{P} \text{ or } y = \frac{clAr}{Pv}.$$

According to this, y increases with A, and with r, and decreases with P, all of which results are absurd.

"The straight line formula is, therefore, wholly empirical. The same is true of the parabolic formula.

"Rankine's formula, while not accurate, is the nearest approach to a rational formula. Eccentricity of load may be taken account of by adding another term.

"It is true that a constant, if entirely empirical, may have a dimension, but a constant in a formula correctly deduced from scientific principles is a mere number.

"The straight line formula or the parabolic formula may very likely fit the results of tests as well as Rankine's, but many tests are inaccurate and the results conflicting. We must have a formula which is *in the proper form.*

"The straight line or parabolic formulae, being irrational in form, cannot be used in any theoretical investigations involving column action, as for instance, in studying beam flanges or plate girder webs, which certainly are columns of peculiar kinds. The Rankine formula can be so used.

"All these facts justify the use of the Rankine formula. Moreover, it is as easily applied as any other.

"Euler's formula may sometimes be used to advantage, as a check, for large slenderness ratios; but such ratios should not occur in practice."

21. Relations between the Constants.—For the ideal columns of the different types and length l, the relations between the constants are as shown in the following table:

Type of column	f is the same in all, or the max. allowable stress. Values of coefficient of $\frac{l}{r}$ or $\frac{l^2}{r^2}$ are,			
	Rankine	Straight line	Parabolic	Coefficient of $\frac{\pi^2 E}{\frac{l^2}{r^2}}$ in Euler's
Ideal pin-ended	c	k	k_1	1
Ideal fixed-ended	$\frac{c}{4}$	$\frac{k}{2}$	$\frac{k_1}{4}$	4
Ideal one fixed, one pin; pin end restrained	$\frac{4}{9}c$	$\frac{2}{3}k$	$\frac{4}{9}k_1$	2.25
Ditto with pin end unrestrained	$4c$	$2k$	$4k_1$	0.25

22. The Ideal Column Never Exists in Practice.—If the load is applied to a steel column through pins, the friction on the pins restrains in some degree the rotation of the ends, and makes the column something between the ideal pin-ended and the ideal fixed-ended column. The frictional moment increases as the lubrication decreases and as the diameter of the pin increases. A flat-ended column is a pin-ended column with a pin of infinite radius. This frictional moment has been shown in Art. 3 to be at least $M = 1.27Pkr$ if the bar slides and does not roll on the pin. If k, the coefficient of friction, is 0.25, which might be a proper value for an unlubricated surface, $M = 0.32Pr$.

It is interesting to ascertain what radius of pin would suffice to fix the ends of an actual (not ideal) pin-ended column. This may be done as follows:

According to Rankine's formula, the deflection of a pin-ended column of length l is $d = \frac{cl^2}{v}$, and from Fig. 372d the deflection of a fixed-ended column of length l is twice that of a pin-ended column of half its length, or it is $2 \cdot \frac{cl^2}{4v} = \frac{cl^2}{2v}$. The moment at the end of a fixed-ended column is $\frac{Pd}{2} = \frac{Pcl^2}{4v}$. Placing this equal to the frictional moment with $k = 0.25$

$$0.32Pr = \frac{Pcl^2}{4v}$$

or, using $\frac{1}{3}$ instead of 0.32,

$$r = \frac{3}{4} \cdot \frac{cl^2}{v}. \qquad (17)$$

Using a common value for ideal pin-ended columns, $c = \frac{1}{9{,}000}$,

$$r = \frac{l^2}{12{,}000v}$$

For a column 30 feet long, if $v = 6$ inches,

$$r = \frac{360 \times 360}{12{,}000 \times 6} = 1.8 \text{ inches.}$$

Hence, if these data are correct, a 4-inch pin would suffice to fix the ends of such a column. Clearly, the frictional moment makes the ideal pin-ended column unrealizable unless the pin is well lubricated. Ordinarily both pin and hole are lubricated with tallow before erection, to facilitate putting in the pin and to prevent rusting, but there is no means of ensuring lubrication afterward. The impact, vibration, and alternation of load in a railroad bridge when a heavy train passes over it may cause a rotation of some bars on the pins, but in that case when the load passes off a portion of this rotation is likely to remain, causing in the unloaded structure stresses of the opposite kind to those caused by rigid joints under loading.

Equally impossible is it to obtain the ideal fixed-ended column. Even with the most rigid connections possible, there will be some yielding and rotation at the ends. Moreover, when a frame deflects, the joints at the end of any bar, such as a chord bar or diagonal, do not deflect alike, and the ends do not remain with the axis parallel and in the same line. Generally the two ends are rotated unequally from their original position, and also moved laterally. Further, loads are generally more or less eccentric, which complicates the problem.

It is thus obvious that *no formula which is correct for an ideal column will be correct for a column in a structure.* Not only is the latter different from the ideal column, but we never can know what the real condition of the actual column is. This is true no matter what the character of the structure.

It should further be observed that a column in a testing machine is never in the precise condition that it would be in a structure. Consider a pin-ended column: it might be supposed that it would be the same, but in a testing machine its tendency to deflect laterally and the restraint exercised by the friction on the pin are due entirely to the action of the opposed compressive forces, whereas in a bridge the deflection of the structure would cause an *active* bending moment at the ends tending to rotate them, or to rotate the pins in the holes, due to the change of shape of the structure as a whole, and independent of the compressive forces acting on the column.

It is therefore safe to say that an actual column connected by pins at the ends is always stronger than the ideal pin-ended column which has been studied.

Similarly, a flat-ended column in a testing machine is subject to no outside influence tending to rotation or lateral movement of the ends, while in a structure it is.

23. Actual Column with Slight Bend.[1]—One way in which the practical column differs from the ideal column is that it often has a slight bend. Let the actual column, before loading, have the shape ACB (Fig. 379). Then its action under a load will be entirely different from that of the ideal column. It will not retain its original shape until the load P attains the value given by Eq. (9), but any load, however small, will produce at any point a bending moment Py, which will cause additional

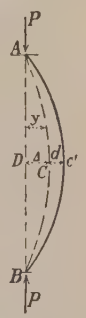


FIG. 379.

[1] See "A Practical Treatise on Bridge Construction," by T. CLAXTON FIDLER, London, Charles Griffin & Co.

bending and so lead to a greater value of y and of the moment. Suppose a load P causes the column to take the shape $AC'B$, with the additional deflection d at the center. Then the bending moment there will be $P(\Delta + d)$, and the curve $AC'B$ will represent the curve of moments. These moments will cause the elastic deflection d.

Following the area-moment method (Art. 15) the moment of the area ADC' about A represents the elastic deflection d; or taking the curve as a parabola,

$$d = \frac{5}{48} P(\Delta + d) \frac{l^2}{EI}$$

$$P = \frac{48}{5} \frac{\Delta}{\Delta + d} \cdot \frac{EI}{l^2} = P' \frac{d}{\Delta + d} \tag{18}$$

if P' is the value given by Eq. (9a) in Art. 15.

Also, from Eq. (18), $$d = \Delta \frac{P}{P' - P}. \tag{19}$$

From this it is clear that the elastic deflection, d, will have a definite value, depending on the load P. It is also clear that if P is very nearly equal to P' the value of $\frac{d}{\Delta}$ will be very large; or, in other words, a very small value of Δ may result in a large value of d.

The action is somewhat similar when the column is not homogeneous, in which case even a uniformly distributed load may cause some bending. A small difference in the value of E on the two sides of a column will produce the same effect as if there were an initial curvature, as shown in Art. 2. If the bend is local, instead of extending the entire length as in Fig. 379, the case cannot well be treated mathematically; but in any of these cases the load, however small, will produce some increased bending.

It thus seems clear that an actual column, since it will not be perfectly straight or homogeneous, will not follow the law of Euler's equation, remaining straight until a certain load is reached, and then failing suddenly; but that it will more likely bend gradually. From this it would appear that the most reliable formula is not Euler's; but one which takes account of the crushing strength of the material, provided the proper constants may be found. The strength of the column must depend upon the compressive stress on the material on the concave side, and this will depend upon the deflection.

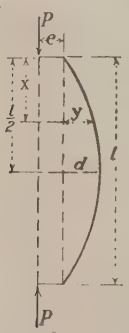


Fig. 380.

24. Theoretical Formula for Eccentric Load.—Let the compressive forces on a pin-ended column free to rotate at the ends act at a distance e (= eccentricity) from the line of the original axis (Fig. 380). Then, as in Euler's method,

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}(e + y) = -m^2(e + y). \tag{20}$$

The general integral of this is

$$y = A \cos mx + B \sin mx - e. \tag{21}$$

The conditions are:

when $x = 0$, or $x = l$; $y = 0$;

when $x = \frac{l}{2}$; $\frac{dy}{dx} = 0$, and $y = d$;

$x = 0$, $y = 0$, gives $A = e$;

$x = l, y = 0$, gives $e = e \cos ml + B \sin ml$;

$$\therefore B = \frac{e(1 - \cos ml)}{\sin ml} = e \tan \frac{ml}{2};$$

$x = \frac{l}{2}$; $\frac{dy}{dx} = 0$, gives the same;

$x = \frac{l}{2}$; $y = d$, gives $d = e\left(\sec \frac{ml}{2} - 1\right)$;

Hence,
$$f = f_c + f_b = \frac{P}{A} + \frac{P(d+e)v}{Ar^2} = \frac{P}{A}\left(1 + \frac{ev}{r^2}\sec \frac{l}{2}\sqrt{\frac{P}{EI}}\right). \qquad (22)$$

This formula is considered by some good authorities as the proper one to be used. It is correctly based on the addition of f_c and f_b, and it holds, if at all, only within the elastic limit, because the value of f_b only holds within that limit. When the end eccentricity $e = 0$, the formula becomes $f = \frac{P}{A}$; in other words, it ignores any bending except that due to end eccentricity. But when the angle $\frac{l}{2}\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$ (that is, when $P = \frac{\pi^2 EI}{l^2}$, which is Euler's formula) the secant is infinity, and the result is indeterminate if $e = 0$, or $f =$ infinity if e is not zero. The angle is, of course, to be expressed in radians. To use this formula, e must be known, though it never is known in an actual column. Those who advocate the use of this or a similar formula endeavor to find the eccentricity which will make the formula agree with actual tests, or which will give lower and upper limits. A column may have a known and intended eccentricity, but in addition to this there will be accidental or unknown eccentricities, as will be explained in the next article. If a column with a known intended eccentricity (which may be zero) is tested and the deflections carefully observed, the formula may enable the accidental eccentricity to be found, by computing the eccentricity necessary to produce the observed deflections and subtracting from this the intended eccentricity. A series of such studies, if the phenomena were regular, might perhaps help to determine a maximum accidental eccentricity which should be used in designing by use of the formula. Such is the argument of those who advocate the use of the formula.[1]

In his valuable paper on "The Practical Column under Central or Eccentric Loads,"[2] J. M. Moncrieff uses a formula based upon an initial eccentricity.

Mr. Moncrieff assumes an initial end eccentricity, c, and uses the moment-area method, assuming the shape of the deflected axis to be a parabola. The center of gravity of the area ACD (Fig. 381) lies at a distance $\frac{5}{8} \cdot \frac{l}{2}$ from A and the resulting formulae are found as follows. The moment at any point will be P times the distance between FG and ADB. The deflection, Δ, of A from the tangent at D will be the moment of the area $FADE$ times P divided by EI, or

[1] Professor O. H. Basquin, in *Journ. Western Soc. of Engineers*, 1912, gives an interesting discussion of this subject. See also Professor C. T. Morris, in *Eng. News*, Nov. 2, 1911.

[2] *Trans. Am Soc. E.C.*, vol. XLV, p. 334, 1901.

$$EI \cdot \Delta = \frac{Pe \cdot l}{2} \cdot \frac{l}{4} + \frac{2}{3} \cdot \frac{l}{2} \cdot P\Delta \cdot \frac{5l}{16}$$

whence

$$\Delta = \frac{Pl^2 e}{8EI - \frac{5}{6}Pl^2} \tag{23}$$

The *maximum intensity of compression*, if v is the distance from the axis to the inside fiber at D, calling $\frac{P}{A} = f_d$, and $\frac{l}{r} = R$, is

$$f_c = \frac{P}{A} + \frac{P(e+\Delta)v}{Ar^2} = f_d\left[1 + \frac{ve}{r^2}\left(1 + \frac{f_d R^2}{8E - \frac{5}{6}f_d R^2}\right)\right]$$

$$= f_d\left[1 + \frac{ve}{r^2}\left(\frac{48E + f_d R^2}{48E - 5f_d R^2}\right)\right] \tag{24}$$

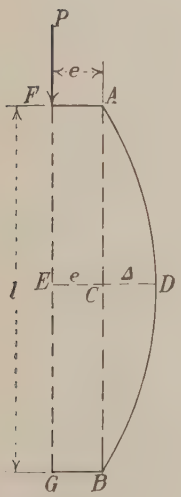


Fig. 381.

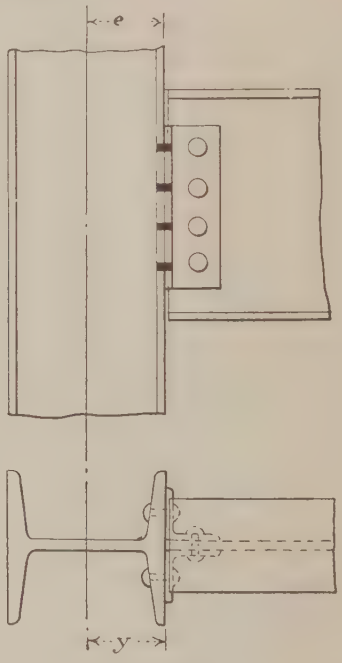


Fig. 381a.

From this the value of $R = \frac{l}{r}$ corresponding to any given value of f_c is found to be

$$R = \sqrt{\frac{48E}{5f_c + f_d\left(\frac{ve}{r^2} - 5\right)}\left(\frac{f_c}{f_d} - 1 - \frac{ve}{r^2}\right)}. \tag{25}$$

If values are assumed for E, f_c, and $\frac{ve}{r^2}$, this equation gives values of $R = \frac{l}{r}$ corresponding to any given $f_d = \frac{P}{A}$, and a curve can be drawn representing these

values, which will give $\frac{P}{A}$ for any $\frac{l}{r}$, consistent with assumed maximum f_c and eccentricity. The values of E and f_c may be satisfactorily chosen, the latter being the strength of short struts, but the value assigned to $\frac{ve}{r^2}$ must be arbitrary, though, of course, for any given set of tests, values may be found which will give curves showing upper and lower limits. Mr. Moncrieff concludes that the lower limit of column strength is fairly given when $\frac{ve}{r^2} = 0.6$, and that the upper limit, rarely exceeded, corresponds to $\frac{ve}{r^2} = 0.15$, and that "these values are applicable alike to cast iron, wrought iron, mild steel, hard steel, and several kinds of timber."[1]

24*a*. If the load on a column is entirely eccentric, as, for instance, when the only load is that which comes from a beam riveted to the flange of a broad-flanged beam which forms the column (Fig. 381*a*), the effect of eccentricity, may be approximately allowed for by multiplying the load by a factor greater than unity, and considering it as a central load.

Let P = the load,

e = its eccentricity = half the depth of the column = y.

P' = equivalent central load.

The max. fiber stress is

$$f = \frac{P}{A} + \frac{P}{A} \cdot \frac{l^2}{cr^2} + Pm + \frac{Py^2}{Ar^2}.$$

The second term of the right-hand member is the stress due to bending if the load were central. The third term, Pm, is the stress due to the central load acting with a lever arm which is the additional deflection, over that due to a central load alone, equal to the *additional* deflection produced by the eccentric moment Pe.

The stress due to the equivalent central load P', which must equal f, is

$$f = \frac{P'}{A} + \frac{P'}{A} \cdot \frac{l^2}{cr^2}.$$

Equating the two values of f, we find

$$P' = P\left(1 + \frac{Amr^2 + y^2}{r^2 + \frac{l^2}{c}}\right).$$

If the terms $\frac{l^2}{c}$ and Amr^2 are omitted, the effects tend to offset each other, and we have

$$P' = P\left(1 + \frac{y^2}{r^2}\right).$$

Suppose, for instance, that the column is a 15-inch Bethlehem girder beam for which the radius of gyration, r, is 6.41 inches, and $y = 7.5$ inches, then $P' = 2.37\ P$. Tables of the coefficients of P, apparently calculated by this formula, are given in the English "Handbook for Structural Steel," of R. A. Skelton and Co.

[1] For results of tests on steel angles, some with flat ends, and some with bolted ends, similar to the riveted connections used in transmission towers, see *Tech. Paper* 218, *U. S. Bureau of Standards*, by A. H. Stang and L. R. Strickenberg, 1922.

The above formula, however, takes no account of the moment at the connection of beam to column, which would still further increase the stress. It only assumes an axial load with an eccentricity y. If the eccentricity in general is e, as when there is a bracket under the beam, making the load applied at a greater distance than y from the axis, the formula becomes

$$P' = P\left(1 + \frac{ey}{r^2}\right).$$

25. If a column consists of a single shape, like a rolled I-beam, or of a single solid made up by riveting shapes together (Fig. 382) it acts as a whole in resisting compression and in deflecting under an axial load. If, on the other hand, it consists of two or more parts, connected together at intervals (Fig. 383) the column acts as a whole as a column of length l, but each part acts separately as a column of length l_1. Figure 383 is a column composed of two channels latticed together; Fig. 384 of the same shapes with "batten plates" or "battens" at intervals.

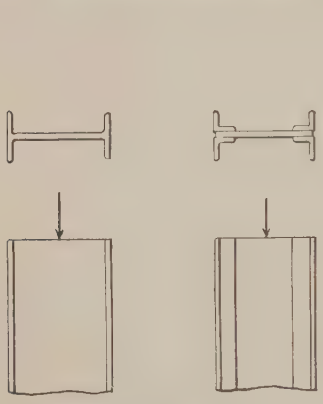


FIG. 382.

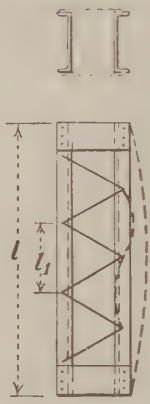


FIG. 383.

FIG. 384.

Obviously each channel acts by itself to resist compression as a column having a length equal to l_1, the distance between connections of the latticing or between batten rivets.

The principle in this case is that *each part, acting separately, as a column of length* l_1, *must be as strong as the column as a whole of length* l. The separate parts would be very weak unless the length is short, because the radius of gyration of each part is small compared with that of the column as a whole.

It is often supposed that this principle merely requires that $\frac{l}{r}$ for the column as a whole must equal $\frac{l_1}{r_1}$ for each separate part, but this is not sufficient by any means, for the end conditions may not be the same for the column as a whole as for the separate part. Considering that the separate part, between lattice connections, may take the shape shown by the dotted line, it is reasonable to say that *the separate part should be considered as the ideal pin-ended column*, even if the column as a whole is fixed-ended. This condition will be met if

$$\frac{l^2}{cr^2} = \frac{{l_1}^2}{c_1{r_1}^2} \qquad (26)$$

c_1 being the constant for pin-ends and c that for the column as a whole; r_1 is the least radius of gyration of the separate part.

There is a further consideration regarding latticed columns: In the column as a whole, using Rankine's formula, $\frac{P}{A}$ is the average stress and the maximum stress f' is

$$f' = \frac{P}{A}\left(1 + \frac{l^2}{cr^2}\right).$$

This maximum stress occurs at one edge, but practically over one of the two parts. It contains no allowance for the action, as a column, of the part between two lattice connections. If the column is of two channels, one of these has more compression over it than the other, and f' is nearly the average over this one channel.

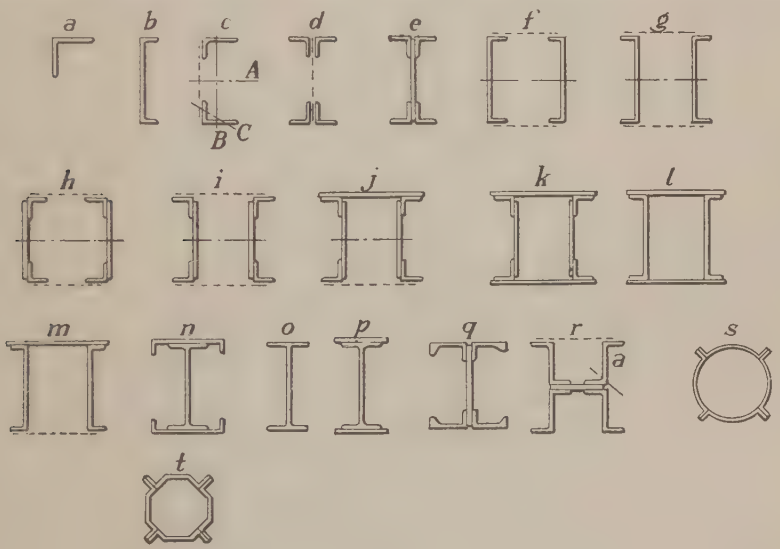


FIG. 385.

Hence the maximum on this channel, which should equal the allowable unit stress, f, for a short strut, is

$$f = \frac{P}{A}\left(1 + \frac{l^2}{cr^2}\right)\left(1 + \frac{{l_1}^2}{c_1{r_1}^2}\right).$$

Hence it would appear that in such a column

$$\frac{P}{A} = \frac{f}{\left(1 + \frac{l^2}{cr^2}\right)\left(1 + \frac{{l_1}^2}{c_1{r_1}^2}\right)} \tag{27}$$

and if Eq. (26) is fulfilled

$$\frac{P}{A} = \frac{f}{\left(1 + \frac{l^2}{cr^2}\right)^2}. \tag{27a}$$

Similarly, if the straight line formula is used

$$\frac{P}{A} = f - k\frac{l}{r} - k'\frac{l'}{r'}. \qquad (27b)$$

However this may be, this formula has never been used, to the writer's knowledge; yet it is reasonable to conclude that *a column composed of two parts latticed or connected by battens is, other things equal, weaker than one composed of a solid section.* Nevertheless, such columns frequently must be used.

26. Forms of Column Sections.—Figure 385 shows the forms of column sections that have been most frequently used; latticing being indicated by dotted lines. Very large sections are not here shown, but will be discussed in a later volume.

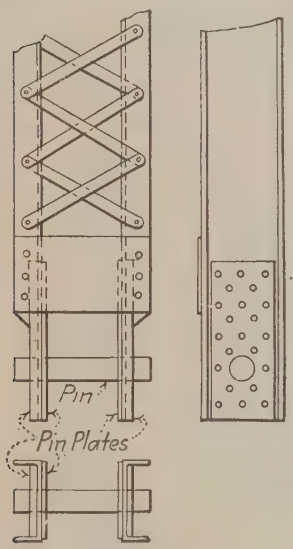


FIG. 386.

The construction of a latticed column of shape *g*, with pin ends, is shown in Fig. 386.

A single rolled shape, like an angle or a channel, may be used as a column. A channel is a bad form, because its radius of gyration is small. Two angles latticed, *c*, are often used, but four are better, *d*. An I-beam, *o*, may be used, forming a "beam column," or shape *e* may be used; the beam column saves two lines of rivets as compared with *e*. The other forms are for heavier columns, *f*, *g*, *h*, and *i* being latticed (Fig. 386). The disadvantage of solid hollow shapes, like *k* and *l*, are that the inside is inaccessible for painting. Shape *s* is the "Phoenix" column, much used 50 years ago by the Phoenix Bridge Co. Considered solely as a column, this was probably the best column ever made, as the material is concentrated as far as possible from the axis; but its inaccessibility for painting on the inside and the difficulty of connecting with other members have led to its disuse. Shape *t* was the Keystone column, also used 50 years ago, but not now.

In using Eq. (26) for a column such as *c*, the value of r is that about axis A or B, while the value for r_1 is about an axis such as C.

27. Ultimate Strength of Columns. Value of f in Column Formulae.—The value of f in all the column formulae is the value of the average stress $\frac{P}{A}$ when the slenderness ratio is zero, that is, for pure compression.

For wood columns, which are generally rectangular in section, and which fail by buckling of the cell walls, the value of f may be the ultimate as given in Vol. 2.

For cast iron, the same is true. Failure is here generally by shear. Values of the ultimate are given in Vol. 2.

For steel and wrought iron, which are ductile materials, the case is not so simple. As explained in Chap. IV, it is impossible to say what the strength in pure compression is, but it is very large. Indeed, unless the metal can shear or flow laterally (bulge) it is difficult to see how it could ever fail by pure compression. A cylinder of somewhat greater length than its least transverse dimen-

sion could fail by shearing on a diagonal plane, in which case its strength would be somewhat greater than twice the shearing strength (greater because of internal friction; see Art. 3 Chap. VI). These, however, are not columns, strictly speaking. In columns, as soon as the yield point is reached, the material begins to flow. If there is a slight local bend, as there nearly always is, the greatest stress will be at the inside of the bend, and the flow will increase the bend and cause local buckling and failure, either with or without the bending and failure of the column as a whole. Thin outstanding material will be apt to buckle and fail If there is no thin outstanding metal, as in a Phoenix column, the thinnest material will "wrinkle."

For a short column, with $\frac{l}{r}$ below say 20, the ultimate value of f will approach the ultimate in pure compression, but for longer columns it is obvious

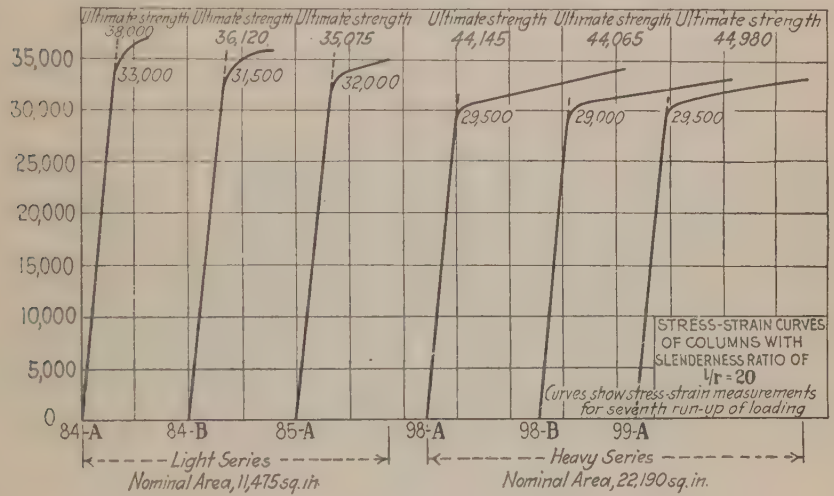


FIG. 387.—(*Trans. Am. Soc. C. E.*, 1920, p. 2430.)

that *the true ultimate for a ductile material should be assumed as the yield point*, or but little above it. It was long customary to take f as the ultimate in pure compression, or about 50,000, but it is now very generally recognized that it should be taken as about the yield point, or for structural steel about 38,000, or at what has been called the "useful limit point" (U.L.P.).

The Useful Limit Point.—The phenomenon of flow at the yield point is not so marked in compression as in tension. The stress-strain diagram in compression is generally of the shape shown in Fig. 388. Hence the Committee on Steel Columns of the A.S.C.E., which submitted its report in 1918, determined the equivalent useful limit point as the point where the slope of the stress-strain diagram (with the axis of strain) was one-half that of the straight or nearly straight part of the line, and took this to correspond to the yield point as the value which should be taken as the ultimate. It is considered to be the value of the stress intensity which limits the practical usefulness of the column. It is somewhat above the elastic limit.

28. Elements Affecting the Strength of Columns.—Column formulae all assume that the only element affecting the strength, aside from the value of f, is the slenderness ratio. This is not the case. There are other elements, some of which will now be considered.

(*a*) *Shape.*—Figure 385 shows that there are many different shapes, and each shape may have material of different thickness. Failure frequently occurs by the *local buckling* of some thin outstanding part, like the leg of an angle in Fig. 385*e*. Two columns may have the same slenderness ratio, yet very different strengths. Consider the shape in Fig. 385*e*: now keeping the same area, but making all the parts thinner, the depth and width will be increased, and consequently r, so that the slenderness ratio may be decreased, and yet if the parts are made very thin the strength may be decreased instead of increased. Thus

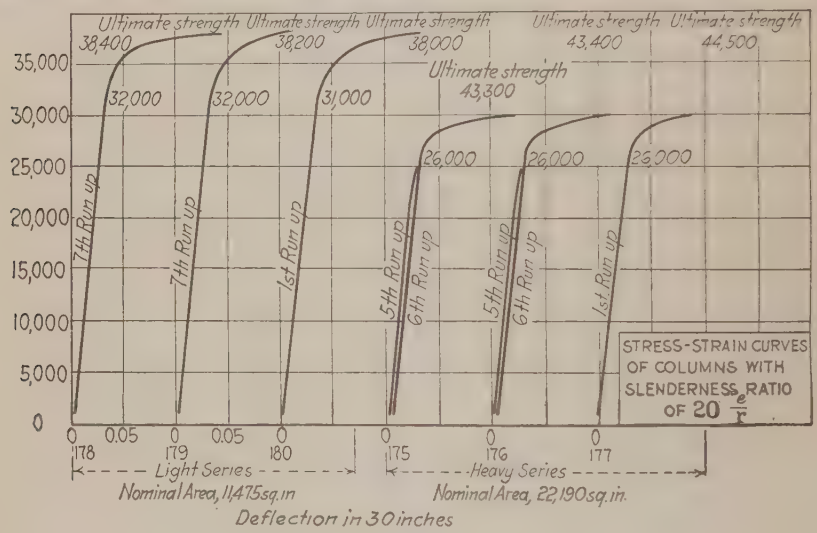


FIG. 388.—(*Trans. Am. Soc. C. E.*, 1920, p. 2431.)

the form of column and the thickness of parts is important. Other things equal, the thicker the material, the greater should be the strength. In the tests made by the Committee of the A.S.C.E., light and heavy sections were used of each form, the two being alike except that in the heavy sections the metal was much thicker. In every case but one, the heavy sections were the *weaker* (for the same slenderness ratio). The opposite result was expected. The attempt had been made to secure material of uniform quality, and the shop tests had shown substantial uniformity, the thick and thin material showing the same yield points in tension. Subsequent tests in compression, however, showed that the compressive yield point of the thick material was considerably lower than that of the thin material, and this, as explained above, was no doubt the explanation of the greater weakness of the thick material. In one type of column, for $\frac{l}{r} = 50$, the *U. L. P.* of the light section averaged 34,700 pounds per square inch, while that of the heavy section averaged 19,700 pounds per square inch, or but 57

per cent of the former. These variations were perhaps abnormal, but the material was specially chosen, and supposed to be the best that the mills could produce. Figure 387 shows the stress-strain diagrams for parts of longer columns which had been previously tested, which were cut out and tested with $\frac{l}{r} = 20$. In these the *U.L.P.* had no doubt been raised by the previous tests. Figure 388 shows similar tests of the same material which had not been previously tested. The ultimate strength of these short columns was greater per square inch for the heavy sections than for the light sections, because they were too short to show the phenomena of flexure; but the *U.L.P.* was lower, and this is the limit which becomes effective in long columns where flexure acts, as these tests show clearly. The difference between the thick and thin material no doubt arose from the greater mechanical working to which the thin material had been subjected in the rolling, but it is difficult to understand why the mill tests in tension did not show the same difference. As a matter of fact, they gave in most cases, higher yield points for the thick material. The conclusions seem inevitable that the usual mill tests do not give a correct indication of the compressive strength of the

FIG. 389.—Failure of Z-bar column.

material, and that the present processes of manufacture do not produce a uniform material.

Some sections have outstanding parts which may buckle; others, like hollow tubes, have none, and yield by wrinkling or collapse of the shell. It is very important to distinguish a failure of the column as a whole, though that may be accompanied, at the end, by buckling of some part, from a failure arising from buckling of some part without failure as a whole. The column formula can only apply to failure as a whole. Many tests, as repeated, do not describe the precise manner of failure, and some are no doubt erroneously assumed to represent the strength of the column as a whole. Each test must be studied individually, and should not be used in a study of column formula if the failure was primarily by local buckling. The column should be designed, if possible, with such details that it will fail as a whole, and not locally, and this can best be done by specifying certain minimum thicknesses, but, as shown above, this is quite uncertain unless the thick and thin material has the same yield point.

Some columns, also, have parts whose center of gravity is to one side of a web which supports those parts, and may fail by the buckling of that web. The Z-bar column, for instance (Fig. 385*r*) has the center of gravity of the part *a* to one side of the vertical web, and may fail by the bending of that web at the diagonal

dotted line and the buckling of the part *a* toward the right. Figure 389 shows such a failure of a Z-bar column, and this should be set down to local buckling, and not to failure of the column as a whole. This action would be prevented by latticing as shown by the top dotted line. Figure 390 shows a local failure by buckling of an outstanding angle leg. Figure 391 shows another local failure.

It is obvious that the strength of different forms of section will be represented by different formulae. No one formula or set of constants will apply correctly

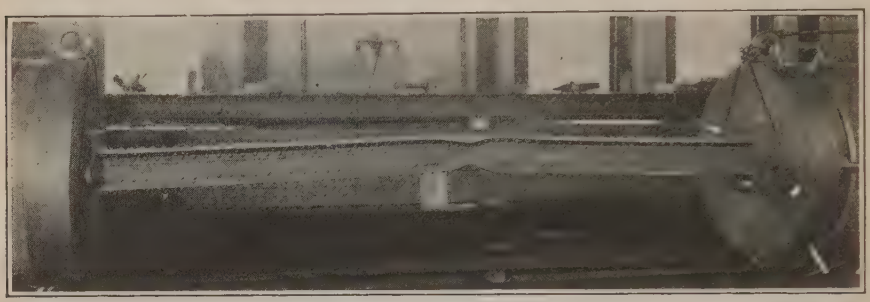


FIG. 390.—Failure of column.

to all forms. Yet we must have such a formula in order to design, even if it is not accurate in all cases.

(*b*) *Quality of Material; Yield Point; Lack of Homogeneity.*—Uncertainty regarding the yield point has been above referred to. Lack of homogeneity of the different parts composing a column will cause slight bends (see Art. 23) and consequent eccentricity, all being uncertain.

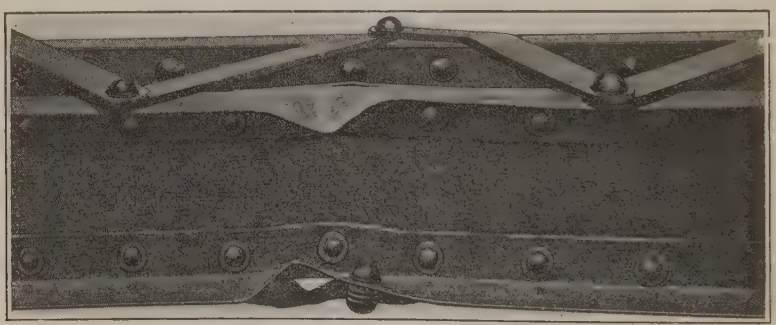


FIG. 391.—Local failure by buckling.

(*c*) *Initial Bends.*—These often occur, and cannot be allowed for in a formula except by an arbitrary assumption of the eccentricity. Bends may be caused by handling, especially of light members or their outstanding parts.

(*d*) *Initial Stresses.*—These may be due to unequal cooling, to the operations of rolling, punching and shearing, or to cold straightening. Most angles and channels have to be straightened cold. If a line of rivets is punched in one leg of an angle or in a flange of a channel, the lateral pressure generated will stretch the piece along the line of rivets, and it will become curved.

When straightened cold it must be stressed above the elastic limit, in tension on one side and in compression on the other side, and when the straightening is completed it will be under initial stress, compression where the tension limit of elasticity was exceeded and tension where the compression limit of elasticity was exceeded. Now when the column is exposed to compression, the portion of the section, at one side or edge, where the elastic limit in tension had been exceeded in straightening, is not only in initial compression, but its elastic limit in compression has been lowered by the previous overstress in tension, and it will therefore fail or flow under a smaller load than if it had not been previously overstressed (see Art. 7 Chap. XXI). This effect is entirely uncertain, and depends upon the previous overstress.

(*e*) *End Conditions: Eccentricity.*—End conditions are often uncertain in the testing machine, and always so in the structure. There may be friction on the pins; the ends for flat ends may not be cut truly at right angles to the axis; shimming may be necessary in testing; pin holes may not be cut precisely at right angles to the length, and the bearing may be more on one side of the column than on the other; and any of these things will cause an unknown eccentricity.

(*f*) *Workmanship.*—This may be good or poor. Rivets may not fill the holes, and stresses may be caused in the process of manufacture, other than those already referred to.

29. Column Tests.—A study of column tests will prove very disappointing to the student seeking for something definite, because there are so many discrepancies and uncertainties, many of which have been discussed in this chapter. The following remarks from the paper by James Christie[1] are enlightening. He made an extensive series of tests on columns with flat ends, fixed ends, hinged ends and rounded ends, and he remarked (the italics being ours):

. . . Every known precaution was observed in order to obtain accurate results, and the disparity in these results seems to be due to unavoidable inequalities in the condition of the bars, as hereafter noted. Very minute changes in the position of the center of pressure produced greater differences in the resistance of the bars than we anticipated.

Judging from previous experience, the result of any few indiscriminate tests would yield no satisfactory conclusions; it was, therefore, determined to make such a number of tests that the range of maximum and minimum could be fairly established, and a proper average deduced therefrom. For reasons not always evident, occasional results were obtained either abnormally high or low, as will be found illustrated on the diagrams; but there is little doubt that the principal cause of low resistance was eccentricity of axes, or non-coincidence between the center of pressure and the axis of greatest resistance of the specimen.

This was most evident on hinged-end struts of a symmetrical cross-section, such as I-beams and welded tubes, with which it was occasionally found that the highest resistance was not always obtained when the centers of balls or pins were placed in exact line with the centers of the section, as described on the ends of the bar. Sometimes, by moving the specimen apparently slightly out of center, the resistance was vastly increased. This was, no doubt, due to minute bends or inequalities in the distribution of the metal, which caused the axis of greatest resistance of the strut to disagree with the apparent axis of symmetry. A few examples of this character are described in the tables.

[1] *Trans. Am. Soc. C.E.*, vol. XIII, p. 85, 1884.

When the bar was straight and accurately centered on the ball or pin, *the strength of the hinged-ended strut of any length was fully equal to that of the flat-ended.*

In fact, *the resistance of the best specimens of hinged-ended bars exceeded the best of the flat-ended.* This behavior, as illustrated on the diagrams, occurred in so many instances that it could not be considered exceptional, but the slightest deviation from the requisite accuracy of centering rapidly reduced the resistance; and as no amount of care could always insure the precise conditions required, it is not believed that in practice, with the most careful workmanship, or the best form of strut or column, could these high resistances be uniformly obtained.

When the bars were very long, in proportion to radius of gyration, the ultimate load could be applied without any permanent injury to the bar. The strain could thus be released and applied at will without any change in resistance. Several bars were tested in this manner, and the effect of change in end conditions noted, a few examples of which will be found in the tables.

The high resistance on balls or pins could always be obtained by trial, that is, by observing the direction of deflection under strain, then releasing the strain and moving the bar in the proper direction; when the point of greatest resistance was passed, the deflection would be reversed in direction.

When the point of greatest strength was reached, the behavior of the specimen was peculiar. Under ordinary circumstances the bar, while bending under strain, rotated from the start on its hinged ends. When correctly centered, no such rotation occurred at the beginning of deflection, but the bar bent, like a flat-ended strut, until the point of failure was reached, when it rotated on its ends suddenly, and with so much force as sometimes to spring from the machine. In fact, the deflection under these conditions more resembled that of fixed-ended than of flat-ended specimens, *for the latter frequently showed indications of rotation on their ends before ultimate resistance was attained.*

In no instance could the effect described be produced on a round-ended strut, and the smaller the pin or ball and socket the more rarely it occurred, or the more difficult to produce it by trial, just as an egg may be stood erect in a very minute cup, but never on a flat table. *The diameter of the ball or pin exercised a marked influence on the resistance of the bar, as also did the fit of the pin.* A few examples are given in the tables illustrating the effect of a change in pin diameter. The test, No. 125, of a hinged-end angle was made on new pins, which were an absolute fit to the semi-cylindrical seat which capped the specimen, the test being made with the pin hinges dry.

The caps were immediately eased with a file and the bearing on pins lubricated, when it was found impossible in the same bar to reproduce the high resistance first obtained. The hinged-ended tests varied all the way from the value of round-ended up to flat-ended. As can be seen on the diagrams, the lowest range of hinged-ended tests approximated very closely to the average of round-ended.

The ends of the shortest flat-ended struts remained solidly seated at the time of failure, whilst, on the contrary, *the longest struts always rotated on their ends, and, as stated before, sometimes showed a tendency to turn before ultimate resistance was attained.*

The following examples illustrate the above remarks: "Two bars of angle iron $2\frac{1}{2} \times 2\frac{1}{2} \times \frac{9}{32}$ inches, 5 feet $4\frac{1}{16}$ inches long, were cut from the same original bar, and adjusted in the testing machine, as near as possible, with the centers of gravity of section in line with centers of the balls which capped both ends."

Experiment 91.—With 2-inch balls and sockets, failed with 36,500 pounds ultimate load.

Experiment 92.—With 1-inch balls and sockets, failed with 24,010 pounds ultimate load.

These bars were then straightened cold and changed on the balls, centers precisely as before.

Experiment 91.—With 1-inch balls and sockets, failed with 17,500 pounds ultimate load.

Experiment 92.—With 2-inch balls and sockets, failed with 36,500 pounds ultimate load.

Angle $2 \times 2 \times \frac{5}{16}$ inches, 8 feet, $3\frac{1}{8}$ inches long.

Experiment 82.—First test with 1-inch balls and sockets, failed with 4,500 pounds.

Experiment 83.—Second test with same balls moved 0.08 inch failed with 8,100 pounds.

The results of Christie's tests will be found plotted with his paper, and the reader should study them. They will be found to contain numerous discrepancies; for instance, in the tests of hinged angles:

No. 106; $\frac{l}{r} = 103$; 1-inch ball and socket; ultimate 38,600 pounds per square inch.

No. 105; $\frac{l}{r} = 105$; 1-inch ball and socket; ultimate 36,250 pounds per square inch.

No. 98; $\frac{l}{r} = 110$; 1-inch ball and socket; ultimate 15,826 pounds per square inch.

No. 103; $\frac{l}{r} = 119$; 1-inch ball and socket; ultimate 38,600 pounds per square inch.

The tests on round-ended ties and angles were the most regular; but even here the rounded ends became somewhat flattened, probably causing an increase in the strength.

Early American experiments were 22 tests of Phoenix wrought iron columns in 1881 (Fig. 385*S*) by T. C. Clarke;[1] 20 of these being 4-segment columns 8.04 inches diameter, 12 square inches, 8 inches to 28 feet long, and 2 being 6-segment columns, 11.8 inches diam., 18.38 square inches, 8 feet $9\frac{1}{2}$ inches and 25 feet 2.65 inches long, all with flat ends. Some ends were not squared smooth and had to be wedged to a bearing. The thickness of material was $\frac{5}{16}$ inche. The shortest columns, 8 inches long, had an ultimate strength of 57,215 pounds per square inch; the next length 4 feet had an ultimate of 50,370. Bouscaren found that these tests were well represented by the formula

$$\frac{P}{A} = \frac{38{,}000}{1 + \frac{l^2}{100{,}000r^2}} \tag{28}$$

while Merriman found that those for $l > 12d$ were mathematically best represented by the formula

$$\frac{P}{A} = \frac{37{,}200}{1 + \frac{l^2}{158{,}500r^2}}. \tag{29}$$

[1] *Trans. Am. Soc. C.E.*, vol. XI, p. 1–120, 1882.

Since some of the ends were not squared smooth, however, too much reliance should not be placed on these tests; though the same faults of workmanship would exist equally in a structure, and there the ends would not be wedged up.

Mr. Clarke said, in describing the tests:

> . . . The behavior of the columns, after taking a deflection of several inches, was remarkable. As the details of the tests will show, they still retained a superior sustaining power, and herein they differ materially from lattice columns. The latter form, after deflecting slightly, suddenly give way by tearing out the riveting of the lattice bars, after which but little strength, as a column, remains.

In connection with the construction of the Cincinnati Southern Railway, G. Bouscaren tested 43 wrought-iron columns,[1] and his paper describing them is a model of clear and logical discussion. He concluded that Rankine's formula was more accurate than Gordon's, clearly proved the importance of firmly riveting the parts together, and laid down a rule still in use, that the thickness of a plate should not be less than one-thirtieth of the distance between its supports transversely to the stress.

The experiments of Christie (see *ante*) were a very complete set. The first were reported in 1883, and were on wrought-iron struts, mostly of single shapes, with ends fixed, flat, hinged, and round; the later tests, reported in 1884, were on steel.[2] Flat ends abutted on the plates of the testing machine without further restraint; fixed ends were in a pocket designed to prevent rotation of the ends. Up to slenderness ratios of 20 there was practically no difference between the four end conditions; the struts with hinged ends, it was shown, could be as strong as those with flat ends, depending on the diameter of the pin. The conclusion was reached that the hinged ends should fall in strength up to a ratio 250, at which point they were about two-thirds that of flat ends; above this ratio the hinged ends gained up to a ratio 500, when they were equal to flat ends, which could only be because the flat ends were not fixed, but rotated, or that the hinged ends did not rotate. Struts with fixed ends gained in strength as compared with the others up to a ratio 500, when they were twice as strong as those with flat or hinged ends. Those with round ends at a ratio 160 were about half as strong as those with flat ends, and at a ratio 450 about half as strong as those with hinged ends.

In the later tests, the value of E for steel in tension was found to average 30,135,000; and in compression 20,478,000! This result casts some doubt upon the measurements, as the two have generally been found substantially equal.

Christie gave the following table showing average strength of wrought-iron struts:

[1] *Trans. Am. Soc. C.E.*, vol. IX, p. 447, 1880.

[2] Both papers are in *Trans. Am. Soc. C.E.*, vol. XIII, 1884.

COMBINED AVERAGE RESULTS OF TESTS OF WROUGHT-IRON STRUTS. ULTIMATE IN POUNDS PER SQUARE INCH[1]

(Christie)

Length divided by least radius of gyration	Flat ends	Fixed ends	Hinged ends	Round ends
20	46,000	46,000	46,000	44,000
40	40,000	40,000	40,000	36,500
60	36,000	36,000	36,000	30,500
80	32,000	32,000	31,500	25,000
100	29,800	30,000	28,000	20,500
120	26,300	28,000	24,300	16,500
140	23,500	25,500	21,000	12,800
160	20,000	23,000	16,500	9,500
180	16,800	20,000	12,800	7,500
200	14,500	17,500	10,800	6,000
220	12,700	15,000	8,800	5,000
240	11,200	13,000	7,500	4,300
260	9,800	11,000	6,500	3,800
280	8,500	10,000	5,700	3,200
300	7,200	9,000	5,000	2,800
320	6,000	8,000	4,500	2,500
340	5,100	7,000	4,000	2,100
360	4,300	6,500	3,500	1,900
380	3,500	5,800	3,000	1,700
400	3,000	5,200	2,500	1,500
420	2,500	4,800	2,300	1,300
440	2,200	4,300	2,100	
460	2,000	3,800	1,900	
480	1,900	.....	1,800	

Professor L. Tetmajer of Zürich, later of Vienna, made an extensive series of tests of columns,[2] using single angles, T's, channels, I-beams, 2 angles and 4 angles riveted together, 2 T's riveted together, 2 channels closely riveted together, and rounds. Hinged ends were made by abutting the test piece against the heads of the machine, which were pivoted; and some flat ends were used. He discovered no effect of form of section, though there must be such effect. He found that parts riveted together acted as one if the rivet pitch was less than 22 inches, if the rivets filled the holes, and if the holes in a section were less than 12 per cent of the gross section. He found steel more sensitive than wrought iron to weakening by rivet holes, and concluded that rivet holes should be deducted if they formed more than 10 per cent of the gross section. He found that, if the slenderness ratio exceeded about 100, bending generally occurred gradually, but with a smaller ratio generally suddenly, which does not agree with the Euler theory. He found

[1] *Trans. Am. Soc. C.E.*, p. 117, 1884.

[2] *Mittheil der Anstalt zur Prüfung v. Baumaterialen, Heft.* IV, *Zürich*, 1890. The results of Tetmajer's tests are plotted in the paper by J. M. MONCRIEFF, "The practical column under central or eccentric loads," *Trans. Am. Soc. C.E.*, vol. XXVI, pp. 320–417, 1900.

the strength for hinged ends represented by Euler's formula under the following conditions:

for steel, if $\frac{l}{r} > 105$; $$\frac{P}{A} = \frac{315{,}350{,}000\, r^2}{l^2} \text{ (pounds per square inch)} \quad (30)$$

for wrought iron, if $\frac{l}{r} > 112.5$; $$\frac{P}{A} = \frac{287{,}323{,}000\, r^2}{l^2}. \quad (31)$$

Below these limits the tests did not follow Euler's theory, but were represented by a straight line, as follows

for steel, $\frac{l}{r}$ from 20.4 to 105; $$\frac{P}{A} = 45{,}340 - 164\frac{l}{r} \quad (32)$$

for wrought iron, $\frac{l}{r}$ from 18.5 to 112.5; $$\frac{P}{A} = 43{,}026 - 184.6\frac{l}{r}. \quad (33)$$

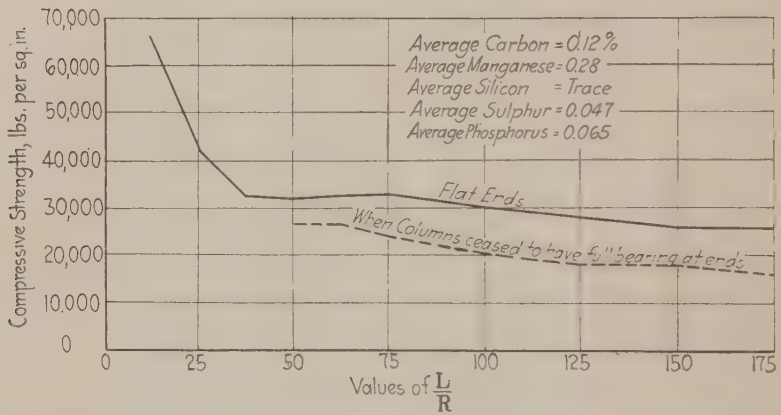


FIG. 392.—Compressive strength of annular sections, lap-welded steel tubing. (From "*Tests of Metals, etc.*," *Watertown Arsenal*, 1908, p. 177.)

He found the mean results represented with sufficient accuracy for practical needs by the Rankine formula

for steel; $$\frac{P}{A} = \frac{42{,}170}{1 + \frac{l^2}{14{,}500r^2}} \quad (34)$$

for wrought iron; $$\frac{P}{A} = \frac{39{,}600}{1 + \frac{l^2}{16{,}667r^2}}. \quad (35)$$

Many tests have been made in this country in recent years, on full-sized columns. A few of these will be referred to briefly.

At the Watertown Arsenal many tests have been made, and are reported in "Tests of Metals" for 1881, 1883, 1884, and in Lanza's "Applied Mechanics." In 1907, an extensive series of tests was planned by the Officer in Charge, with the advice of a committee of which the writer was a member; but after making tests of steel tubing and rolled H-sections, which were reported in 1908 and 1909, the program was abandoned.

Figure 392 shows the results of tests on lap welded steel tubes, and the effect of columns ceasing to have full bearing at ends. Figure 393 shows results for *H*

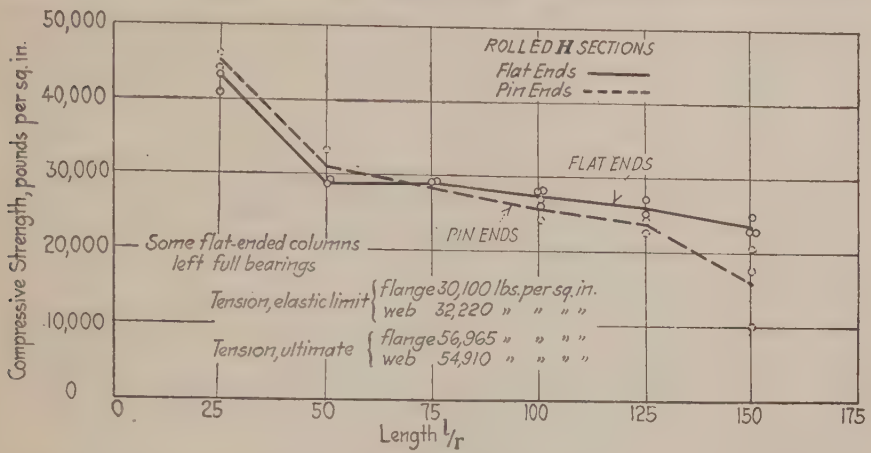


FIG. 393.—(From "*Tests of Metals*" 1909.)

sections, and that pin ends were not always weaker than flat ends. Figure 394 shows the appearance of a built-up column 3 feet 5½ inches long, the loads having been carried considerably beyond the point of maximum resistance.

Technologic Paper 101 *of the U. S. Bureau of Standards* describes tests of some large bridge columns, with an interesting discussion thereon.

The A.S.C.E. appointed in 1909 a committee on steel columns, which made a series of tests on columns of different shapes (Fig. 385*e*, *l*, *k*, *n*, *p*, *q*, *r*, and Bethlehem *H*), with a heavy and a light section of each type (to observe the effect of increasing the thickness of metal), and with slenderness ratios of 50, 85, and 120. Progress reports were made in 1910, 1915, 1916, 1917, and a final report in 1920. In cooperation with this committee, the A.R.E.A. appointed in 1912 a similar committee which prepared a schedule of tests on eight sections commonly found in bridges, arranged to supplement and not duplicate the schedule of the first committee (Fig. 385*f*, *g*, *h*, *i*, *j*, *d* and *c*). All the tests were of flat-ended columns, except one type of the A.R.E.A. schedule. All tests were made at the Bureau of Standards in Washington, with the collaboration of that Bureau. Three columns of each kind were tested, so that for three slenderness ratios there were 9 light sections and 9 heavy sections of each kind. The A.R.E.A. results are reported in the Proceedings, Vols. 16 and 19. The material desired and ordered for these tests

FIG. 394.—Watertown Arsenal Tests, 1909. Failure of column 3 feet 5¼ inches long.

was one having an ultimate tensile strength of 60,000 ±1,500 pounds, and an elastic limit of 38,000 ± 1,000 pounds.

These tests, being on only three slenderness ratios, shed no light on the proper column formula to use. They showed that the ultimate strength was somewhat below the yield point; that is, that the ultimate was reached when the average stress $\frac{P}{A}$ was below the yield point. The committee determined the point at which the tangent to the stress-strain diagram made an angle with the axis of strain one-half that in the straight portion, and called this the "Useful Limit Point" (U.L.P.), as stated in Art. 27. This point was above the proportional limit, but below the yield point. The average U.L.P. of all the A.S.C.E. tests for $\frac{l}{r} = 50$ and 85 was 27,200, but in one extra heavy section it was 19,700, the lowest value found. If it is assumed as 24,000, and a factor of safety of two taken, the maximum value of $\frac{P}{A}$ would be 12,000 pounds per square inch. Both committees finally agreed to recommend the formula

$$\frac{P}{A} = 15{,}000 - 50\frac{l}{r} \tag{36}$$

with a maximum of 12,500.

The average ultimate of all the tests was about 32,000 pounds per square inch. If this is taken with a factor of two, the maximum $\frac{P}{A}$ is 16,000.

The tests showed the unexpected result that in every case but one the heavy columns (thick material) had a lower $\frac{P}{A}$ than the light columns. The thick material had had a smaller amount of mechanical work put upon them, in rolling, than the thin material, and its elastic limit and yield point were lower than in the thin material. This same result was also found in Dagron's tests, in 1889[1] (see Art. 28).

In the A.R.E.A. tests the effect of using batten plates instead of lattice was investigated (Fig. 384). It was shown conclusively that battens, at usual distances, were inferior to lacing, and should not be used. It depends obviously upon the distance between battens and between lacing connections.

The tests therefore showed, as a whole, that the ultimate $\frac{P}{A}$ was between the elastic limit and the yield point, and probably the real unit ultimate stress on the most stressed fiber was about at the yield point (f in Rankine's formula); that thick material was apt to sustain a lower unit stress than thin material because of lower yield point; and that specimen tension tests of the material were not a criterion of the strength of a column (see Art. 28).

30. Remarks on Column Tests.—It will probably be obvious to the reader that column tests are unsatisfactory, and do not furnish an accurate basis for precise theoretical results. Among the reasons are:

(1) The ends of columns are not always plane and parallel, hence flat-ended columns, as tested, are not always the ideal fixed-ended column. The ends

[1] *Trans. Am. Soc. C. E.*, vol. XX, p. 254, 1889.

may leave the blocks of the machine at one edge. The report of the column committee states:

> The Bureau found that the heads and bases of the columns were not planed truly parallel, and they were then scraped at the Bureau to bring the bearing surfaces true. The channels making up the columns were also found to be not in complete and full bearing on the base plates, and, owing to the details of the ends, this could not be corrected after fabrication. The bearings, however, usually closed into full contact after the application of a loading of from 1,000 to 5,000 pounds per square inch.

It was also found that the points of inflexion were not far from the ends, which proved that the ends were not truly fixed.

(2) The material may have been overstressed in tension before being put into the column, as by cold straightening. This would lower the elastic limit for compression, and so might greatly reduce the compressive strength. Baker[1] found that solid mild steel rods with a length of 30 diameters, which, when annealed, carried in compression 14.5 tons per square inch, when previously cold straightened carried 11.8 tons, and when previously stretched 10 per cent carried 12.6 tons.

(3) Small initial bends may much reduce the strength.

(4) Initial stresses may do the same.

(5) There may be errors in setting the column in the machine.

(6) Differences in homogeneity of material may affect the results. The experiments of Christie and others show that a column may be stronger if eccentrically loaded than when centrally loaded.

(7) The actual sections cannot be accurately found by calipering. In the tests of the column committee the sections were "found by carefully weighing the channels, as it was found impossible to get very accurate results by calipering, or even by using the planimeter on imprints of the ends of the sections."

(8) Extensometer measurements, though made with instruments reading to $\frac{1}{50{,}000}$ inch, are most unsatisfactory. The report of the A.R.E.A. committee states, for instance, that in the test of a column from the New York Central Railway, taking E as 29,500,000, "it appears that the stresses in the column corresponding to the 10-inch gage measurements are almost all higher than the average load on the column," which of course is impossible, as strain measurements were taken all around the section. It also appeared, in the same test, that the results of tests of lattice bars were "very erratic, some of the greatest changes occurring with the smaller loads" and that in these lattice bars "adjacent bars both carry compressive stresses in most cases, and that it is rather exceptional where one bar showed compression and the adjacent one showed tension, as might be expected, at least at the higher loads when appreciable deflection commenced." If the lattice bars show the same kind of stress it indicates that the column is not tending to deflect as a whole but that the two parts are bowing out or bowing in, which is quite possible.

These remarks will perhaps justify to the reader the writer's distrust of the results of many tests (see Art. 28 and also Chap. XXIV).

31. Curves of Tests.—Many results of tests have been plotted, and curves representing column formulae drawn. The paper by Moncrieff (see *ante*) has

[1] *Proc. I.C.E.*, vol. 92, p. 44.

many such charts, and in the *Proc. A.R.E.A.*, vol. 21, p. 530, 1920, Prof. Turneaure gives a very able discussion of column tests and formulae. The reader is referred to these. Only one set of tests will here be given, namely, Mr. Christie's tests on round-end wrought-iron tees and angles (Fig. 395). The Euler curve with $E = 27{,}000{,}000$ is drawn, and fits the results for high slenderness ratios

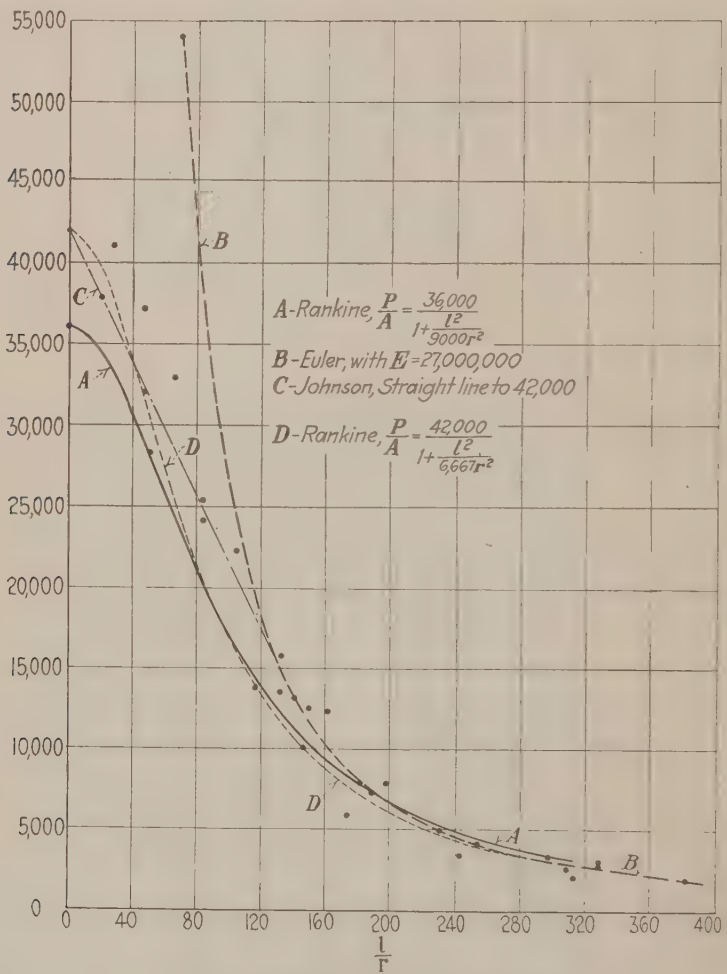


FIG. 395.—Christie's Tests, wrought-iron columns.

quite well, but not those for low ratios. The Johnson straight line tangent, to the Euler curve, is shown, and two Rankine curves. The object of this figure is to show that a Rankine curve can be found that will fit a series of tests reasonably well, and also to show the general shape of this curve.

Figure 396 shows a number of formulae which have been used, with the plotted curves.

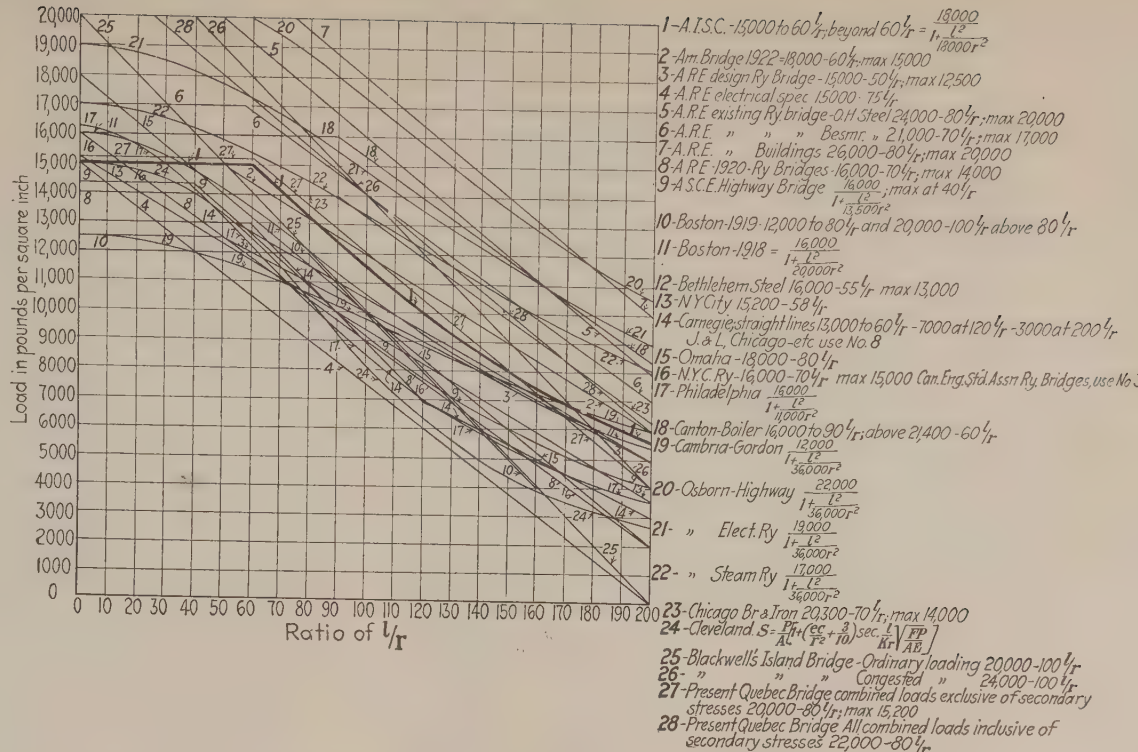

Load in pounds per square inch
Ratio of l/r
1-A.I.S.C.-15,000 to 60 l/r; beyond 60 l/r = 18,000 / (1 + l²/18000r²)
2-Am. Bridge 1922=18,000-60 l/r; max 15,000
3-A.R.E. design Ry. Bridge-15,000-50 l/r; max 12,500
4-A.R.E. electrical spec. 15,000-75 l/r
5-A.R.E. existing Ry. bridge-O.H. Steel 24,000-80 l/r; max 20,000
6-A.R.E. " " " Besmr. " 21,000-70 l/r; max 17,000
7-A.R.E. " Buildings 26,000-80 l/r; max 20,000
8-A.R.E.-1920-Ry. Bridges-16,000-70 l/r; max 14,000
9-A.S.C.E. Highway Bridge 16,000 / (1 + l²/13,500r²); max at 40 l/r
10-Boston-1919-12,000 to 80 l/r and 20,000-100 l/r above 80 l/r
11-Boston-1918 = 16,000 / (1 + l²/20,000r²)
12-Bethlehem Steel 16,000-55 l/r max 13,000
13-N.Y. City 15,200-58 l/r
14-Carnegie; straight lines 13,000 to 60 l/r -7000 at 120 l/r -3000 at 200 l/r
J. & L. Chicago-etc. use No. 8
15-Omaha-18,000-80 l/r
16-N.Y.C. Ry.-16,000-70 l/r max 15,000 Can. Eng. Std. Assn. Ry. Bridges, use No. 3
17-Philadelphia 16,000 / (1 + l²/11,000r²)
18-Canton-Boiler 16,000 to 90 l/r; above 21,400-60 l/r
19-Cambria-Gordon 12,000 / (1 + l²/36,000r²)
20-Osborn-Highway 22,000 / (1 + l²/36,000r²)
21- " Elect. Ry. 19,000 / (1 + l²/36,000r²)
22- " Steam Ry. 17,000 / (1 + l²/36,000r²)
23-Chicago Br. & Iron 20,300-70 l/r; max 14,000
24-Cleveland S = P/A [1 + (ec/r² + 3/10) sec (l/2r)√(P/AE)]
25-Blackwell's Island Bridge-Ordinary loading 20,000-100 l/r
26- " " " Congested " 24,000-100 l/r
27-Present Quebec Bridge combined loads exclusive of secondary stresses 20,000-80 l/r; max 15,200
28-Present Quebec Bridge All combined loads inclusive of secondary stresses 22,000-80 l/r

FIG. 396.—Column formulae and diagrams. (*Am. Inst. of Steel Construction.* L. H. *Miller*, Jan. 7, 1924.)

Rankine's formula, if adopted, would be used with a table which would give $\frac{P}{A}$ for any value of $\frac{l}{r}$. Figure 397 shows a graphical method of using Rankine's formula. Figures 396 and 397 have been prepared by L. H. Miller, Secretary of the American Institute of Steel Construction, which has adopted Formula 1 *for buildings*.

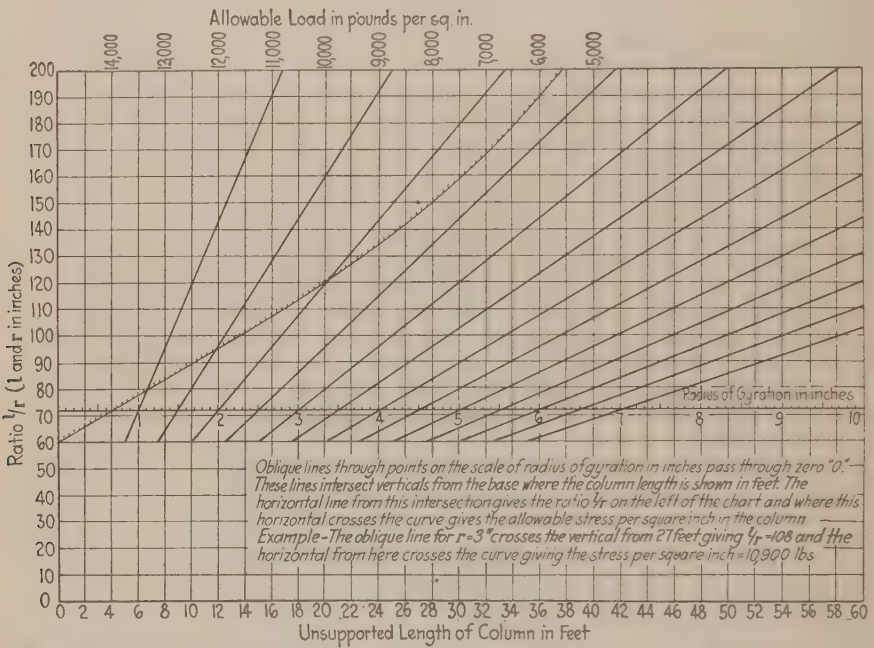

Allowable Load in pounds per sq. in.
14,000
13,000
12,000
11,000
10,000
9,000
8,000
7,000
6,000
5,000
Ratio l/r (l and r in inches)
Radius of Gyration in inches
Oblique lines through points on the scale of radius of gyration in inches pass through zero "0." These lines intersect verticals from the base where the column length is shown in feet. The horizontal line from this intersection gives the ratio l/r on the left of the chart and where this horizontal crosses the curve gives the allowable stress per square inch in the column
Example-The oblique line for r=3" crosses the vertical from 27 feet giving l/r=108 and the horizontal from here crosses the curve giving the stress per square inch=10,900 lbs
Unsupported Length of Column in Feet

FIG. 397.—Diagram for columns. (*Am. Inst. of Steel Construction. L. H. Miller*, Jan. 7, 1924.)

PRACTICAL FORMULAE

32. Steel Columns.—For buildings, the formula

$$\frac{P}{A} = \frac{18{,}000}{1 + \frac{l^2}{18{,}000r^2}} \tag{37}$$

with a maximum of 15,000, has been adopted by the A.I.S.C., and allows a factor of safety of about 1.8 on the ultimate as shown by the tests of the A.S.C.E. Committee (see Art. 29). There are seldom pin-ended columns in buildings.

It was formerly customary to use different constants for pin ends and fixed ends, but of late it has become recognized by many engineers that it is sufficient to use one constant for all columns.

For bridges, the straight line Formula (36), specified by the A.R.E.A., may be used; or the Rankine formula

$$\frac{P}{A} = \frac{15{,}000}{1 + \frac{l^2}{20{,}000r^2}} \tag{38}$$

may be used; or 18,000 may be used instead of 20,000. The Rankine formula recommended by the A.S.C.E. Committee on Specifications for Bridge Design and Construction (1922) is

$$\frac{P}{A} = \frac{16{,}000}{1 + \dfrac{l^2}{13{,}500r^2}}. \tag{39}$$

Mention has been made of the fact that some engineers, including Mr. G. Lindenthal, maintain that rivet holes should be deducted in compression pieces as well as in tension pieces, because, owing to shrinkage of the hot rivet, the cold rivet will not fill the hole, but will have a small vacant ring around it. It may be, however, that, in driving, a circumferential compression is produced, which causes an initial radial compressive strain, and that the shrinkage in cooling merely reduces this, and leaves the rivet still filling the hole. Certainly, many well-made rivets have been cut through rivet and plate, and no vacant space around the rivet is discoverable with the eye or with an ordinary magnifying glass. Of course, there may be some defective rivets which do not fill the holes. It is not customary, however, to deduct rivets in columns, and the writer does not consider it necessary.

33. Wooden Columns.—Wooden columns are almost always rectangular or round, with flat or fixed ends. They are subject to considerable eccentricity, due to the nature of the connections, and this eccentricity is seldom or never known, even approximately. Such columns are subject, not only to the uncertainties involved in columns, but to the great variations in quality of timber, so that they require large factors of safety.

In view of these uncertainties, it is perhaps more justifiable in wood than in steel to use an entirely empirical formula, such as the straight line formula, especially as wooden columns having a large slenderness ratio are uncommon; yet even here the Rankine formula appears to the writer preferable. The Gordon formula is, *for ultimate strength,*

$$\frac{P}{A} = \frac{f_c}{1 + \dfrac{l^2}{250d^2}} = \frac{f_c}{1 + c_1 \dfrac{l^2}{d^2}} \tag{40}$$

in which d is the least dimension; and with a rectangular section, since $r^2 = \frac{d^2}{12}$, Rankine's formula is

$$\frac{P}{A} = \frac{f_c}{1 + \dfrac{l_2}{3{,}000r^2}} = \frac{f_c}{1 + c \dfrac{l^2}{r^2}}. \tag{41}$$

These values of the constants were originally given for columns with fixed ends. But by Art. 17, in order to agree rationally with Euler's formula for large slenderness ratios, the constant c for hinged ends should be $\frac{f_b}{\pi^2 E}$, and taking $f_b = 5{,}000$ and $E = 1{,}500{,}000$, $c = \frac{1}{2{,}961}$, *so that the above formulae should be for hinged ends, and for fixed ends the constant in eq.* 21, *should be* 12,000.

Boyd's tests[1] showed these last constants to be reasonably correct, with f_c = 5,200 for spruce.

The Bridge Specifications of the Canadian Department of Railways and Canals (1908, last) specifies the formulae for *working stresses:*

$$\frac{P}{A} = \frac{1{,}200}{1 + \frac{l^2}{1{,}000d^2}} \text{ for southern yellow pine and Douglas fir,} \tag{42}$$

$$\frac{P}{A} = \frac{1{,}000}{1 + \frac{l^2}{1{,}000d^2}} \text{ for white oak,} \tag{43}$$

$$\frac{P}{A} = \frac{800}{1 + \frac{l^2}{1{,}000d^2}} \text{ for white pine and spruce.} \tag{44}$$

The Massachusetts Department of Public Utilities (1922) specifies

$$\frac{P}{A} = 1{,}000 - 10\frac{l}{d} \text{ for longleaf yellow pine.} \tag{45}$$

The A.R.E.A. specifies

$$\frac{P}{A} = f_c\left(1 - \frac{l}{60d}\right) \tag{46}$$

the value of f_c varying with the timber, from 1,300 for longleaf pine to 800 for Norway pine.

The above formulae are for static loads, and impact is to be added to live loads to reduce them to static loads.

The Division of Forestry of the Department of Agriculture gave in 1896 the formula

$$\frac{P}{A} = \frac{f_c}{1 + \frac{l^2}{d^2}\left(\frac{1}{700 + 15\frac{l}{d}}\right)}. \tag{47}$$

This is still given in some of the handbooks for the use of timber.

34. Cast-iron Columns.—The first important experiments on cast-iron columns were made by Eaton Hodgkinson, an English engineer, and are reported in the *Philosophical Transactions of the Royal Society of London*, for 1840 and 1857. They were on columns with hemispherical rounded ends, and on columns with flat ends, of various shapes, but generally solid or hollow cylindrical, or solid square.[2] Hodgkinson deduced the following from his experiments on round solid or hollow columns, P being the ultimate strength, h_1 the external diameter and h_0 the internal diameter, and c a constant depending upon the conditions:

When $l \geq 30h_1$;

For solid columns; $P = c\frac{h_1^{3.6}}{l^{1.7}}$ (48)

For hollow columns; $P = c\frac{h_1^{3.6} - h_0^{3.6}}{l^{1.7}}$. (49)

[1] J. E. Boyd, on "Strength of spruce struts," *Technologic Paper* 152, U. S. Bureau of Standards.

[2] See also, for these experiments, Moncrieff on "The practical column," in *Trans. Am. Soc. C. E.*, Mch., 1900.

The values of c were given as follows:

Solid columns, round ends;	33,376 pounds
Solid columns, flat ends;	98,918 pounds
Hollow columns, round ends;	29,120 pounds
Hollow columns, flat ends;	99,232 pounds

When $l < 30\ h_1$;
let P be the value given by the preceding formulae,
let P_1 be the strength of a short block = 109,760.4; then the ultimate strength P_2 is

$$P_2 = \frac{PP_1}{P + \frac{3}{4}P_1}. \tag{50}$$

These formulae are empirical, but have been much used. At present, however, the Rankine and the straight line formulae are used, and, as the reader will anticipate, the writer favors the former.

Cast iron is not considered reliable enough for use in important structures, especially those exposed to impact, on account of its brittleness and its low resilience (see Chap. XXI) or power of resisting shocks, and its uncertain quality, danger of hidden defects, initial stresses, and the liability of unequal thickness of hollow columns due to "floating" of the core. In buildings, however, where the load is almost all static, it is still much used.

A relatively large factor of safety is necessary. As compared with wood, cast iron occupies less space, and it is less expensive than steel, and resists fire better.

According to the building laws, the straight-line formula used is either

$\frac{P}{A} = 11{,}300 - 30\frac{l}{r}$ { Bridgeport, District of Columbia, New Orleans, Portland, Me.

or $\frac{P}{A} = 9{,}000 - 40\frac{l}{r}$ { Boston, Minneapolis, New York, New Haven, and the National Fire Underwriters

or $\frac{P}{A} = 10{,}000 - 60\frac{l}{r}$ (Chicago, Milwaukee, Omaha, St. Louis)

while Rankine's formula is of the form

$$\frac{P}{A} = \frac{14{,}000}{2 + \frac{l^2}{600d^2}}$$ (d = least dimension) (Buffalo);

or for hollow square columns $\frac{P}{A} = \frac{10{,}000}{1 + \frac{l^2}{1{,}067d^2}}$

and for hollow round columns $\frac{P}{A} = \frac{10{,}000}{1 + \frac{l^2}{800d^2}}$

and for other hollow columns $\frac{P}{A} = \frac{10{,}000}{1 + \frac{l^2}{6{,}400r^2}}$

where d is the least dimension, and r the least radius of gyration (Cincinnati, Louisville):

$$\frac{P}{A} = \frac{10{,}000}{1 + \frac{l^2}{800d^2}} \text{ (Detroit)}$$

$$\frac{P}{A} = \frac{8{,}500}{1 + \frac{l^2}{11{,}000r^2}} \text{ (Providence)}$$

and other forms. The minimum thickness of hollow columns is generally limited to $\frac{3}{4}$ inch, the diameter to be over 5 or 6 inches, $\frac{l}{r}$ to be not over 60 or 70.

CHAPTER XVIII

HOLLOW CYLINDERS AND SPHERES

1. Hollow cylinders, and sometimes spheres, are in common use, exposed to pressure from the inside or the outside, as in boilers, pipes, and tanks. If the thickness of the containing wall is so thin that the stress may be considered uniform throughout the thickness, the theory is very simple; if so thick that the variation must be taken into account, the theory is more complicated.

(*a*) THIN VESSELS SUBJECT TO INTERNAL PRESSURE

2. The internal pressure, whether of a liquid or a gas, is a fluid pressure. The following principles are fundamental:

(*a*) Fluid pressure always acts normally to the surface against which it presses.

(*b*) If the fluid pressure on any surface be resolved into two components in any two rectangular directions, either of these will be the normal pressure per square unit of surface multiplied by the projection of the surface on a plane perpendicular to the direction chosen; *i.e.*, the horizontal component in any given horizontal direction will be the normal intensity multiplied by the vertical projection of the surface on a plane normal to that direction; and the same for the vertical component.

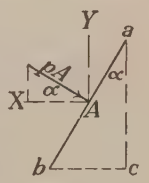


FIG. 398.

This is proved as follows: let ab (Fig. 398) be any surface of area A perpendicular to the paper, exposed to a normal pressure of intensity p. The total pressure will be pA. Resolve this into two components along any rectangular axes X and Y. These components will be

$$\text{Along } X;\ pA\cos\alpha = p\cdot ac$$
$$\text{Along } Y;\ pA\sin\alpha = p\cdot bc$$

ac and bc are the projections of ab perpendicular to X and Y respectively.

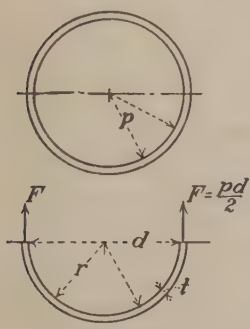


FIG. 399.

3. Hollow Cylinders.—Let Fig. 399 represent a ring of a cylinder, one unit in length perpendicular to the paper exposed to a uniform normal pressure of intensity p from the inside. Then the component, perpendicular to any diameter, of the pressure on a semi-circle, by the preceding article, which must be resisted by the two equal forces F, is pd, if d is the internal diameter, hence

$$F = \frac{pd}{2} = pr. \qquad (1)$$

This force is the *hoop tension*, or the uniform tension on any section of the ring whose area is t, the thickness of the shell, since the length is unity. Hence the intensity of hoop tension is

$$f_t = \frac{pr}{t} \qquad (2)$$

or

$$t = \frac{pr}{f_t}$$

if for f_t is substituted the allowable tensile stress.

This is but an example of the general principle that if a thin curved piece, exposed to uniform normal stress on every section, is exposed to normal pressure, the stress in the piece at any point is $F = pr$, if p is the intensity of the normal pressure at that point, and r the radius of curvature.

This simple formula is all that is needed for the computation of stresses in *thin* cylinders such as water pipes, boilers, and tanks, aside from the computation of the riveted joints.

In a boiler, there is a head at each end consisting of a plate, either flat or stayed in some way. The total pressure on the head is $\pi r^2 p$, and this is resisted by a *longitudinal* tension in the shell, uniformly distributed over an area $2\pi rt$, hence the longitudinal tension is, per square inch

$$f'_t = \frac{\pi r^2 p}{2\pi rt} = \frac{pr}{2t} \tag{3}$$

or one-half the intensity of hoop tension.

The head itself is a circular plate firmly riveted to the shell all around its circumference, and the stresses in it are very complicated, and belong in another chapter of this work.

4. Tanks.—Tanks for holding water, oil, molasses, and other liquids are generally circular cylinders, having either flat bottoms resting on a suitable foundation or on a tower, or curved bottoms of various shapes. Only cylindrical tanks are here considered. Standpipes for water-works systems come under this head, and are sometimes of great height and comparatively small diameter. Tanks for storing oil and other liquids, where storage and not pressure is the object, are generally lower and of greater diameter. Gas holders also belong under this heading.

Such tanks for liquids are exposed to a hydrostatic pressure varying in intensity from zero at the surface of the liquid to a maximum at the base. They differ from boilers in that the latter are exposed to a uniform pressure throughout, while each narrow horizontal ring of the tank is exposed to a practically uniform outward pressure; boilers may therefore have a shell of uniform thickness, while tanks and standpipes are composed of horizontal rings of varying thickness, increasing from the top downward.

At any depth h below the top, or highest surface of the liquid, if w is the weight of the liquid per cubic foot, the hydrostatic pressure will be wh per square foot, and the hoop tension will therefore be whr, and the required thickness in inches, if h and r are in feet, will be

$$t = \frac{wh}{144} \cdot \frac{12r}{f_t} = \frac{whr}{12f_t}. \tag{4}$$

5. Joints.—A tank or boiler must generally be made of a number of separate rings, lapped over each other, or butted, and riveted at the transverse or circumferential joints. In a boiler, such a joint has to carry the longitudinal tension. In a tank it has to carry the weight of the rings above and of the roof, if there is a

roof supported on the shell, and also, perhaps, a certain hydrostatic pressure on the surface of ends of rings lapped over other rings.

Each ring is generally composed of a number of plates connected by longitudinal joints.

Such joints have been sufficiently treated in the chapter on Riveting, and need not be discussed here, except to say that a plate and joint in a tank or boiler under normal pressure, though curved, is precisely the same as if plane, structurally and statically, for it is the hydrostatic pressure which requires the curved form. Such a joint is therefore the same as a joint between two plane plates exposed to a load equal to the hoop tension.

On account of the weakening of the plates by the joints, the values of the thickness of shell plates should always be, practically, greater than the value given by the preceding formulae.

6. Design.—There are many interesting and important points to be discussed regarding the design of tanks, aside from the joints, and these will be taken up in a later volume.

7. Spherical Shells.—A thin spherical shell of internal diameter d, subjected to an internal pressure of intensity p, will have the resultant pressure on a hemisphere equal to $\pi r^2 p$. This is resisted by the tension on a section $2\pi rt$, hence

$$f_t = \frac{\pi r^2 p}{2\pi rt} = \frac{pr}{2t} \tag{5}$$

or the same as the longitudinal tension intensity in a cylinder of equal thickness, radius, and pressure.

8. Thin Cylinders and Shells under External Pressure.—If the external pressure is normal and of uniform intensity, the same formulae apply as for internal pressure, except that d and r are here outside dimensions. The ring or shell has a hoop compression instead of a hoop tension, and will fail by collapsing.

Theoretically, if a cylindrical ring were of exactly uniform thickness, and absolutely homogeneous and perfect in shape, it should never fail under such compression. Professor Bridgman, of Harvard University, has experimented upon cylinders of ductile material, having considerable thickness, under very high pressures, and has found that they would not fail, but would close up tight, forming a rod instead of a cylinder. With thin cylinders, however, and since perfect shape and homogeneity never occur, such cylinders wrinkle and then collapse.

The principal case of external pressure on cylinders is in boiler tubes. The formula for ultimate strength is empirical, since, as explained, they should never fail if the conditions were ideal as assumed. Fairbairn's formula for wrought-iron tubes is

$$R = \text{collapsing pressure in pounds per square inch} = \frac{9{,}600{,}000t^{2.18}}{ld} \tag{6}$$

where l is the length and d the diameter of tube, all in inches. Fairbairn's experiments on wrought-iron tubes gave the following results for the collapsing pressure:

Length, inches	37	60	61
Diameter, inches	9	24.5	18.75
Thickness, inches	0.14	0.125	0.25
Unit pressure, pounds per square inch	378	125	420

Fairbairn's[1] formula agrees closely with experiment in some cases, but not in others. Hutton's formula, which is considered more reliable by many engineers, is

$$R = \text{collapsing pressure, pounds per square inch} = \frac{Ct^2}{d\sqrt{l}} \qquad (6a)$$

where d is the diameter and l the length, both in inches, t the thickness in *thirty-seconds* of an inch, and C a constant given by Hutton as 600 for iron and 660 for steel. If t_1 is the thickness in inches, this would become

$$R = \frac{614{,}400t_1^2}{d\sqrt{l}} \text{ for iron} \qquad (6b)$$

$$R = \frac{675{,}840t_1^2}{d\sqrt{l}} \text{ for steel.} \qquad (6c)$$

9. Oval Cylinders.—If the cylinder is not circular, but oval, the formulae that have been given do not apply. These cases come under the head of curved beams, and may be treated by methods that will be explained in the chapters on Curved Beams and on Arches in a later volume.

10. Cylinders subject to external pressure which is not uniform and normal also come under the head of Curved Beams. Such are sewer or culvert pipes exposed to earth pressure or perhaps to concentrated loads. These cases are important for the engineer, and will be treated in appropriate chapters.

11. Principal Stresses and Strains in Cylinders.—As shown above, at any point in the shell of a thin cylinder exposed to internal pressure, and closed at the ends, there is a longitudinal tension of intensity

$$n_y = \frac{pr}{2t}$$

and also a circumferential tension of intensity

$$n_x = \frac{pr}{t}.$$

The first of these acts on a plane at right angles to the axis of the cylinder, the latter on a plane containing the axis.

There is also a compression on a plane perpendicular to the radius, varying in intensity from p at the interior surface to zero at the outer surface if there is no exterior pressure. This may be neglected with reference to the other two, if the thickness of the shell is small.

There is no shearing stress on either of these planes, hence *the above stresses are the principal stresses.*

The circumferential unit *strain* will be

$$e_x = \frac{n_x}{E} - \frac{n_y}{mE}.$$

The longitudinal unit *strain* will be

$$e_y = \frac{n_y}{E} - \frac{n_x}{mE}$$

where $\frac{1}{m}$ is Poisson's ratio.

12. Thin Hollow Cylinder with Internal or External Pressure, and with Torsion.—If a hollow cylinder exposed to internal pressure be exposed also to

[1] For further discussion of boilers, see Peabody and Miller, "Steam boilers," (Wiley, 1913); Haven and Swett, "Design of boilers and pressure vessels," (Wiley, 1915).

direct tension or compression, and also to torsion, there will be, in addition to longitudinal tension (or compression) and circumferential tension, a shearing on the plane on which the longitudinal tension acts, and, in consequence, also an equal shear on a longitudinal plane. The principal stresses will therefore be inclined to both of these. Using the notation in Fig. 400, the principal stresses will be

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{n_x + n_y}{2} \pm \frac{1}{2}\sqrt{(n_y - n_x)^2 + 4s_y^2}.$$

By varying the internal and external pressure, longitudinal stress, and torsion, it is possible to get almost any combination of principal stresses. With pure torsion and no radial pressure we have principal stresses equal and opposite; with pure longitudinal tension and no torsion or radial pressure the principal stresses are tension and zero; but we cannot, with longitudinal tension and torsion only, get both principal stresses tension. Similarly, with longitudinal compression combined with torsion we may get principal stresses equal and opposite (pure torsion), or one compression and the other zero; but not both compression. With no torsion, but with varying longitudinal force combined with internal pressure, we may get variations of principal stresses

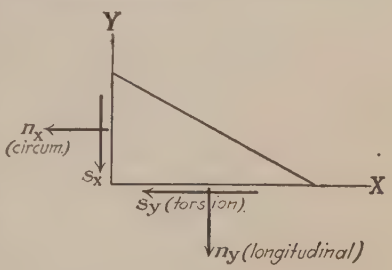


FIG. 400.

from n_x tension and n_y zero
to n_x tension and n_y an equal tension or compression.

With external pressure and axial force we may get
from n_x compression and n_y zero
to n_x compression and n_y an equal compression or tension.

For these reasons, some experimenters have selected such experiments as the best means of studying the effects of varying ratios of principal stress.

(*b*) THICK HOLLOW CYLINDERS

13. If the cylinder is thick, the hoop tension is not distributed equally over the thickness, but is very different at the outside and inside surfaces. This is of great importance in the case of heavy guns, which are thick hollow cylinders exposed to very great internal pressures.

14. Lamé's Theory for Thick Hollow Cylinders.[1]—Figure 401 is a section of a thick cylinder one unit in length. Consider an infinitesimal rectangular particle distant x from the center, forming part of the ring with thickness dx shown in Fig. 402. The radial compressive stress per unit on the inside of this ring will be p_x; on the outside it will be $p_x + dp_x$. On the radial sides the tensile stress will be the same all the way around the ring, p_y; the hoop tension on the ring is $p_y dx$, and from Eq. (1),

$$2p_y \cdot dx = 2x \cdot p_x - 2(x + dx)(p_x + dp_x)$$

or

$$p_y dx = -p_x dx - x dp_x \tag{7}$$

$$p_y = -p_x - x\frac{dp_x}{dx} = -\frac{d(xp_x)}{dx}. \tag{8}$$

[1] This demonstration closely follows that in MORLEY'S "Strength of Materials."

It is now assumed that the longitudinal unit stress f_1, which is assumed to be tension, is constant over the section. This is the first assumption made. It seems reasonable, but is not necessarily true.

The unit longitudinal elongation at the point considered will then be, if $\frac{1}{m}$ is Poisson's ratio,

$$e = \frac{f_1}{E} + \frac{p_x}{mE} - \frac{p_y}{mE} = \frac{1}{E}\left(f_1 - \frac{p_y - p_x}{m}\right). \tag{9}$$

It is now assumed that this elongation is constant, or independent of x. *This is a pure assumption.* It means that plane sections remain plane and unchanged in direction. If they did not remain plane, it would follow that there were shearing stresses on a transverse section. If the ends of the cylinder are closed by heads, since the diameter tends to increase under internal pressure, there must be radial tensile stresses in the heads, and it seems as if there could be shearing stresses on

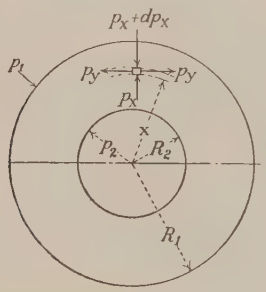


Fig. 401.

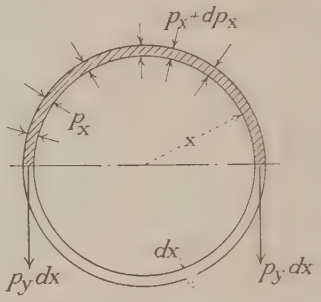


Fig. 402.

parts of the transverse section of the cylinder, though these must all balance, and the resultant shearing force in any direction on a transverse section, must be zero. This assumption therefore, does not seem to be necessarily true, at all events near the heads, but it is made. It means, since E, f_1, and m are constants,

$$p_y - p_x = \text{constant} = 2a \text{ (say)}. \tag{10}$$

Hence, from (8)

$$-2p_x - x\frac{dp_x}{dx} = 2a$$

whence

$$-\frac{2dx}{x} = \frac{dp_x}{p_x + a}.$$

Integrating,

$$-\log x^2 + C = \log (p_x + a)$$

$$p_x + a = \frac{b}{x^2}. \tag{11}$$

where b is a constant.

or

$$p_x = \frac{b}{x^2} - a. \tag{12}$$

From (10)

$$p_y - p_x = p_y - \frac{b}{x^2} + a = 2a$$

$$p_y = \frac{b}{x^2} + a. \tag{13}$$

From Eqs. (12) and (13) the values of p_x and p_y may be found. The conditions which determine a and b are

$$\text{for } x = R_1;\ p_x = p_1 = \text{external pressure};$$
$$\text{for } x = R_2;\ p_x = p_2 = \text{internal pressure};$$

or
$$p_1 = \frac{b}{R_1^2} - a;\ p_2 = \frac{b}{R_2^2} - a.$$

From this,

$$b = \frac{R_1^2 R_2^2 (p_2 - p_1)}{R_1^2 - R_2^2};\ a = \frac{p_2 R_2^2 - p_1 R_1^2}{R_1^2 - R_2^2}. \tag{14}$$

These values, substituted in Eqs. (12) and (13), give p_x and p_y.

Case I. Internal Pressure Only.—In this case $p_1 = 0$, hence

$$p_x \text{ (compression)} = \frac{p_2 R_2^2}{R_1^2 - R_2^2}\left(\frac{R_1^2}{x^2} - 1\right) \tag{15}$$

$$p_y \text{ (tension)} = \frac{p_2 R_2^2}{R_1^2 - R_2^2}\left(\frac{R_1^2}{x^2} + 1\right). \tag{16}$$

The circumferential tension p_y and the radial compression p_x both have their maximum values for $x = R_2$, or at the inner surface, and

$$\text{Max. } p_x = p_2$$
$$\text{Max. } p_y = p_2 \frac{R_1^2 + R_2^2}{R_1^2 - R_2^2}. \tag{17}$$

Taking, for example, $R_2 = 10$ inches, $R_1 = 20$ inches, the values of $\frac{p_x}{p_2}$ and $\frac{p_y}{p_2}$ are as follow:

$x =$	10	11	12	13	14	15	16	17	18	19	20
$\frac{p_x}{p_2} =$	1.0	0.768	0.592	0.456	0.347	0.259	0.187	0.128	0.078	0.036	0
$\frac{p_y}{p_2} =$	1.667	1.434	1.259	1.122	1.014	0.926	0.854	0.795	0.745	0.703	0.667
$\frac{eE}{p_2} =$	1.917	1.626	1.407	1.236	1.101	0.991	0.901	0.827	0.764	0.712	0.667

The maximum unit elongation is

$$e = \frac{p_y}{E} + \frac{p_x}{mE}$$

and the equivalent single stress is

$$eE = p_y + \frac{p_x}{m}. \tag{18}$$

Calling $m = 4$, the values of $\frac{eE}{p_2}$ are given in the table.

Figure 403 shows the values of $\frac{p_x}{p_2}$ and $\frac{p_y}{p_2}$. If there is a direct axial stress, due to closed ends or to applied axial force, this would be a third principal stress.

By this theory the difference between p_x and p_y is numerically constant, and equal to $\frac{2p_2R_2^2}{R_1^2 - R_2^2}$ for any value of x; but since p_x is compression and p_y is tension, the algebraical difference is their numerical sum. Since the maximum shear intensity equals half the sum of the two principal stress intensities, p_x and p_y, taken algebraically, the shear is a maximum at the inside, like the principal stresses.

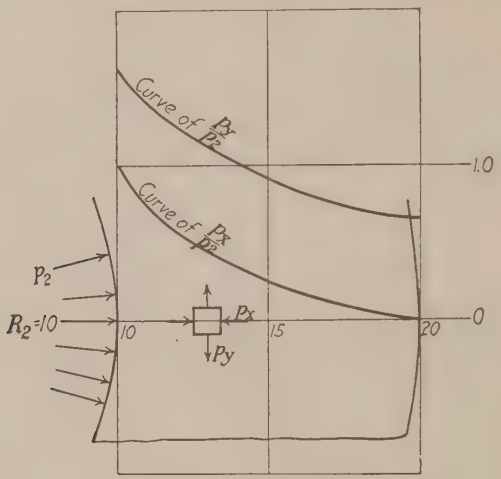


FIG. 403.

Case II. External Pressure Only.—In this case $p_2 = 0$, hence

$$p_x = \frac{p_1R_1^2}{R_1^2 - R_2^2}\left(1 - \frac{R_2^2}{x^2}\right) \tag{19}$$

$$p_y = -\frac{p_1R_1^2}{R_1^2 - R_2^2}\left(1 + \frac{R_2^2}{x^2}\right). \tag{20}$$

Here the circumferential stress p_y is compression, and its maximum value, when $x = R_2$, is

$$\text{Max. } p_y = -\frac{2p_1R_1^2}{R_1^2 - R_2^2}. \tag{21}$$

15. Applications.—These principles are applied in the construction of hydraulic cylinders exposed to large internal pressures, and also in heavy guns, which are thick cylinders exposed to very large internal pressures. It is desirable to counteract the large tension at the inside of a gun. This was done at one time by casting the tube on a core so as to chill the inside. The inside becoming solid while the outside was still liquid, the contraction of the latter produced an initial circumferential compression at the inside and an initial tension at the outside, so that under an internal pressure the stress would be more uniformly distributed than if there were no initial stresses. Of late a similar result has been accomplished by making the tube of two tubes, and shrinking the outer one on the inner one; or by winding an outer tube of wire around the inner tube, the wire being under regulated tension in such manner as to produce the desired initial stresses.

This subject does not concern the structural engineer, and will not be treated here. Formulae may be found in Morley's Strength of Materials, and in works on Ordnance. Thick hollow spherical shells are also treated there.

16. Rotating Cylinders or Discs.—If a hollow cylinder rotates about its longitudinal axis, or if a disc with a hole in it rotates about an axis through its center and perpendicular to it, the condition is not a static one, for each particle of weight wdV has a centripetal force acting toward the axis of rotation equal to $\frac{wdV}{g} \cdot \frac{v^2}{x}$, if dV is the volume of the particle, w its weight per unit of volume, v its velocity and x its radial distance from the axis. By d'Alembert's principle the case may be reduced to one of statics by applying to each particle a force equal and opposite to this. If the cylinder or disc is *thin* in a radial direction, with a thickness t and radius r, and if its length is unity, a particle having a unit length around the circumference will have a volume equal to t; and, if the angular velocity is α, the actual velocity will be αr, hence the condition will be the same as a thin cylinder exposed to an internal pressure of $\frac{wt\alpha^2 r}{g}$, and its hoop tension will be $\frac{wt\alpha^2 r^2}{g}$, and the stress intensity will be $\frac{w\alpha^2 r^2}{g}$. This will be the "centrifugal tension" in a belt running around a pulley.

If the ring is thick, the velocity of each particle will vary with the distance from the axis and it will not be close enough to consider that the velocity is uniform. The case will be similar to that of a thick cylinder. Formulae for this case are given in Morley's Strength of Materials. It is sufficient here to call attention to the way in which d'Alembert's principle is applied (see Chap. II, Art. 11).

17. Examples.—

(1) A tank containing molasses to a depth of 48 feet 9 inches is 90 feet in diameter and 50 feet high. If molasses weighs 88 pounds per cubic foot, and if the lower ring is $1\frac{1}{16}$ inch thick, what is the intensity of hoop tension at the bottom, not allowing for riveted joints?

(2) A standpipe for a water-works system is 200 feet high and 20 feet in diameter. How thick should the lower ring be if the allowable stress per square inch is 16,000 pounds and the efficiency of the vertical joints 75 per cent for tearing of plate?

(3) If the greatest allowable thickness of metal is $\frac{5}{8}$ inch, and efficiency of joint 80 per cent, how high could a standpipe 18 feet in diameter be if the allowable stress is 20,000 pounds per square inch?

(4) A cast-iron water pipe 48 inches in inside diameter is under a head of 200 feet. What should its thickness be if 1,500 pounds per square inch is allowed, considering it as a thin cylinder?

(5) A seamless tube 6 inches in diameter is exposed to an internal pressure of 200 pounds per square inch. What should its thickness be if a stress of 12,000 pounds per square inch is allowed?

(6) A cast-iron water pipe is $\frac{1}{2}$ inch thick, and 12 inches in diameter. What head of water will produce a stress of 1,500 pounds per square inch?

(7) A hydraulic cylinder is 12 inches in diameter inside, and is subjected to an internal pressure of 2,000 pounds per square inch. Find the necessary thickness if the maximum stress is not to exceed 3,000 pounds per square inch.

(8) A steam boiler 4 feet in diameter has a pressure of 200 pounds per square inch; find the necessary thickness if 10,000 pounds per square inch is the allowable stress, and the efficiency of the joints for tearing is 75 per cent.

CHAPTER XIX

INITIAL STRESSES IN METALS

1. Steel and other metals, as received and before being exposed to load, often have initial internal stresses of considerable magnitude. In such a case, since there is no outer load, the internal initial stresses on any section must balance. If there is tension over a certain part of the section, there must be equal compression over the remaining part. Further, in order that the moments may balance, there must be two tension areas and two compression areas, in order that the internal stresses may form two equal and opposite couples; or else, if there is but one tension and one compression area, the resultant tension must be in the same line with the resultant compression.

2. These initial stresses may reduce the capacity of the piece to resist load. If the load produces tension on the section, the initial tension is increased and the initial compression reduced, and the opposite if the load produces compression. Initial stresses are most important in columns, in which the yield point is the ultimate, for with high initial stresses the column may fail under a load much less than that for which it is supposed safe. A piece having initial stresses will act under load just as it would were there no initial stresses; except that the elastic limit may be reached sooner, and even under a uniformly distributed load the elastic limit will not be reached simultaneously in all parts of the section.

3. Cause of Initial Stresses.—Initial stresses are caused by *heat treatment* or by *mechanical treatment.*

A casting, in cooling, cools first at the outside, and the outside becomes solid while the interior is still liquid. As the liquid cools, it contracts, and thus brings the outside into compression while the inside is in tension. Generally speaking, the part which cools first will be in compression, and the part which cools last, in tension. Similarly, thin parts cool faster than thick parts, and the former are apt to be in compression and the latter in tension. Initial cooling stresses, however, are all liable to be modified by the stresses produced by mechanical work.

Processes of mechanical work, such as rolling, forging, hammering, wire drawing, and cold-straightening, all cause initial stresses, and this will depend upon the temperature at which the work is done. These stresses are due to the fact that the work done is not applied uniformly to every part of the piece. Cold-working particularly produces initial stresses. Even if work or heat treatment is applied uniformly to all parts, if the material is not homogeneous, initial stresses will be set up.

In cold-rolling, the metal is reduced in thickness between the rolls, and the material so squeezed must go somewhere, and must flow backward. The friction of the rolls on the surface of the metal hinders the backward flow of the metal at and near the surface, so that the backward flow must be greatest at the center of the thickness, and decrease toward both surfaces. After the rolling, therefore, the mid-thickness will be in compression, decreasing and becoming tension at the top

and bottom. The stresses will increase with the amount of reduction per pass. Howe and Groesbeck[1] have shown that if two plates are placed one on the other and the two cold-rolled together, they will curve away from each other in rolling, becoming curved in opposite directions. The stresses may be inferred from the amount of curvature.

In drawing through a die the opposite effect may be produced, the outer layers being in compression, the inner in tension; though it is sometimes stated that the outside is in tension and the inside in compression.[2]

While the distribution is as yet uncertain, there seems no doubt that these operations produce internal stresses. Cold-straightening has a similar effect, and will be referred to later.

4. Measurement of Initial Stresses.—There are two ways of measuring the initial stress; one is to set out an accurate measured length in the direction of the stress to be measured, then to cut out the piece containing this length, and then to measure it again: if it is less than before, there was initial tension, if greater, there was compression, and from the value of E the stress can be calculated. J. E. Howard has applied this method to structural shapes.

The other method is to cut or etch away a part of the piece, and from the change of length or shape of the part remaining to calculate the initial stress. Thus, with a tube or rod, if a thin outside annular ring is cut away, and the remainder increases in length, the part cut away was in tension. By cutting away successive rings, the distribution of stress may in some cases be found. In the case of a thin sheet a small part may be cut out, all but one surface covered with paraffine, and, with acid, one-fourth of the thickness etched away. If there were internal stresses, the remaining part will become curved, and the initial stresses computed from the curvature. This method has been used at the Bureau of Standards.[3]

5. Initial Stresses Due to Cold-straightening.—A great part of the pieces used in structures are bent in the process of fabrication, and must be straightened cold. This is called "gagging." This means exposing a piece to stresses above the elastic limit both in tension and compression, and giving it a permanent set or bend. If a line of rivet holes is punched in one leg of an angle or in one flange of a channel, that leg or flange will be stretched by the punching, and the piece will become curved, and must be straightened. In riveted lap joints, as in tanks, it is frequently necessary, in the shop or in erection, to bend the plates at the joints.

Such bending stresses a plate to or above the yield point in tension at one edge of a section, and above the yield point in compression at the opposite edge. When the bending force is removed, the fibers that have not been stressed above the elastic limit tend to resume their original lengths, but are prevented from doing so by the resistance of the fibers which have been stressed above the elastic limit. The result is that the fibers that were overstressed in the bending in tension will be left with an initial compression, and those overstressed in compression will be left with an initial tension. Having in mind the principles of Art. 1, it is clear that the distribution of stress will be somewhat as shown in

[1] *Proc. A.S.T.M.*, p. 31, 1920; also *Tech. Paper* 163, *Bureau of Standards.*
[2] MERICA and WOODWARD, *Proc. A.S.T.M.*, p. 167, 1918.
[3] *Tech. Paper* 82 of the *Bureau of Standards.*

Fig. 404. If the bending in straightening would produce theoretical stresses shown by AB, SA and S_1B being above the yield point, SD and S_1G representing the yield points, the fibers from E to D and from F to G will tend to stretch or contract, throwing more stress on adjoining fibers, and the distribution will be changed (see Chap. X, Art. 35). When the bending is discontinued and the set given, the central fibers will tend to return to their original lengths. Equilibrium will be established if there are an equal tension and compression T_1 and C_1, and an equal tension and compression T_2 and C_2 forming two equal couples. The exact condition must remain unknown, and the distribution will be only approximately as shown by $MNOP$. J. R. Worcester[1] has attempted to analyse this case, and while his assumptions are no doubt inaccurate they may be approximately true. He concludes that "in a bar which is quite straight and wholly innocent in appearance there may exist a compressive stress along

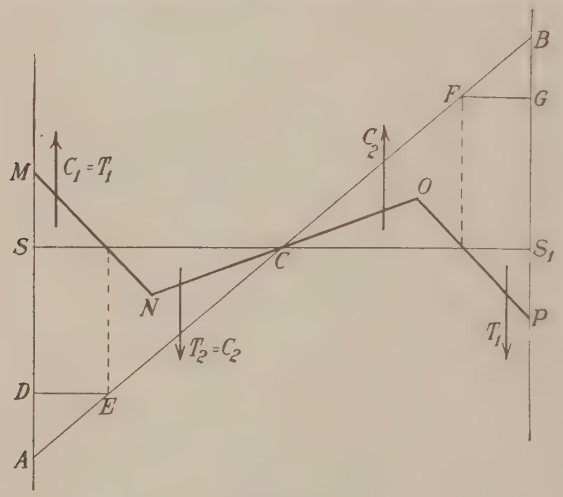


Fig. 404.

one edge of 15,000 pounds per square inch, while along the opposite edge there is a tensile fiber stress of an equal amount." This stress may not be wholly due to cold-straightening.

6. Magnitude of Initial Stresses.—Measurements have shown initial stresses even greater than the estimate of Mr. Worcester. Merica and Woodward state that cooling of heterogeneous materials may cause stresses of 16,000 pounds per square inch or more. They found in brass, by measurement, stresses of 18,000 and 25,000 pounds per square inch. Howard[2] has found, in steel, in one flange of an 8 inch I-beam a compression of 12,200, and at another point of the same beam a tension of 3,300, making a range of 15,500 pounds. In a plate he found at one edge a compression of 8,700 pounds and at the other a compression of 10,500, corresponding to a very different distribution from that of Fig. 404. A channel showed a range of 24,300 pounds. Evidently the combination of cooling stresses with those due to working may produce very complex conditions. After a piece

[1] "Initial Stresses in Structural Steel," *Jour. Assoc. Eng. Soc.*, p. 129, April, 1913.
[2] *Trans. Am. Soc. C. E.*, p. 879, Dec., 1921.

is put in service it may be subject to mechanical work that will produce internal stress, as, for instance, a railroad rail, in which the cold-rolling due to wheel pressures has this effect.[1] In rails, Mr. Howard has found a range of stress from a compression of 44,100 pounds per square inch at the top of the head to a tension of 15,300 pounds in the web, a total range of 59,400 pounds.[2]

The cold-rolling effect of a locomotive wheel on the top of a rail is different from that of the rolls in cold-rolling, and it seems that it should tend to cause longitudinal compression at the surface. The net result in a railroad rail is very complex. The cooling stresses should be compression around the surface and tension in the interior, but not uniform around the surface; for the thin web would cool faster than the thicker head and base, and the cooling of these would tend to cause compression all through the web, in the interior and on the surface. The gagging or cold-straightening would mainly affect the head and the base, causing tension in one and compression in the other, or tension on one side of the axis of symmetry and compression on the other, according to the direction of the original bend. Any rolling at too low a temperature, in manufacturing, since the rail is rolled on its side, would leave tension at the two sides of the head and base, and compression on and near the axis of symmetry, including the top surface of the head. The cold-rolling of the wheels would cause compression on the top of the head and tension at the interior. Howard found cooling stresses alone of 18,000 pounds in the head, and he states that cold-rolling stresses in the top of the head "reach or exceed 20,000 pounds per square inch, the interior elements of the head acquiring a state of tension."

If a round bar is suddenly quenched from a high temperature, the outer part will be left in compression, the inner in tension. Measurements have shown that a one-inch round brass bar quenched from 450° C. in water has an average initial compression on the outer half of 14,500 pounds per square inch.

7. Removal of Initial Stress.—Unless disturbed by heat treatment or mechanical work, initial stresses are permanent. They may be partially or perhaps wholly removed by annealing at a proper temperature and slow cooling, and this is the only practical method for steel. Annealing of bronze for ½ hour at 200° C. reduced the initial stress from about 37,000 pounds per square inch to about 6,000 pounds per square inch, and at 300° C. to about 3,000 pounds.[3] It is thus possible to nearly remove the initial stress at such a low temperature that the hardness is practically unaffected. In the case of brass rods a process of "springing" or cold bending has also been used.

8. "Corrosion Cracking" of Brass.—A singular phenomenon is shown by brass and some other alloys, which crack without load, primarily on account of initial stresses, but in general only when surface corrosion occurs. They will not crack if the surface is protected from corrosion, by oiling or otherwise, though variation of temperature, instead of corrosion, may also produce it. This is known as "corrosion cracking." Brass in which the surface is under initial compression does not crack in this way; and if there is surface tension it does not crack unless the initial tension exceeds about 25,000 pounds per square inch.[4] In the construc-

[1] HOWARD, "The internal strains in steel rails," *Proc. N. E. Railroad Club*, Jan. 8, 1918.

[2] "Report on the formation of transverse fissures in steel rails," Interstate Commerce Commission, 1923.

[3] *Proc. A.S.T.M.*, p. 174, 1918.

[4] *Id.*, p. 171.

tion of the Catskill aqueduct, for the water supply of New York, there were many failures of brass bolts from this cause. It is important to know what specifications and tests to impose The usual test has been the mercury salt solution test. The specifications of the New York Board of Water Supply (1915) require sheets and tubing less than ¼ inch thick to be "tested by immersion in a saturated solution of mercuric chloride for one hour and then kept under observation for two weeks." The International Aircraft Standards Board Specification for Naval Brass provides; "the rod shall stand an immersion in a mercurous nitrate solution containing 100 grams of mercurous nitrate and 13 cubic centimeters of nitric acid (specific gravity 1.42) for 15 minutes without cracking." The effect of the solution is that of corrosion, and if the metal resists the solution it is supposed that it will not crack. This result is not assured, however, for it will not crack anyway unless the initial stress reaches a certain limit, which Merica and Woodward place at 25,000 pounds per square inch, and any initial stress reduces by so much the capacity to carry load, so that even if the metal resists the solution it may still crack under a smaller load than it is supposed to carry. The solution test seems therefore reliable for material that is not to be exposed to stress, but not necessarily for material that is to be stressed. The condition of the surface has some effect, the cracks generally occurring at some surface defect. Polishing greatly reduces the liability to crack.

FIG. 405.—I-beam broken by a light hammer blow.

In view of the liability of initial stresses it is clear that the high ultimate strengths given for bronzes is likely to be misleading as to the available strength in service.

9. Initial Stresses in I-beams.—An I-beam of the usual dimensions has a comparatively thin web, and much thicker flanges. The thin web, on account also of its large surface, cools faster than the flanges, or at least faster than the interior of the flanges. This causes stresses which are probably greatest where

the web and flanges meet. If such a beam were cut in two longitudinally at the center of the height of the web, and the two parts allowed to cool separately, each would become curved, separating from each other toward the ends. Cooling as one beam, therefore, there should be transverse tensile stresses in the web. It is said that it was formerly not uncommon for beams to split horizontally at the ends, on account of such tension,[1] especially the old shapes of wrought-iron beams, which had thicker flanges than the modern steel shapes. The modern broad flange beams, in which the flanges are still thinner, are claimed to be still less subject to internal stresses.

10. The danger of initial stresses in metals is obvious. No doubt it explains some failures. It may also be a source of strength and increased safety, as, for instance, if cold-straightening or bending leaves tensile stress on one side and compression on the other side of a piece in which the loads cause the opposite stresses. Instances are not wanting in which a steel member has broken when dropped in unloading. Figure 405 shows a Bessemer steel beam which was broken in the shop when struck a light blow with a hammer. This may have been due to bad chemical composition, but initial stress is a factor that the engineer should bear in mind. It is one element in the factor of safety required.

That initial stresses may, and often do exist in considerable magnitude, is undeniable. Professor Upton says, "Cooling stresses within the material may be of the same magnitude as the extrinsic strength, and so reduce the available strength to nearly nothing."

It must not be supposed, however, that initial stresses always reduce the available strength. They may greatly increase it. If the initial stress, at a certain point, on a certain plane, is tension, and if the actual loads cause a tensile stress on the same plane, at the same point, then the initial stress reduces the available strength; but if the actual loads cause compression at that point, then the initial stress increases the available strength. If the initial stress causes tension at one edge of a section, and compression at the opposite edge, then, if the loads produce a bending moment which produces tension at the first edge and compression at the second, the available strength is reduced; but if the actual loads cause a bending moment which produces compression at the first edge, and tension at the second edge, the available strength is increased. In this way, initial stresses may be the ruin or the salvation of a structure, depending upon circumstances.

[1] PRITCHARD, "Faults in the theory of flexure," *Trans. Am. Soc. C. E.*, vol. 75, pp. 895–981, 1912.

CHAPTER XX

FLEXURE OF CURVED BARS

1. The bars in structures which are exposed to flexure are not always straight. The simplest case of a curved bar is a hook (Fig. 406); other curved bars are solid arches of steel or concrete.

The complete theory of such bars is quite complex, and will be deferred to a later volume of this work, where it will be given in detail. The complexity arises from the fact that the outer forces are not statically determined. In the arch with hinges at the ends, for instance, as shown in Fig. 407*a*, if there is a single load, the two reactions must act through the hinges and must meet on the line of action of the load, but there are no other statical conditions, and these may be satisfied by any two reactions which meet on the load and form in the force diagram a closed triangle with the load itself. In the arch without hinges (Fig. 407*b*) the points of application of the reactions are not known, and the case is still more indeterminate.

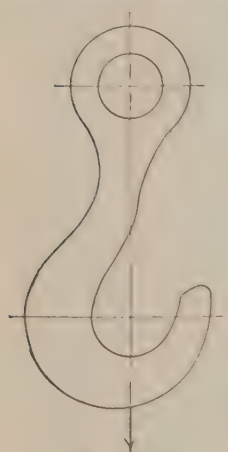


Fig. 406.

In case 407*a* there are four unknown quantities, namely, two components of each reaction, and there are but three statical equations.

In case 407*b* there are six unknown quantities, namely two components and the point of application of each reaction, and but three statical equations.

In the case of the hook, the load being given, there is of course no uncertainty about the reaction.

Assuming that the outer forces are all known, the resultant outer force on any cross-section may be found. It may then be resolved into normal and tangential components, the former being a direct force of tension or compression, acting at a known point of the cross-section, and the latter being a shearing force in the plane of the cross-section.

So far there is no difficulty if the outer forces are known.

2. Stresses Due to the Normal Outer Force.—The normal outer force acting on any section being known, and its point of application, then *if planar distribution of the stress is assumed*, the intensity of the normal stress at any point may be found by the same formulas which are used for straight beams and which have been discussed in Chap. IX. For, if the force acting on a given section is known, as well as the point in which it cuts the plane of the section, the distribution of its normal component is fixed if planar distribution of stress is assumed, that is, a distribution such that the intensity of the stress at a point is proportional to the distance of the point from some straight line in the plane of the section. Within the elastic limit, since intensity of stress varies as intensity of strain, a planar

distribution of stress means that plane sections remain plane only when one originally plane section remains plane and when the original plane sections are everywhere the same distance apart, *i.e.*, when the axis of the bar is straight.

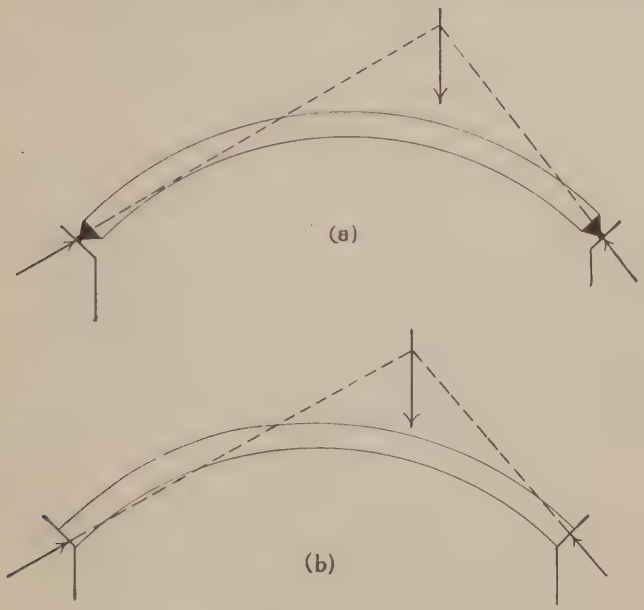


FIG. 407.

Thus, in Fig. 408, if the axis is straight, two parallel sections originally ab and cd, if they remain plane, would become respectively $a'b'$ and $c'd'$, and the strains would vary directly as the distances from the neutral axes through m and n; and

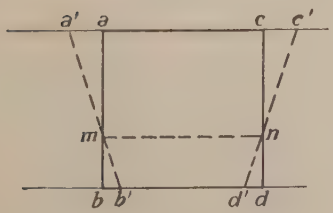


FIG. 408.

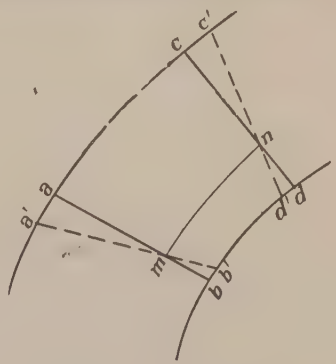


FIG. 409.

since every fiber has the same length originally, the strain per unit length, and therefore the stress intensity, varies in the same manner. Planar distribution of stress is here a necessary result, therefore, if plane sections remain plane.[1] If the

[1] The converse, however, is not true, as shown in Chap. X. That is, planar distribution of normal stress does not *necessarily* mean that plane sections remain plane.

axis of the bar is curved, however, (Fig. 409), two sections ab and cd are not originally parallel, and although they may remain plane and take the positions $a'b'$ and $c'd'$, so that the strain of each fiber would vary uniformly from m and n, yet since the fibers are of different lengths originally, the strain per unit length and hence the intensity of stress, would not vary in the same manner.

In a curved bar, therefore, the supposition that plane sections remain plane is inconsistent (within the elastic limit) with a planar distribution of stress; and the converse is also strictly true.

In Chap. X the question whether sections which are plane before bending remain plane after bending was discussed, and it was seen that the reason why such a supposition cannot be true arose from the presence of shearing forces. Now the shearing forces in an arch or curved beam are generally of much less magnitude and importance than in a straight beam; hence the assumption that plane sections remain plane is more justified in curved beams.

Let us therefore work out the problem of the distribution of normal stress in curved beams on the supposition that plane sections remain plane.

3. Stresses Supposing Plane Sections to Remain Plane.—In Fig. 410 let ED be the plane of the section, D a point on the axis (not neutral axis) of the curved

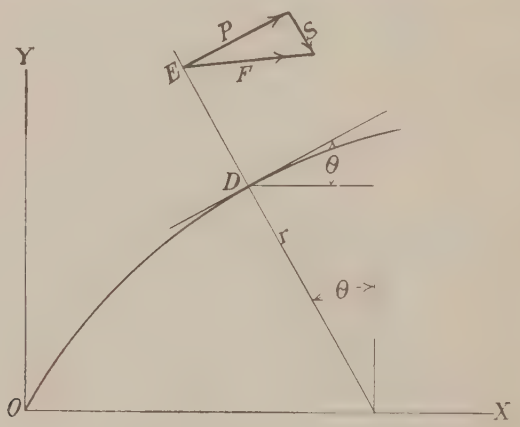


Fig. 410.

beam, E the point where the resultant, EF, of the outer forces on the left of the section cuts the plane of the section. Let EF be resolved into normal and tangential forces P and S. The moment about D is $P \cdot ED$. S acts at E. Let the forces act in the plane of the paper, and let one principal axis lie in that plane. Let θ be the angle that the tangent to the axis at D makes with the horizontal, or that the radius makes with the vertical, and let θ be positive when the angle between OX and the tangent is measured from OX in an anti-clockwise direction around to the tangent; that is, in the figure θ is about $+30.°$

The normal force P is taken positive when it acts toward the section, considering the part of the bar to the left of the section, *i.e.*, when it is compression. M is positive when it is clockwise on the left, or causing compression outward from the axis along the radius.

The equations which connect the outer forces with the internal stresses on the section are, if we call f_n = intensity of normal compressive stress at a point distant v from the axis, measured positive outward from the axis (Fig. 411); f_s = intensity of shearing stress in a direction parallel to the paper and acting as shown, and f_s' = intensity of shearing stress in a direction perpendicular to the paper, at any point:

$$\left.\begin{aligned} \int f_n dA &= P \\ \int f_s dA &= S \\ \int f_s' dA &= o \\ \int f_n v dA &= M \end{aligned}\right\} \qquad (1)$$

the integrations being extended over the entire section, and dA being an infinitesimal area.[1]

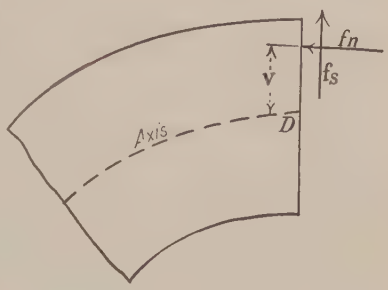


FIG. 411.

These equations, as in the case of a straight bar, do not suffice to enable us to find the shearing stresses. To find these, an indirect method must be used (see Chap. X, Art. 9), based on a previous determination of the normal stresses. The *normal stresses* may be found as follows:

Imagine two plane sections which before the deformation due to stress are at a distance ds apart along the axis. Then at a distance v from the axis their distance apart is

$$ds_v = ds - vd\theta \text{ (note sign of second term).}$$

Before the deformation, the angle between these two sections is numerically $d\theta$. Under the action of the loads and stresses, these sections change their relative position. If they remain plane, the following relations hold:

ds becomes $ds + \Delta ds$
$d\theta$ becomes $d\theta + \Delta d\theta$
ds_v becomes $ds_v + \Delta ds_v = ds + \Delta ds - vd\theta - v\Delta d\theta$.

Hence

$$\Delta ds_v = \Delta ds - v\Delta d\theta$$
$$\frac{\Delta ds_v}{ds_v} = \frac{\Delta ds - v\Delta d\theta}{ds - vd\theta} = \frac{\Delta ds - v\Delta d\theta}{ds} \cdot \frac{r}{r+v}.$$

According to Hooke's law

$$f_n = -E\left(\frac{\Delta ds}{ds} - \frac{v\Delta d\theta}{ds}\right)\frac{r}{r+v} \qquad (2)$$

the negative sign being prefixed because f_n is compression when positive.

The first and last of Eq. (1), together with Eq. (2), give the following:

$$\frac{P}{E} = \frac{\int f_n dA}{E} = -r\frac{\Delta ds}{ds}\int \frac{dA}{r+v} + r\frac{\Delta d\theta}{ds}\int \frac{vdA}{r+v}$$
$$\frac{M}{E} = \frac{\int f_n v dA}{E} = -r\frac{\Delta ds}{ds}\int \frac{vdA}{r+v} + r\frac{\Delta d\theta}{ds}\int \frac{v^2 dA}{r+v}.$$

[1] *Question.*—Why are S and P taken positive in Eq. (1)?

But $\int dA = A$ = total area of cross-section, and if we call $\int \frac{v^2 r dA}{r+v} = I_o$, it is clear that

$$r\int \frac{dA}{r+v} = \int dA - \int \frac{v dA}{r+v} = \int dA - \frac{\int v dA}{r} + \frac{1}{r}\int \frac{v^2 dA}{r+v}$$
$$= A + \frac{I_o}{r^2}.$$

The integral $\int v dA = 0$ because the axis is at the center of gravity.

Also
$$r\int \frac{v dA}{r+v} = \int v dA - \int \frac{v^2 dA}{r+v} = -\frac{I_o}{r}.$$

Therefore

$$\frac{P}{E} = -A\frac{\Delta ds}{ds} - \left(\frac{\Delta ds}{r ds} + \frac{\Delta d\theta}{ds}\right)\frac{I_o}{r} \tag{3}$$

$$\frac{M}{Er} = \left(\frac{\Delta ds}{r ds} + \frac{\Delta d\theta}{ds}\right)\frac{I_o}{r}. \tag{4}$$

From these we find

$$\frac{\Delta ds}{ds} = -\frac{P}{AE} - \frac{M}{AEr} = \textit{increase} \text{ of length per unit length} \tag{5}$$

$$\frac{\Delta d\theta}{ds} = \frac{M}{EI_o} + \frac{M}{AEr^2} + \frac{P}{AEr}. \tag{6}$$

Substituting these values in Eq. (2), we find, finally,

$$f_n = \frac{P}{A} + \frac{M}{Ar} + \frac{Mrv}{I_o(r+v)} \tag{7}$$

which gives the intensity of stress at any point under the assumption that plane sections remain plane.

If the bar is straight ($r = \infty$), $I_o = I$, and

$$\frac{\Delta ds}{ds} = -\frac{P}{AE}$$
$$\frac{\Delta d\theta}{ds} = \frac{M}{EI}$$
$$f_n = \frac{P}{A} + \frac{Mv}{I}$$

as in the usual demonstrations.

4. Value of I_o.—From the equations following Eq. (2),

$$I_o = r^3 \int \frac{dA}{r+v} - Ar^2. \tag{8}$$

For a rectangular section of width b and height h, the center at a distance r from the center of curvature, $dA = b dv$, and

$$I_o = br^3 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dv}{r+v} - Ar^2 = br^3 \log_e \frac{2r+h}{2r-h} - Ar^2. \tag{9}$$

For a circular section of radius r_1,

$$I_o = 2r^3 r_1^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 \alpha d\alpha}{r + r_1 \sin \alpha} - Ar^2$$
$$= \pi r^2 (2r^2 - 2r\sqrt{r^2 - r_1^2} - r_1^2). \tag{10}$$

In most practical cases, r is large, and I_o may be taken as equal to I. If the section is symmetrical about an axis through the center of gravity and perpendicular to the plane of the forces, it follows that for every little area dA at a numerical distance v outward there is an equal area at an equal numerical distance v inward. The value of the element of I_o for the first will be $v^2dA \cdot \frac{r}{r+v}$, and for the second $v^2dA \frac{r}{r-v}$. The sum of these will be nearly $2v^2dA$ if r is large compared with v, but always a little greater. For instance, taking even a small radius for the axis, say 5 inches, with a depth of section of 2 inches, then for the extreme fibers, one value would be $v^2dA \cdot \frac{5}{6}$, and for the other $v^2dA \cdot \frac{5}{4}$; the sum being $.2\frac{5}{12}v^2dA$. It follows that I_o is always greater than I.

5. Equation (7) shows that for the curved beam, if plane sections remain plane, the distribution of stress is not planar, since f_n is not linear with reference to v; the curve in which the plane of the paper cuts the surface produced by erecting at each point of the section a perpendicular proportional to the intensity of stress at that point is a hyperbola of which one asymptote goes through the center of curvature and is parallel to the tangent to the axis at D. The intensity of stress does not vary directly with v, so that the neutral axis does not go precisely through the center of gravity of the section.

$f_n = 0$ when

$$v = \frac{-I_o r(M + Pr)}{I_o(M + Pr) + MAr^2}$$

or, if $P = 0$, when

$$v = \frac{-I_o r}{I_o + Ar^2}$$

that is, the neutral axis deviates from the center of gravity in a direction toward the center of curvature. It will now be evident why the term with M in Eq. (5) is negative, for there is compression at the center of gravity due to the moment alone.

6. Stresses Assuming Planar Distribution.—Under this assumption, Eqs. (1) hold good, but Eq. (7) becomes

$$f_n = \frac{P}{A} + \frac{Mv}{I}. \tag{7a}$$

In the previous paragraph, the value of f_n had to be obtained, in terms of $\frac{\Delta ds}{ds}$ and $\frac{\Delta d\theta}{ds}$, by considering the relative motion of two plane sections remaining plane. After this was done, Eqs. (1) were used, and from them, by elimination, the values of $\frac{\Delta ds}{ds}$ and $\frac{\Delta d\theta}{ds}$ were found, which, substituted in Eq. (2), gave finally Eq. (7).

In the present case, the assumption of planar distribution gives at once Eq. (7*a*), and the relative deformation of two sections must be obtained *after* this equation is found, *not before*, in order to find $\frac{\Delta ds}{ds}$ and $\frac{\Delta d\theta}{ds}$. Thus, consider two radial sections, originally ds apart along the axis; this distance is decreased by $\frac{P}{AE} ds$,[1] since the fiber stress in compression at the axis ($v = 0$) has the intensity $\frac{P}{A}$. Hence

$$\frac{\Delta ds}{ds} = -\frac{P}{AE}. \tag{5a}$$

[1] The reader should draw a sketch to illustrate this demonstration.

At a distance v outward from the axis, the intensity of stress is given by Eq. (7*a*); hence that fiber, whose length was originally $ds - vd\theta$, is *shortened* by

$$\left(\frac{P}{EA} + \frac{Mv}{EI}\right)(ds - vd\theta)$$

$$v \cdot \Delta d\theta = \left(\frac{P}{EA} + \frac{Mv}{EI}\right)(ds - vd\theta) - \frac{Pds}{AE}$$

or

$$\Delta d\theta = -\frac{Pd\theta}{EA} + \frac{Mds}{EI} - \frac{Mvd\theta}{EI}$$

and, remembering that $rd\theta = -ds$

$$\frac{\Delta d\theta}{ds} = \frac{P}{EAr} + \frac{M}{EI} + \frac{Mv}{EIr}$$

$$= \frac{P}{EAr} + \frac{M}{EI}\left(\frac{r+v}{r}\right). \tag{6a}$$

For $r = \infty$, Eq. (6*a*) becomes

$$\frac{\Delta d\theta}{ds} = \frac{d^2y}{dx^2} = \frac{M}{EI}$$

which again is the familiar result for straight beams.

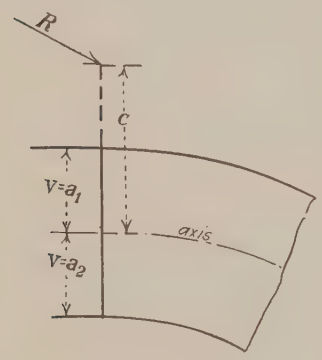


FIG. 412.

7. Kernel of the Section (Plane Sections Remaining Plane).—As in the straight beam (Fig. 412) the maximum stress occurs at the outside fibers. At the upper outer edge, distant a_1 from the axis, the compression is a maximum, and is given by the equation (from Eq. (7))

$$\text{Max. compression} = \frac{P}{A} + \frac{M}{Ar} + \frac{Mra_1}{I_o(r + a_1)}. \tag{11}$$

At the inner edge distant $-a_2$ from the axis, the tension is a maximum, and is given by the equation

$$\text{Max. tension} = -\frac{P}{A} - \frac{M}{Ar} + \frac{Mra_2}{I_o(r - a_2)}. \tag{12}$$

(Note that a_1 and a_2 are both positive quantities.)

Calling c the distance outward from the axis to the point where R cuts the section, these equations become

$$\text{Max. comp. at upper fiber} = \frac{P}{A}\left(1 + \frac{c}{r}\right) + \frac{Pcra_1}{I_o(r + a_1)} \tag{13}$$

$$\text{Max. ten. at lower fiber} = -\frac{P}{A}\left(1 + \frac{c}{r}\right) + \frac{Pcra_2}{I_o(r - a_2)}. \tag{14}$$

The stress will be compression over the entire area, when, c being positive and P compression, the max. tension at lower fiber is zero, or

$$c \leq \frac{I_o}{\frac{Ara_2}{r - a_2} - \frac{I_o}{r}} \tag{15}$$

or when, c being negative, the max. tension at top fiber is zero, or

$$c \geq -\frac{I_o}{\frac{Ara_1}{r + a_1} + \frac{I_o}{r}}. \tag{16}$$

If c has just the positive value given by the right-hand member of the first inequality, there will be just zero stress at bottom fiber. If c has just the negative value given by the second inequality, there will be just zero stress at top fiber. In both cases there will be compression over the entire area; and for any value of c between the two points just fixed there will be compression over the entire area. These values of c, then, are the upper and lower limits of the kernel of the cross-section, as described in Chap. X.

For a straight beam, $r = \infty$, and we find, in place of Eqs. (11) to (16), the following, which agree with values found in Chap. IX;

$$\text{Max. comp.} = \frac{P}{A} + \frac{Ma_1}{I} \tag{11a}$$

$$= \frac{P}{A} + \frac{Pca_1}{I} \tag{13a}$$

$$\text{Max. ten.} = -\frac{P}{A} + \frac{Ma_2}{I} \tag{12a}$$

$$= -\frac{P}{A} + \frac{Pca_2}{I}. \tag{14a}$$

Limits of kernel

$$c \leqq \frac{I}{Aa_2}; > -\frac{I}{Aa_1} \tag{15a and 16a}$$

or

$$c < \frac{\rho^2}{a_2}; > -\frac{\rho^2}{a_1}$$

where ρ is the radius of gyration.

The locus, for all sections of a curved bar, of the points distant c from the axis, where the resultant on one side of the section cuts the plane of the section (whether within or without the section itself) is called the *line of pressure.* It is therefore clear that *if the line of pressure, at each point of its length, lies within the kernel of the corresponding section, then there is only compression in the curved beam or arch* (P being compression); *if it lies outside the kernel at any section, then at that section there is both compression and tension, the former over a portion of the section nearest to P, the latter over the remaining portion, on the other side of the axis from P.*

To find the intensity of stress at the upper kernel point, substitute the value of c from Eq. (15) for v in Eq. (7), and we obtain

$$\text{Intensity of } \textit{compression} \text{ at upper kernel point} = f' = \frac{P}{A} + \frac{M}{Aa_2}$$

$$= \frac{P(c + a_2)}{Aa_2}$$

$$= \frac{\text{moment of } P \text{ about lower fibre}}{Aa_2} \tag{17}$$

Similarly, we find

$$\text{Intensity of tension at lower fibre} = f'' = \frac{\text{Moment of } P \text{ about upper fibre}}{Aa_1}. \tag{18}$$

It is thus shown that the intensity of stress at either edge may be found by taking moments about the opposite kernel point, as in the case of a straight beam.

8. Shearing Stresses.—As in a straight beam, the intensity of the shearing stresses on a section of a curved beam is greatest at the neutral axis, and decreases to zero at the extreme fibers. The intensity at any point can only be determined by the same indirect method that was used in the case of straight beams. This

will not be given here. It is enough to say that since the shearing forces in an arch are small, they are frequently not considered at all; and if considered, they may be found by the same formulae that are used for straight beams.

9. The remainder of the theory of curved bars consists of a study of deflection and slope. This, together with a full discussion of curved bars, is reserved for a later volume.

10. *Example.*—Consider the hook shown in Fig. 406. Let the radius of the center line or axis at the horizontal section be 2 inches, and the diameter of the section 2 inches; and let the load be 10,000 pounds. $I = \frac{\pi r^4}{4} = \frac{\pi}{4} = 0.7854$; $A = 3.1416$ square inches.

Then by the usual method, of planar distribution,

$$\text{Max. } f \text{ (tension at inside)} = \frac{P}{A} + \frac{Mv}{I}$$

$$= \frac{10{,}000}{3.1416} + \frac{10{,}000 \times 2 \times 1}{0.7854} = 28{,}648 \text{ pounds per square inch.}$$

By Eq. (7), first find I_0 from Eq. (10); it will be found to be 0.2864π (comparing with $I = 0.25\pi$). Then

$$f \text{ (compression)} = \frac{P}{A} + \frac{M}{Ar} + \frac{Mrv}{I_o(r+v)}$$

$$= -\frac{10{,}000}{\pi} + \frac{10{,}000 \times 2}{\pi \times 2} + \frac{10{,}000 \times 2 \times 2 \times (-1)}{0.2864\pi(2-1)}$$

$$= -\frac{40{,}000}{0.2864\pi} = 44{,}444 \text{ pounds per square inch tension.}$$

11. The great discrepancy in the two values found in the preceding section shows that either the assumption of planar distribution or that of plane sections remaining plane is very much in error. It also shows that with such great differences in the results of fundamental assumptions, mathematical hair-splitting in this—as in many other structural matters—is out of place. Which is correct? The case taken is of course extreme, with a very small radius, but the principle is the same in all cases.

Now while the shearing forces in a curved beam are generally less than in straight beams, they need not be, and there is some shear in almost all cases; it follows that plane sections cannot remain plane, as shown in Chap. X. In the above example, however, there is no shear. The question still remains, therefore —which of the two suppositions is the nearer to the truth. This subject is discussed in a later volume, and the conclusion there reached is that the supposition of planar distribution is more in conformity with the principle of least work. This is the writer's belief, and, if correct, the stress in the above example should be taken as 28,648 pounds. It must be admitted, however, that there may be some error in this.

CHAPTER XXI

TESTS OF MATERIALS

1. This is not intended to be a book on the testing of materials. That is a specialized subject, and the engineer should study it in the following or similar works:

UNWIN: "The Testing of Materials of Construction," Longmans, Green & Co., 3rd edition, 1910.

MARTENS: "Handbook of Testing Materials," *trans.* by Henning, John Wiley & Sons, 1899.

JOHNSON: "Materials of Construction," rewritten by Withey and Aston, Wiley, 1919.

The *Proceedings of the Am. Soc. for Testing Materials* (A.S.T.M.): Yearly volumes since 1901.

Some brief remarks on the subject, however, are necessary in order that the reader may understand the subjects treated in this work.

2. Tests may be either for purposes of *scientific investigation*, to determine fundamental properties and laws, or they may be *commercial or control tests.* After fundamental properties and laws have been determined, the engineer is in a position to decide what properties are necessary for his purposes, and what material to use. He must then make control or commercial tests of the material received, in order to make sure that he gets the material or properties desired. The two kinds of test are radically different in object, and often in the methods, apparatus, and the degree of accuracy obtained.

3. In making scientific tests, simple and fundamental properties should be determined. Facts that can be reasoned out should not be tested. The principle should be observed that but one variable should be investigated at a time. For instance, if the effect of carbon on the ductility or strength of steel is to be studied, all other elements and all physical conditions should be unvaried. Under these conditions any differences found may properly be considered due to the effect of carbon; but otherwise no logical conclusions can be drawn. A very slight change in the conditions may affect the results profoundly. Thus, very small percentages of vanadium may entirely mask the effect of carbon. It is too often the custom to experiment in such a way that there are many variables, and the results of such an experiment may be valueless because they are only properly applicable when the conditions are identical with those under which the experiment was made. It must not be assumed that because conditions are kept *nearly the same*, the results are conclusive. The aim of research should be to determine simple, elementary constants or laws. Reasoning will enable us, in most cases, to combine these elements in complex cases, and reasoning should be relied upon. Too often, testing is resorted to for the purpose of avoiding the necessity of reasoning. If we test to failure a complex structure, such as a model of a roof

truss, we may find how much it will carry, and we can see how and where it fails. If the failure occurs in such a manner that it can be made the subject of exact reasoning and computation—if, for instance, we can find with exactness the stress in the part which fails—we may be able to draw valuable conclusions; otherwise the results may show nothing except that that particular model failed in that particular way. Neglect of these principles makes a great many results of tests worth little or nothing.

All this shows the necessity for standardization. The shape of test pieces should be standardized, as well as the methods of manipulation, such as speed of testing, and the physical treatment of the material. Barba's law of similarity (see Chap. IV, Art. 19) should be observed. The well-known but often neglected laws of scientific investigation should be followed throughout, otherwise the results may be inconclusive and perhaps misleading or quite erroneous, and cannot be logically compared.

4. Commercial tests are not made on the material to be used, but on other material supposed to represent it. Hence care must be taken to secure truly representative specimens.

5. The pieces tested are generally small compared to those actually used. If the law of similarity is observed, reliable results may be secured. The small piece tested, however, such as a piece of timber, would usually contain no obvious defects such as occur in the actual pieces to be used, and the results may be misleading when applied to commercial sizes. Precise similarity can never be attained. This has sometimes led to the attempt to test large specimens, of the actual size to be used, like large steel columns. In this case it is difficult if not impossible to make the conditions of the test the same as the conditions in the structure, and the results may again be misleading. There is today in the opinion of the writer too much testing and too little thinking, because to test a model affords a means of escape from the arduous work of reasoning.

6. Specimens are usually tested to destruction. Possibly some day it will be practicable to test the actual material to be used, with reference to some qualities—not, of course, its strength—before putting it in the structure. Magnetic tests may, for instance, some day lead to the detection of defects or flaws in steel.

7. Tests are sometimes made on actual structures, as beams, bridges, or buildings, to determine such facts as the deflection under given loads, and from such tests important information is frequently obtained.

Testing requires, in many cases, very minute measurements, and is subject to accidental and personal errors: carelessness in the manipulation or faults in the instruments may render the results valueless.

All these facts make the testing of materials a specialized occupation, that of the *testing engineer*. His work and his field are most important. He determines the fundamental properties of materials, and stands between the manufacturer and the user, showing the former how to produce certain properties and holding him up to a proper standard of production, and informing the latter as to the best kind of material to achieve certain results and seeing that he gets it.

8. The qualities of materials which are usually tested are many. A list of methods and tests is given in the report of Committee E-1 on Methods of Testing, page 481 of the Proceedings of the A.S.T.M., 1923. Those which it is desirable to refer to here are the following.

Ultimate strength, in tension, compression, shearing, torsion, and flexure;

Elasticity, including elastic limit, yield point, angle of twist in shafts exposed to torsion, and value of E (determined by computation);

Ductility, including elongation and reduction of area, or ability to resist cold bending;

Hardness, or resistance to abrasion or indentation;

Toughness, meaning resilience, or resistance to impact;

Fatigue, or ability to sustain repetitions or alternations of stress;

Chemical tests, to determine whether objectionable elements are present in amounts over the allowed limit;

Microscopic tests, particularly of metals, to examine the molecular structure;

Miscellaneous tests, particularly of non-metallic materials such as stone and concrete, including tests of absorption (porosity), weathering qualities, specific gravity, consistency, time of setting of concrete or mortar, constancy of volume, etc.

Tests should be made which will truly indicate the properties desired. Sometimes they are of the wrong kind, and show properties not really important.

The best test of a property is experience in its use; but this requires time, and frequently the property cannot be isolated but is complicated with others. Tests should, if possible, be short, when they are control tests.

Mechanical tests are of three kinds: (1) static tests, including tests of the various strengths, and of hardness, in which one application of the load is made; (2) repeated stress tests, in which a load less than the static ultimate is applied many times; (3) impact tests, in which a piece is broken by a blow or by repeated blows.

Static tests show the ability to resist a steady load or one applied a comparatively few times. Repeated stress tests indicate the ability to resist loads many times repeated, as in certain rotating parts of machinery. Impact tests indicate toughness or resilience, and the ability to withstand blows.

The most powerful testing machine in the world is the vertical Olsen compression machine of the U. S. Bureau of Standards, at Pittsburgh, which has a capacity of 10,000,000 pounds. The Emery horizontal machine at the Watertown Arsenal has a capacity of 800,000 pounds.

9. Machines and Appurtenances for Strength Tests.—Machines for strength tests must contain means for applying a load, and means for measuring it. In the usual vertical machine (Fig. 413) when used for tension tests, the specimen is held at the top, by suitable grips or shackles, in the fixed head of the machine, which is supported by four columns resting on the weighing table. The lower end of the specimen is held by similar grips in a lower movable head parallel to the upper head. This lower head is supported by four large screws which pass down through the weighing table near the corners, but inside the columns supporting the fixed head, and through the base casting. Under the base, these screws have nuts upon them which can be uniformly rotated by machinery, thus drawing the lower head down by reacting upward on the base. In this way the load is applied to the specimen, and by varying the speed of the machinery the lower head may be pulled down at any speed desired. The machine is, therefore, a straining machine, which gradually stretches a tension specimen at a desired rate. The load thus applied must be weighed. The weighing table, upon which the

four columns press downward with a force equal to the tension on the screws or the stress in the piece (neglecting its own weight) rests on several horizontal levers (Fig. 414), whose fulcra A and B rest on the base casting, the other ends of the levers being connected to other levers which reduce the force until it can be balanced by a weight sliding upon a graduated arm, which weight is moved farther and farther to the right as the pull on the specimen is increased. The lower platform E rests on the levers at two points, C and D, C resting only on certain levers, and D on the other levers. The knife edges are placed so that, in the figure, $\frac{ac}{ae} = \frac{bd}{be}$, so that all the levers rotate through the same angle, and the load at e is the same proportion of the loads at C and D. Aside from the weight of the

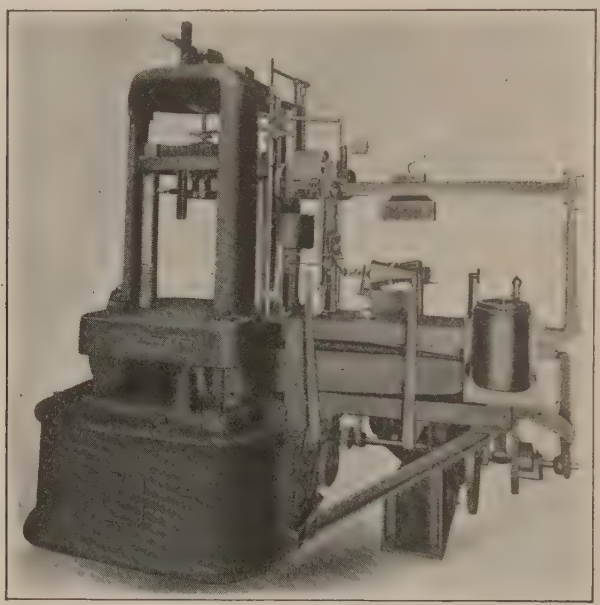


FIG. 413.—Olsen testing machine.

machine, the total downward force exerted on the base is greater than the upward force exerted by the screws by the weight of the specimen tested. Figure 414, which is merely a sketch, will enable the reader to understand the principle. Such a machine is a *lever machine*, the weighing being done by a combination of levers

The straining force may also be applied and weighed by means of fluid pressure. The straining and weighing may both be done at one end. Such machines are *hydraulic machines*. They are liable to error on account of the friction of the packing in the hydraulic cylinders. This friction is avoided in the Emery machine, the first large machine of this type, with a capacity of 800,000 pounds, being the one at the Watertown Arsenal. The books referred to give full details of the various types of testing machines.

The shackles or grips may be of various designs. Often they are wedges with the parallel sides rough, so that they indent and grip the end of the specimen,

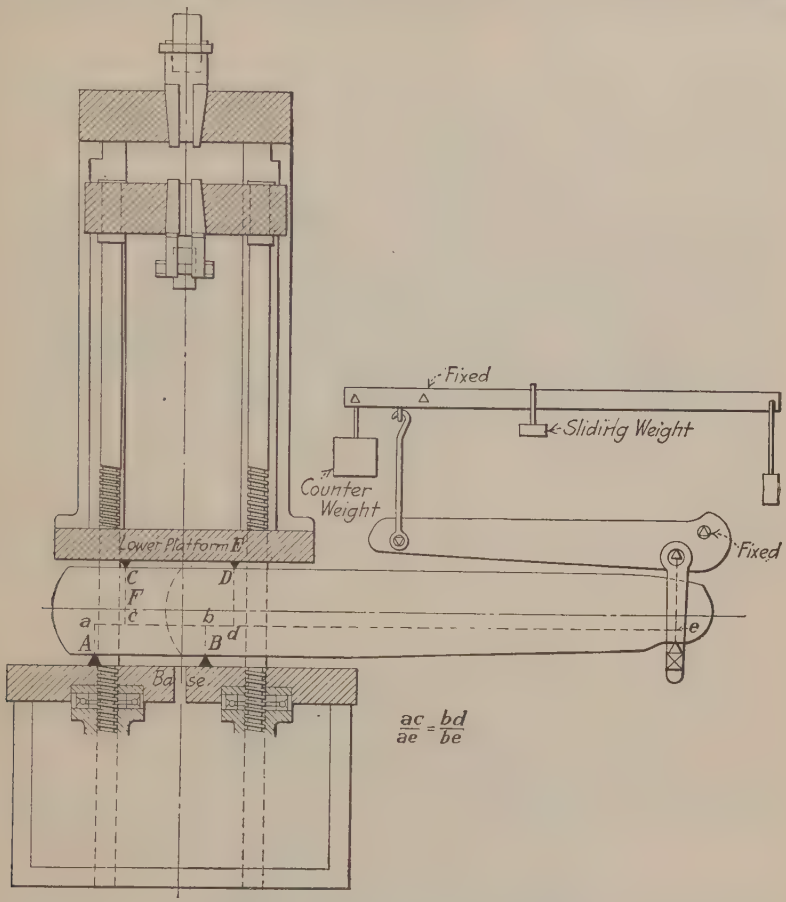


FIG. 414.—Diagram of testing machine.

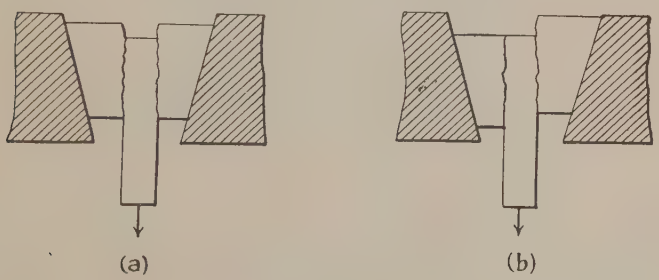


FIG. 415.

which should be slightly enlarged so that fracture will not occur at the grip (Fig. 415*a*). Sometimes the enlarged end is threaded (Fig. 423). It is essential

that the pull on the specimen be exactly central. If, for example, the wedges were displaced, as in Fig. 415*b*, the result would be misleading. Sometimes a spherical bearing is used, so that the grip may adjust itself.

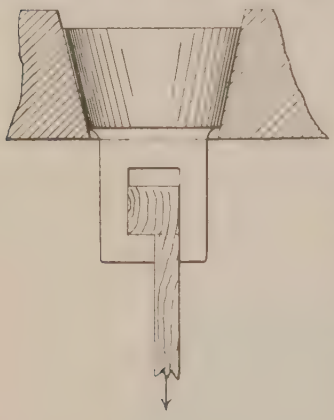


Fig. 416.

For compression tests, the specimen is placed under the movable heads resting on the weighing table.

For shearing tests, shackles like those of Unwin for timber may be used (Fig. 416) or some form of apparatus similar to Fig. 417*a* may be devised, in which the specimen *A* is passed through the supports *C* and the piece *B* is forced down, shearing the specimen along the two surfaces between *B* and *C*, causing shearing on two planes. Shearing on a single plane may be produced, but in that case there will be a tendency for the specimen to tip (Fig. 417*b*). It is extremely difficult, if not impossible, to produce pure shearing in a test. It is generally, or always, accompanied by some flexure. For instance, in Fig. 417*a*, when the piece *B* is pressed downward, the first tendency will be for the forces to be concen-

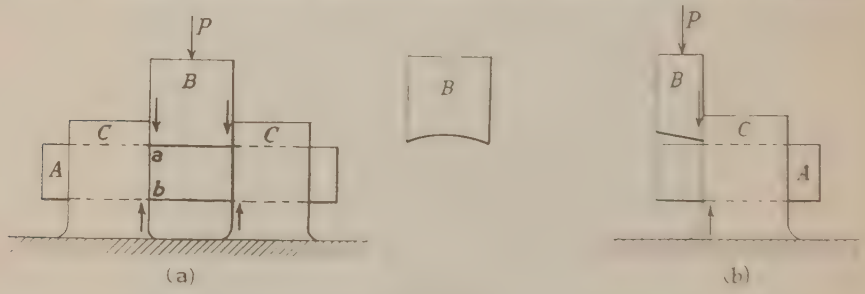


Fig. 417.

trated at the edges of pieces *B* and *C*, causing pure shear (because if the load is distributed the piece *A* will deflect); but this concentration of the force would produce an infinite pressure per square unit, which is impossible, and there must be *some* deformation, however small, of the edges of pieces *B* and *C*, and of the specimen; and the force will be distributed to some extent even if the edges of *B* are sharp, as shown in the small figure. The result will be that the two forces tending to shear the specimen at *ab* will not be *exactly* in the same line, but will have a small lever arm as shown by the small arrows. This small lever arm will result in a moment, causing flexure of the specimen; and if there is flexure, however small, the shearing stress on the surface *ab* will not be uniformly distributed over *ab*, but will be greatest at the neutral axis and zero at the top and bottom. In this case the real maximum stress intensity is considerably greater than the average or *P*

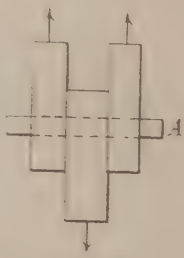


Fig. 418.

divided by the sheared area, which is generally given as the shearing strength. The same is of course true when the tests are made as in Fig. 418). Many tests of shearing strength are therefore inaccurate. All results of tests of shearing strength should therefore be carefully scrutinized, and the method of

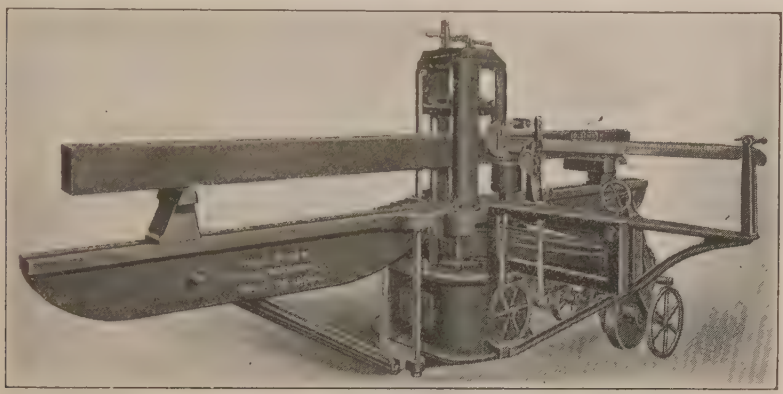


FIG. 419.—Beam Testing Machine. (Olsen.)

making the tests examined, before giving credence to the results, which will generally be less than the true shearing strength.

The best way to produce pure shearing is by torsion. The ultimate shearing stress in torsion, however, as given by Eq. (2) of Chap. VIII, is always larger

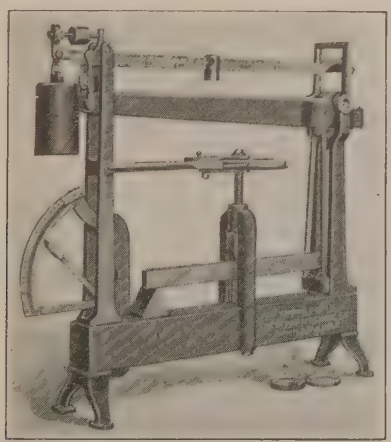


FIG. 420.—Machine for testing small beams. (Olsen.)

than the true shearing strength, for reasons explained in Chap. X, Arts. 35, 36. Professor Upton, in his book on "Materials," gives a demonstration to show that for ductile materials the true shearing strength is nearly three-quarters the value given by the equation, increasing for brittle materials to the value given by the equation. Ductile materials, in torsion, break square across; brittle materials, like cast-iron, having a low tensile strength, break by tension on a spiral surface

making an angle with the axis of about 45° at the outside fiber. Brittle materials are weaker in tension than in shearing; ductile materials the reverse. In comparing shearing tests in torsion with tests of direct shear, the fact that in the latter tests there is almost always bending should be remembered.

For tests in flexure, short beams may rest on supports placed on the weighing table; or that table may be extended on the sides so as to take beams up to 25 feet in length (Fig. 419). For short beams and small loads, as for testing the cast-iron "arbitration bar" the machine shown in Fig. 420 may be used, in which

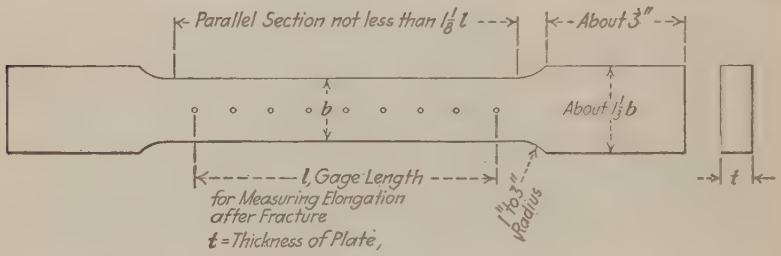


Fig. 421.

the reaction on one end is weighed as shown. There are also special machines for large beams.

For tests in torsion, one end of a shaft has a twisting moment applied to it, and at the other end the equal couple is measured by weighing one of the two equal forces, acting with a known lever arm, by means of levers or calibrated springs.

The determination of the ultimate strength thus requires only the weighing of the load, and is comparatively easy and accurate, provided the rate of application of the load is standardized, which has been shown in Chap. IV to be necessary. The ultimate does not always mean fracture, but may mean a

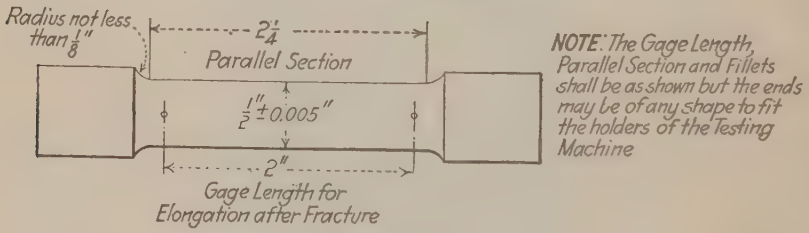

Radius not less than 1"
2¼"
Parallel Section
½" ± 0.005"
2"
Gage Length for Elongation after Fracture
NOTE: The Gage Length, Parallel Section and Fillets shall be as shown but the ends may be of any shape to fit the holders of the Testing Machine

Fig. 422.

given deformation, as, for instance, in compression tests of wood across the grain. In such cases the deformation as well as the load must be measured.

10. Standard Specimens for Strength Tests.—The standard A.S.T.M. test specimen for plate metal in tension is shown in Fig. 421, the thickness being that of the plate from which it is cut, except for thick material, in which case the specimen is machined thinner: a gauge length of 8 inches is recommended by the Committee but 2 inches is sometimes used. For general use in testing metals, the tension specimen of circular section shown in Fig. 422, with a gauge length of 2 inches for measuring elongation is recommended. The important

point is that the gauge length is four times the diameter. For brittle materials, the change of section at the ends should be more gradual, or as shown in Fig. 423.

Increase of speed of testing increases the yield point and the ultimate. The Committee recommends that in no case should the values in the following table be

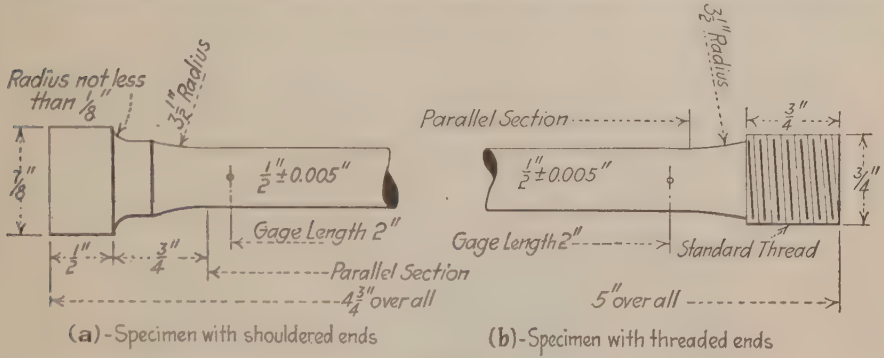


FIG. 423.

exceeded, the speed meaning the speed of the cross-head when the machine is running idle:

Specified minimum tensile strength of the material, pounds per square inch	Gauge length, inches	Maximum cross-head speed, inches per minute	
		Yield point	Tensile strength
80,000 or under	2	0.50	2.0
	8	2.00	6.0
Over 80,000	2	0.25	1.0
	8	0.50	2.0

In determining elastic limit, the speed for a 2-inch gauge length should not exceed 0.125 inch per minute; and in determining proportional limit it should not exceed 0.025 inch per inch of gauge length per minute. The beam of the machine must of course in all cases be kept balanced.

11. Elasticity.—The determination of the yield point for materials, like mild steel, that have a well-defined yielding at that point, may be done satisfactorily by the "drop of the beam" (see Chap. IV, Art. 21). To find the elastic limit accurately is more difficult, and requires plotting the stress-strain diagram; and even when this is plotted, it is often difficult to find the precise point where the limit of proportionality is reached. To find where a permanent set begins requires repeated loading and unloading. Hence in commercial testing the true elastic limit is seldom found, and that reported is the yield point found by the drop of the beam, or, if there is no sudden yielding, with less accuracy as the point at which the elongation reaches a given limit, or begins to increase rapidly.

12. Extensometers or Strain Gauges.—Deformations are measured by an *extensometer*, which is an instrument having two sharp projecting points at a standard or fixed distance apart, one being fixed and the other movable. This is clamped on to the test piece, and as the distance between the two points stretches or shortens, under tension or compression, the movable point moves, and its motion is magnified by levers or other devices, so that it can be read on a scale. The standard length measured in the United States is 8 inches or 2 inches.

A simple extensometer, and one much used, is the *Berry strain gauge* (Fig. 424). The movement of the right-hand or movable point is magnified at a by a lever extending inside the case from b to a and rigidly connected to the movable point at b, and at a is transmitted and further magnified, to the dial, on which the deformation of the standard length, which may be 2, 8, or 20 inches may be

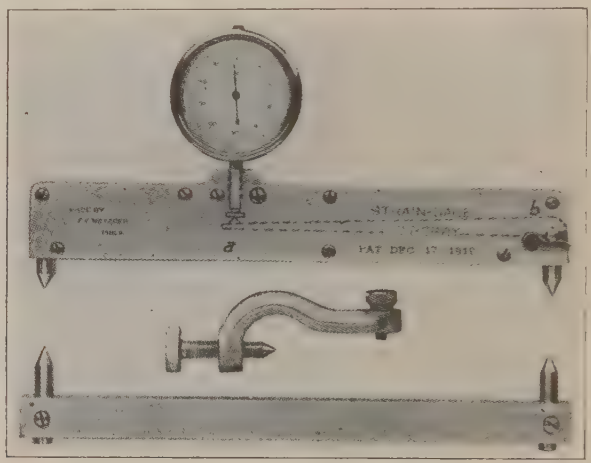


FIG. 424.—The Berry strain gauge.

read. For measurements on bridges, a long instrument is desirable. There are many other forms of extensometers, in some of which the motion is magnified and read by a microscope.

The usual extensometers read to $\frac{1}{5,000}$ or $\frac{1}{10,000}$ inch. If the latter, then with $E = 30,000,000$ and a gauge length of 8 inches, it measures deformations of 375 pounds per square inch. In soft steel, the total stretch to fracture may be as much as 2 inches, depending on the length measured; and the stretch to the elastic limit (say 30,000 pounds per square inch) will be $\frac{1}{1,000}$ inch per inch, or $\frac{1}{125}$ inch in 8 inches. If the accuracy is to be within 1 per cent, the gauge should read to $\frac{1}{12,500}$ inch. If the true elastic limit (not yield point) is to be determined, it should measure closer, and extensometers have been devised which are claimed to measure to $\frac{1}{50,000}$, $\frac{1}{100,000}$ inch, or even more accurately. Measurements as close as this, if indeed possible (which the writer doubts), can only be made under perfect conditions, in the laboratory, and by the most careful and

skilful observers. They are subject to numerous errors. In the first place, they must be made on two opposite sides of the test piece; otherwise, if the piece is not exactly straight, or if the pull is not exactly central, the bending or straightening of the piece will vitiate the results.

Let Fig. 425 represent a curved bar, and let $2c$ be the arc ab, $2a$ the chord ab, and h the versed sine. Then $c = \frac{4\sqrt{a^2 + h^2} - a}{3}$, nearly. If the measurements are made *on the axis*, the error due to straightening alone will be $2c - 2a$; if $ab = 10$ inches, $h = \frac{1}{50}$ inch, this error will be only $\frac{1}{18{,}868}$ inch, or too small to be significant. But extensometers not only do not measure along the axis, but are sometimes set out even from the bar itself, and measure along some line a_2b_2, distant x from the axis. By straightening, the original length a_2b_2 will be increased to the arc ab.

Now
$$\frac{d}{a} = \frac{r - x}{r} = 1 - \frac{x}{r};$$

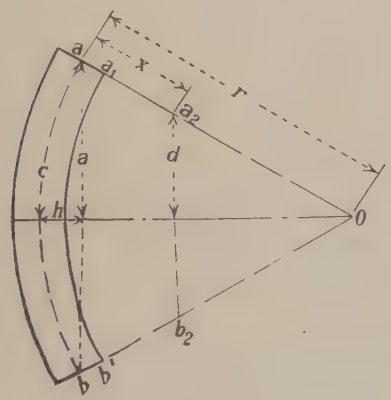


Fig. 425.

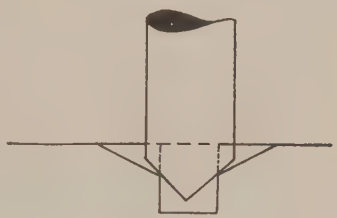


Fig. 425a.

and since $2r - h : a :: a : h$
$$a^2 + h^2 = 2rh$$
and
$$1 - \frac{x}{r} = 1 - \frac{2xh}{a^2 + h^2} = \frac{a_2b_2}{ab \text{ (chord)}}.$$

If, as before, $ab = 10$ inches, $h = \frac{1}{50}$ inch, $x = 2$ inches, the error in measuring on a_2b_2, due to straightening alone, will be 0.0032 inch per inch, or 0.032 inch in 10 inches, or not to be neglected. This shows the necessity of measuring on both sides, for a bend of over $\frac{1}{50}$ inch in 8 or 10 inches may well occur. R. H. Thurston was the first to use two extensometers, one on each side, but all accurate instruments do this now.

The writer believes, also, that there is likely to be slipping of the points of extensometers. Sometimes the points rest in slight indentations made with a punch. These indentations will be smooth and rounded, and a very slight movement of the point would vitiate the measurement. The accurate way to use the strain gauge is to drill a small hole into the test piece, and then to bevel or countersink the hole with a tool having an angle of about 110° (Fig. 425*a*). This ensures a good shoulder for the point of the strain gauge, which is highly essential for

accuracy. It is claimed that experienced operators can secure reliable results within limits of ± 0.0002 inch with a Berry gauge. But even thus, the point of the gauge may tip in the hole. At all events, the writer is quite sure that many reported extensometer measurements are incorrect, from the causes mentioned or others, and he confesses to considerable distrust of them unless made under perfect conditions (see Chap. XVII, Art. 30). Certainly the results of tests often contain obvious discrepancies and errors; and if the strain measurements are correct, as they may be, then the actual state of the structure does not depend directly upon them.

In some extensometers, the motion is read by a microscope. The most accurate instrument is the mirror extensometer of Martens (Fig. 426). In this and other mirror instruments, the movable end has no point, but has between it and the test piece a roller (Bauschinger), or a diamond-shaped piece (Martens) one point of which rests against the test piece and the other against the spring of the extensometer. To the rollers or to the diamond-shaped pieces, one on each side of the bar, are attached mirrors, which are observed through a telescope. A ray or line of light is thrown upon the mirror and reflected back to a scale near the telescope, by looking through which the scale and the line of light can be seen. The motion of the extensometer relative to the bar rotates the mirror and makes the line of light move on the scale, the motion being greatly magnified, and the motion on the scale, seen through the telescope, gives the elongation of the piece. By such an instrument it is claimed that a motion of $\frac{1}{250{,}000}$ inch can be detected.

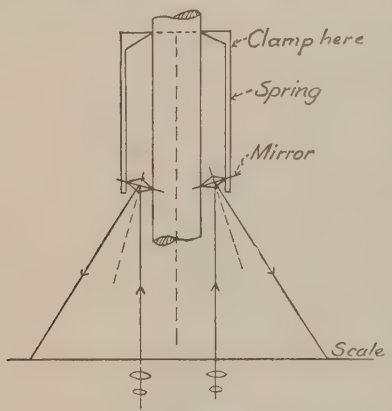


Fig. 426.

Bauschinger's mirror apparatus, described in the fifth Heft of his communications, had, he thought, an accuracy of $\frac{1}{250{,}000}$ inch. In his communication of 1886 he states, however, that this instrument had at one time "a rather large lost motion," which he sought to avoid by warming the springs by holding near them hot pieces of plate. The defect, he stated, was later corrected. Martens discusses in his book the errors of extensometers. It seems hardly safe to rely on the friction of a spring on a roller to measure so minute a deformation.

There are many autographic devices, by which, when a test is made, the stress-strain diagram is drawn on a paper attached to a revolving drum.

Professor W. E. Dalby, of London, has devised three ingenious recorders. In the first, a spot of light draws the stress-strain diagram on a photographic plate. This instrument is free from many errors of other machines (such as those arising from friction and inertia), and allows of very rapid loading. It is stated that "test pieces have been broken and the complete diagram obtained in a second of time." In this instrument it is not the strain of the test piece itself which is measured, but that of a hollow steel bar called a "weigh-bar," which is coupled together with the test piece in the same line, like two links in a chain, so that the

load on the test piece is the load on the weigh-bar, and the stretch of the weigh-bar is a measure of the load on the test piece; while the weigh-bar is so proportioned that a load which will break the test piece will not stretch the weigh-bar beyond the elastic limit. The second of Professor Dalby's instruments shows the first 0.04-inch extension, and the third draws a torque-twist curve, all by the image of a spot of light on a photographic plate.[1]

13. Ductility.—The total elongation in the gauge length taken is measured by putting the parts together after fracture as closely as possible. For this measurement to be significant, the entire length of the necked portion must be included in the gauge length. It is sometimes required that, if the fracture is outside the middle half of the gauge length, a new test may be made at the option of the manufacturer. The A.S.T.M. committee says that the elongation should not be reported if the break is outside the middle third of the gauge length.

The reduction of area is found by measuring with a micrometer the smallest section of the fractured specimen. It is by many considered a better indication of ductility than elongation.

Another test of ductility is the cold bend test, in which a piece is bent around a pin having a diameter which is greater as the thickness of the test piece is greater, or flat on itself (see Vol. 2 for the usual requirements).

A material which has great ductility will show a large angle of twist in a torsion test.

14. Hardness.—The different meanings of hardness have already been referred to, namely: resistance to scratching, to indentation, to wear or abrasion, to cutting by a drill or other tool, to fracture of the edges by a shock, or a high elastic limit. There is no necessary relation between all these different qualities, and a list of materials arranged in the order of one will not necessarily be in the proper order for another.

The most common test for hardness is the *Brinell test* for indentation, in which a hardened steel ball 10 millimeters in diameter is pressed against the clean, smooth metal surface with a standard pressure of 3,000 kilograms for iron or steel, and 500 kilograms for softer metals and alloys. The hardness is denoted by the Brinell number, which is the pressure, in kilograms per square millimeter of the spherical surface of indentation. The greater the hardness the higher the number. The number is found by measuring the depth of indentation (t), or movement of the ball, assuming that the ball is so hard that it does not distort; or by measuring the average diameter of the indentation (d). If P is the pressure, and D the diameter of the ball, the surface of indentation is πDt and

$$\text{Hardness number} = \frac{P}{\pi Dt} = \frac{2P}{\pi D(D - \sqrt{D^2 - d^2})}. \qquad (1)$$

The Brinell hardness numbers are about as follows:

For softest steel	100
For steel 100 per cent pearlitic	200
For sorbite, according to the carbon	250–350
For troostite, according to the carbon	350–400
For martensite, according to the carbon	400–650
For carbon steel quenched	650

[1] See PROFESSOR DALBY'S recent book "Strength and Structure of Steel and Other Metals," Longmans, Green & Co., 1923.

There is nearly a linear relation between the Brinell number and the tensile strength for the same kind of material, as shown by Fig. 427 for carbon steels.[1] Mr. Abbott made 3,932 tests, on 300 types of steel, with different heat treatments, comprising all the modern alloy and carbon steels. The inclined lines represented, with reasonable closeness, the results of the tests, and were calculated by the method of least squares. For carbon steels, the maximum tensile strength in thousands of pounds, M, was connected with the Brinell hardness number, B, by the equation

$$M = 0.73B - 28.$$

Equations are given in Mr. Abbott's paper for the other kinds of steel, but they were not greatly different from the above. It will be noted that the inclined

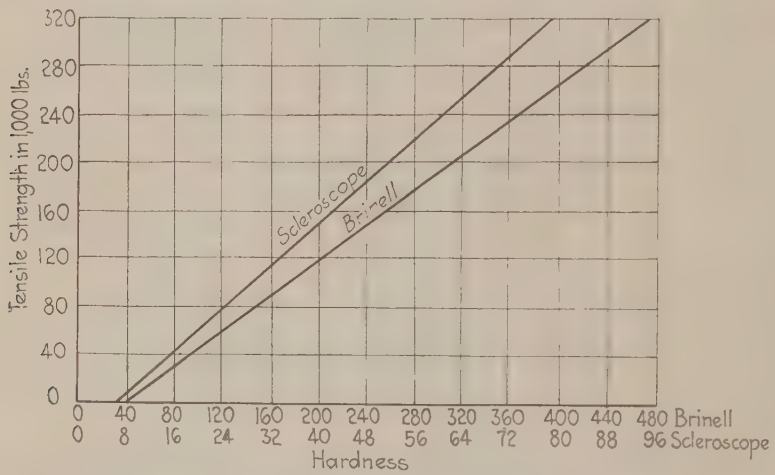


FIG. 427.—Carbon steels. Relation between the tensile strength and the hardness number. (From *Abbott, A.S.T.M.*, p. 44, 1915.)

lines representing the equations do not pass through the origin. In foreign papers it is sometimes stated that for carbon steels

$$M(\text{in kilograms per square millimeter}) = 0.34B$$

which would correspond to

$$M(\text{pounds per square inch}) = 484B.$$

The line representing this relation would pass through the origin, and would not agree with Mr. Abbott's results.

The Brinell hardness number depends on the size of the ball and the pressure, hence these must be standardized as above. The ball will also deform, and the radius of the indentation will depend on the hardness of the ball. This, too, is standardized by the A.S.T.M., which prescribes that the balls should contain from 1 to 1.2 per cent of carbon, and from 1 to 1.5 per cent of chromium, and should be heat treated to make them as hard as possible. "The time of pressure should be at least 30 seconds."

[1] ABBOTT on "Strength and Hardness of Steels," *A.S.T.M.*, p. 42, 1915; see also DEVRIES, "A Comparison of Five Methods of Hardness Measurement," *A.S.T.M.*, p. 709, 1911.

The main source of error in Brinell tests is in measuring the diameter of the indentation; for the material around it flows upward, forming a circular rounded ridge, and the edges of the indentation are not always sharp. A similar error would be involved if the depth of the impression below the top of the ridge were measured, but Devries measures the actual movement of the ball, which is the indentation below the original surface.

In Brinell tests the center of the indentation should be more than two and one-half times its diameter from the edge of the piece.

Figure 428 shows a Brinell machine. The weights seen at the side load the liquid in a hydraulic cylinder with a pressure such that the pressure on the ball is the standard 3,000 kilograms, or whatever standard is desired.

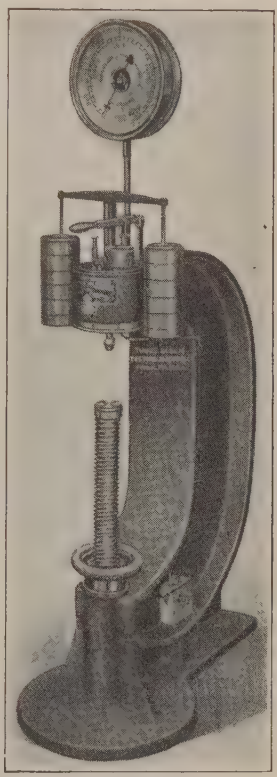


FIG. 428.—Brinell machine. (Olsen.)

Another test of indentation hardness is by using a 90° cone of hardened steel instead of a ball (Ludwik's cone), which it was thought would give a hardness number independent of the load, that is, that the load per square unit of the surface of the indentation would be constant; but this does not appear to be true. The cone is not well suited for hard materials, for its point becomes flattened.

The *scratch test* is used by mineralogists, who have a scale or list of minerals (Mohs' scale) arranged with the softest at the top, so that any mineral in the list will scratch those above it and will be scratched by those below it. At the bottom is the diamond, the hardest known mineral. For metals, the *sclerometer* has been used, in which a diamond is pressed against a polished metal surface, and moved over it, and the hardness is measured by the pressure required to produce a barely visible scratch.

An instrument much used to determine hardness is the *Shore scleroscope*, in which originally a small diamond-pointed hammer (but now often a hardened steel hammer) is allowed to fall from a height of about 10 inches in a close-fitting (but not tight-fitting) tube upon the smoothed or polished surface to be tested, and the hardness is measured by the height of the rebound, which is observed by the eye. The scale on which the rebound is read is about 10 inches in height, and is arbitrarily divided into 140 equal parts. The scleroscope hardness number is the height of rebound on this scale; it is therefore a purely relative number, having no definite physical meaning. The ball makes a small indentation of the surface, too small to serve as a basis of measurement. In making this indentation, some parts of the metal are probably stressed above the elastic limit, and others not, and the hammer itself may be slightly deformed. The height of rebound depends upon the elastic recovery after the deformation, and thus depends upon the shape of the lower end of the hammer, that is, upon the form

and dimensions of the indentation. Figure 429 shows a Shore scleroscope.[1] Tha hammer is raised to the top of the tube by the suction of a rubber bulb, and a pressure on the bulb releases the hammer without causing pressure behind it.

A direct-reading hardness tester, known as the "Rockwell," is made by the Wilson-Maculin Co., 383 Concord Ave., New York. In this instrument the penetrator is a steel ball or a diamond cone, and the hardness number is based on the additional depth to which the ball or cone is driven by a heavy load, beyond the depth to which the same penetrator is driven by a light load.

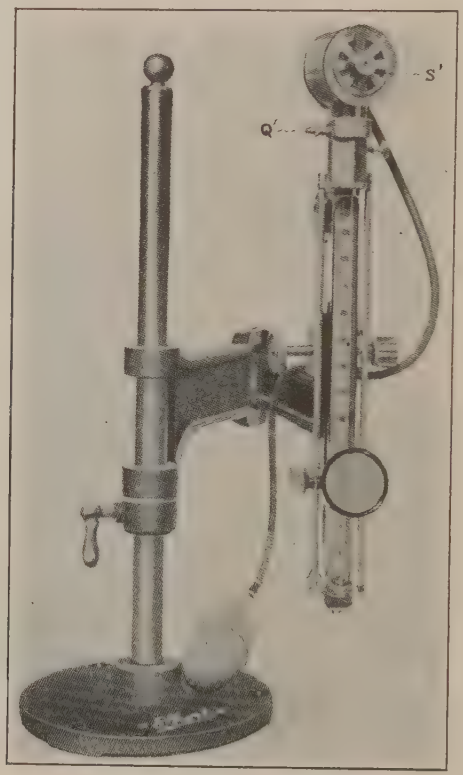


FIG. 429.—Shore scleroscope.

The scleroscope tests the very surface of the material, while the Brinell test goes deeper. The two tests should not be expected, therefore, to agree in all cases, though they will both generally arrange a series of steels in the same order as to hardness. For this same reason, while the relation between scleroscope hardness and tensile strength is approximately linear, it is less closely so than the relation between Brinell hardness and tensile strength. Abbott, in his paper above referred to, gives for carbon steels, if M is the tensile strength in thousands of pounds and S the scleroscope hardness, the equation

$$M = 4.4S - 28.$$

[1] Made by the Shore Instrument & Mfg. Co., Jamaica, N. Y.

Equations are given by Abbott for the other kinds of steel, but they differ more from each other than do the Brinell equations.

As the scleroscope reading depends upon elastic recovery, it depends on elasticity as well as on hardness. A hard material with a low elastic limit may give a smaller rebound than a softer material with a higher elastic limit, all modified by the shape of the indentation and the actual distribution of stress over it. Devries stated[1] that an alloy of 90 per cent copper and 10 per cent tin showed a scleroscope hardness greater than that of tool steel, while the Brinell test showed it not much greater than pure copper. Generally $B = (5.75 \text{ to } 6)S$ for iron and steel.

Cutting hardness is measured by the material removed by a standard drill under constant speed, pressure, and time.

Resistance to abrasion or wear is measured by the loss of material of a standard piece when pressed with a given force and for a given time against a standard abrading surface. This is used in testing road materials.

Abrasive resistance, together with toughness or resistance to blows, is also measured by putting a number of standard samples of the material to be tested, such as bricks, in a revolving drum or "rattler," rotating it for a given time, and measuring the material finer than a standard size which has been abraded or broken off from the edges.

Abrasive hardness is very different from indentation hardness. A material with a low Brinell number may give the best wear. A high Brinell hardness does not therefore indicate high resistance to wear, as of rails and tires.

Strain-hardening, or stressing above the yield point, appears not only to raise the elastic limit and yield point, but to increase indentation hardness. Cold working does this. This is shown by various facts. Devries[2] found that if a Brinell indentation is made, and then the scleroscope hammer dropped into this indentation, the rebound will be greater than when dropped on the unindented surface; also that if the hammer is dropped twice in the same spot, the second rebound is greater than the first. He also stressed samples of vanadium steel to various stresses, below and above the elastic limit, and then measured the scleroscope hardness, which proved to be unchanged as long as the stress had not exceeded the elastic limit, but increased by 18 per cent for the highest stress above the elastic limit, as shown by the following figures:

Vanadium Steel with Elastic Limit of 78,000

Stress	Scleroscope Hardness
0	36.5
10,000	36.8
20,000	36.8
35,000	37.0
60,000	37.0
83,000	40.0
98,000	43.0

Whether the Brinell hardness is increased by strain-hardening has not been conclusively proved, to the writer's knowledge, but probably it is.

[1] Devries, "Hardness in Its Relation to Other Physical Properties," *A.S.T.M.*, p. 725, 1911.
[2] *Id.* p. 726.

The different tests of hardness are obviously not comparable with each other. They test different forms of hardness. Moreover, any one test, such as the scleroscope, is only useful in comparing metals of similar composition.

Tests of hardness are useful and widely used, as, for instance, in controlling a product at the factory. They may even be used to determine tensile strength approximately, when tests of the latter are impracticable, by means of the relations that have been explained.

Hardness, it should be remembered, is not rigidity, nor is it brittleness, though hardness and brittleness generally increase together.

15. Toughness or Resistance to Blows. Dynamic or Impact Tests.—Capacity to resist blows may be the quality most desired in a material. This

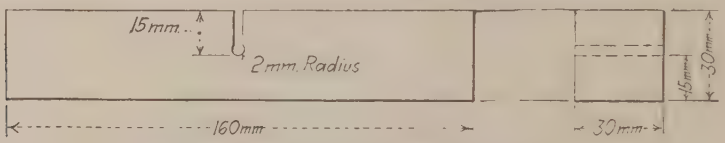


FIG. 430.—Large Charpy impact test piece.

may be estimated from a tension test, if the stress-strain diagram is drawn, but the estimate will be incorrect, because the shape of the diagram under a blow, if it could be drawn, would no doubt be very different from the diagram for a static tension test.

Impact or dynamic tests, by which a test piece is struck a blow, and the energy absorbed in breaking it is attempted to be measured, have come much in use. Many think that it shows qualities not indicated by other tests, and not to be inferred from them. It has long been used in testing rails, which are particularly exposed to blows in service, by the so-called *drop test*. The falling weight, or "tup" weighs 2,000 pounds, and is allowed to drop from a height, varying with

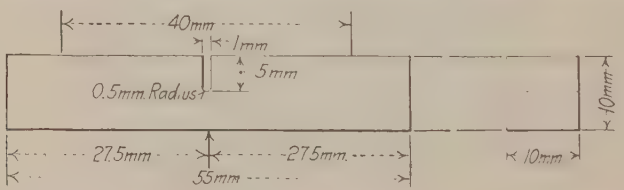


FIG. 431.—Small Charpy impact test piece. (*A.S.T.M.*, 1922.)

the weight of the rail, from 15 feet for a rail of 50 to 60 pounds per yard to 21 feet for a rail of 101 to 120 pounds per yard, upon a piece of rail not less than 4 nor more than 6 feet long, with a span of 3 or 4 feet, depending on whether the rail weighs less or more than 100 pounds per yard. The elongation of the outer fiber is measured. The complete specification is too long to be quoted here, but may be found in the *A.S.T.M. Standards*, pp. 20–26, 1921.

The shock-resisting power of a material is obviously not proportional to the tensile strength, but depends upon a combination of tensile strength and ductility.

The usual tests by impact, aside from the drop test for rails, are the *Charpy test* and the *Izod test*. In each of these a pendulum is allowed to swing, striking the test piece at the bottom of its swing and fracturing it.

In the *Charpy test*, the test piece is a short beam, struck horizontally at the middle of its length. In order to ensure that it shall break at a determined

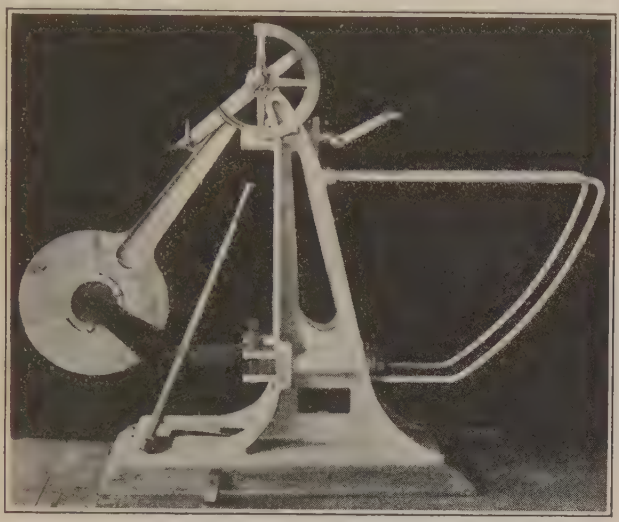


FIG. 432.—Charpy impact machine.

place, namely, at the center, the beam is notched at that point. There are two standard test pieces of different sizes shown in Figs. 430 and 431. Sometimes a hole is drilled through the piece to form the bottom of the notch, and a cut sawed down to this hole. The blow is struck horizontally, directly opposite the notch, by a knife edge on the pendulum. The machine is shown in Figs. 432 and 433. The rather flat pendulum is raised to a standard angle with the vertical, held by a catch, and by releasing the catch the pendulum swings and fractures the specimen, and then swings beyond the lowest point to an angle which is marked by a pointer. If the pendulum is swung freely, with no specimen to fracture, the friction of the bearings and the air resistance will prevent it from swinging as high beyond the lowest point, as the point it started from: this angle is observed, and the loss of energy by these resistances may be computed. When a test is made, the energy of the fall is disposed of as follows:

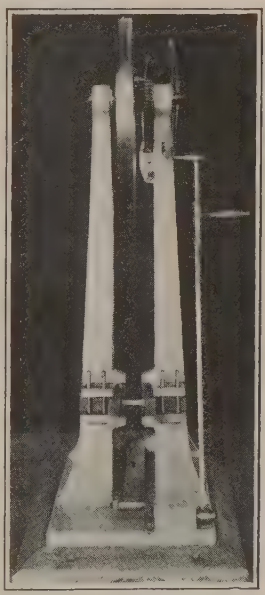


FIG. 433.—Charpy impact machine.

(1) That absorbed in fracturing the specimen, which is the quantity to be determined:

(2) That absorbed by friction, resistance of the air, etc.:

(3) That absorbed by deformation of the pendulum, the anvil against which the specimen rests, the base of the machine, etc.

(4) That remaining in the pendulum, and represented by its swing after fracturing the specimen.

The second element may be computed and allowed for, as above explained; the fourth is observed and measured; the third is uncertain and dependent on many factors, such as the mass and rigidity of the parts, and cannot be found. It follows that the energy absorbed in fracturing the specimen cannot be precisely ascertained. The notch is used to localize the fracture, and make sure that the beam will break where the blow is struck; but the beam deforms somewhat as a whole, and the distribution of stress at the notched section, as well as on each side of it, is highly uncertain (see Arts. 4 and 5, Chap. VI for effect of a hole or notch in a tension piece). Moreover, the material is not perfectly homogeneous, and the bottom of the notch may be in a part which is harder or softer, stronger or weaker, than the part a very little deeper, so that a very slight difference in the depth of the notch may greatly affect the result of the test. Also, the notched section may not be the weakest section of the beam. It cannot be expected, therefore, that the estimated work done in fracturing the specimen will agree with any computation of resilience from a stress-strain diagram, especially as the form of that diagram for the case of a blow is unknown. The test does not measure a single, definite property of the material, but is a complex test. However, it is useful as a control test of material exposed to shock in daily use.

When used as a control or standard test, the measure of the resistance to impact is taken as the work done in fracturing the specimen, which is assumed to be the energy of the pendulum at its lowest point, less a correction for friction and resistance of the air, and less the energy remaining in the pendulum after the fracture, as shown by the angle to which it swings after fracturing the piece. The pendulum is always dropped from the same standard angle of elevation, and the angle to which it rises, after breaking the piece, is measured, and then a prepared table shows the foot pounds of work which has been done in fracturing the piece. The loss due to deformation of pendulum, anvil, and machine is neglected, as it is unknown. Material which, in the standard specimen, does not absorb the required amount of work is rejected. It is found that this test ensures a material that will not fail in the desired service. The amount of work specified must be determined by experiment, and varies with the material and the service required of it.

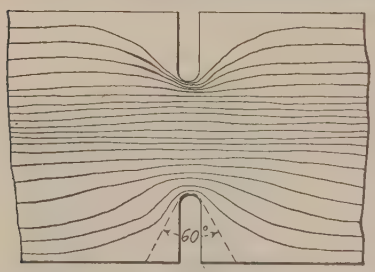


FIG. 434.—Tension impact test specimen.

The Charpy machine may also be used for tests in tension. The pendulum, at its lowest point, swings of course between the two parts of the anvil against which, in a flexure test, the ends of the beam specimen rest. For a tension test, the tension specimen, which is a round piece with a notch cut around it at the center of its length, has one of its enlarged ends screwed into the pendulum so that the specimen is in the plane of the flat pendulum. The other end of the specimen is screwed into a rather heavy transverse block, long enough, at right angles to the pendulum, to strike against the two parts of the anvil. When this block strikes the anvil, the pendulum being at its lowest point, the tension specimen is ruptured, one half continues to swing with the pendulum, while the other half, with the block, is thrown backward to the floor. Figure 434 shows the tension specimen.

In order to measure more accurately the work absorbed in breaking the test piece, that is, to take account of the work of deforming the anvil and machine, it has been proposed to make the anvil a part of a second pendulum, which would be swung by the blow, and the swing of which could be measured. It has also been suggested to mount the anvil on a second pendulum similar to the first, and swinging from the same height, so that the two would meet at the lowest point. For a quite full discussion of the subject see "A Symposium on Impact Testing," in the *Proc. A.S.T.M.*, pp. 5–150, 1922.

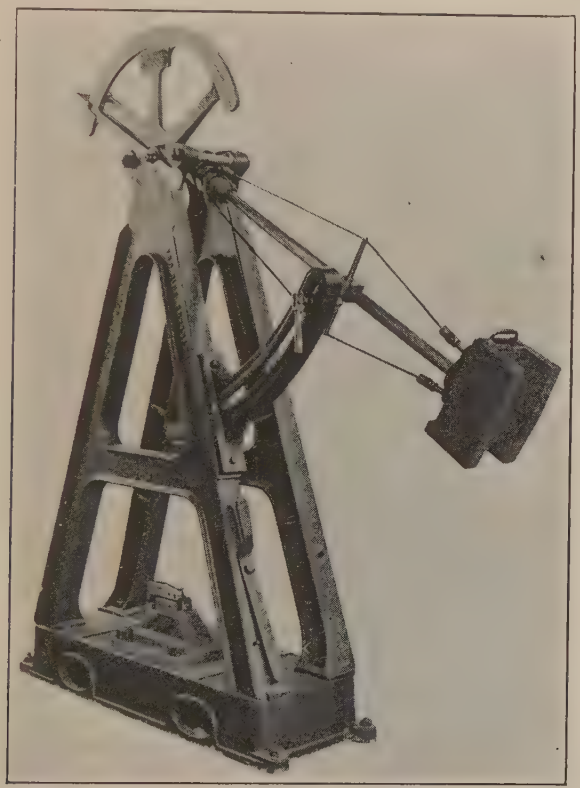


FIG. 435.—Izod impact machine.

In the Charpy tension test, if the test piece strikes its anvil when the pendulum is at its lowest point, the test piece, being at the back of the pendulum, will not be moving horizontally. The surface of the anvil should be at right angles to the direction of motion of the test piece, otherwise some energy will be absorbed in bending the piece. This is not always done; but even if not done, the test will give comparative results.

The *Izod machine* (Fig. 435) much used in England, is similar to the Charpy machine in that a swinging pendulum strikes the specimen, but the specimen is a round notched cantilever rod projecting vertically upward from the anvil. This projecting rod is struck by a knife edge on the pendulum, and either frac-

tured or bent over. If bent over, the pendulum may ride up over it, and damage is sometimes caused to the machine. The specimen generally has a V-shaped notch. It is found that the angle at the root of the V may be increased to 60° without materially affecting the results in impact tests, as indicated in Fig. 434. This machine does not appear, to the writer, as good a one as the Charpy machine.

The *Fremont* machine is one in which a weight is allowed to fall vertically, striking with a knife edge upon a small specimen, and breaking it transversely.

There are machines for repeated blows upon a specimen, which are used in tests of fatigue.

16. Fatigue Tests.—The subject of fatigue is treated in Chap. XXII, and all that it is necessary to state regarding testing is there explained. It may simply be here remarked that a material may fail after a sufficient number of repetitions of a load below the static ultimate. Reversal of stress, many times repeated, is still more destructive. Fatigue tests consist, therefore, in subjecting a piece to repetitions or reversals of stress, and observing the number of repetitions necessary for fracture under various conditions. There are fatigue tests in tension, shear, and torsion, and reversals from tension to compression.

17. Chemical Tests.—Chemical tests of some materials are made, to determine whether certain injurious elements are present in larger proportion than is desirable or above the limit prescribed by the specifications; such as sulphur and phosphorus in steel, and sulphur in cement.

An important question for the engineer to consider is, whether his specifications should require physical properties or chemical composition, or both. It is maintained, on the one hand, that he should specify only the physical properties which he deems necessary for his purposes, leaving to the manufacturer to comply with these requirements as may be for him most economical, and by the use of any chemical composition which will produce the required properties. If this principle is followed, it is necessary to specify every physical property which may be adversely affected by an injurious element, and there may be some properties regarding which the effect of such elements may not be known. It might also allow the substitution, under the specification, of a material substantially different from the one desired. On the other hand, it is claimed that, in order to secure precisely the material desired, the chemical composition, as well as the physical properties, must be specified. It is obvious that it is easy to go too far in specifying chemical requirements, and that unless the engineer who draws the specifications is a manufacturer himself, or an expert on the material in question, he may even specify physical qualities and chemical constitution so as to be incompatible with each other. The manufacturer should generally be allowed, as far as reasonable, to produce the desired qualities in any way he likes. For this reason, chemical constitution is only now specified so far as to designate clearly the kind of material desired (such as silicon steel or nickel steel with certain limiting quantities of silicon or nickel), and admittedly injurious elements are limited in quantity. In this way the physical properties specified and tested may be fewer in number than if there were no chemical requirements.

18. Microscopic Tests.—The use of the microscope in the examination of materials is coming more and more into use as the science of Metallography becomes better known and its importance more generally recognized. It is

often the crucial test in determining the cause of a failure, or the suitability of a material for a specified function. The expert metallographist should be consulted in such cases.

19. Miscellaneous Tests.—It is not the object here to refer to the great number of miscellaneous tests used in controlling the properties of materials. The reader if interested in these, should consult the Book of Standards of the A.S.T.M. He will also find, in the Bulletin of the National Research Council for August, 1923, a description of many tests used for road material.

CHAPTER XXII

REPEATED OR REVERSED STRESSES. FATIGUE

1. It is a matter of common knowledge that it is easier to break a metal by repeating a stress—and still easier by reversing it from tension to compression—than to break it by one application of a tensile stress. If we wish to break a wire, we bend it backward and forward several times.

Materials in actual use are exposed to repetitions and sometimes reversals of stress, an illustration of reversal being a car axle, in which the stress is completely reversed with each revolution.

The first steps to secure accurate experimental evidence on this subject were taken by the Prussian railway mechanical engineer, A. Wöhler, who in 1858 suggested the need of such information, and was enabled to make experiments in tension, bending and shear, between 1859 and 1870. The results were published in 1870, under the title, "*Über die Festigkeitsversuche mit Eisen und Stahl*," Berlin, 1870, Ernst and Korn. These experiments indicated certain laws, which are the subject of this chapter.

Wöhler's experiments were extended between 1871 and 1873 by L. Spangenberg, Professor at the Gewerbe-Akademie at Berlin, and the results published under the title "*Über das Verhalten der Metalle bei wiederholter Anstrengungen*," Berlin, 1875, Ernst & Korn.

Further experiments were made by J. Bauschinger, professor at the technical school at Münich, and published in the thirteenth Heft of his "Mittheilungen," Munich, Ackermann, 1886. Bauschinger continued his experiments until his death, in 1893, and his later results were published by his successor, A. Föppl, in the twenty-fifth Heft of the "Mittheilungen," in 1897.

Until recently, these were the principal papers and experiments on this subject, though many scattered experiments were made and described.

Recently the subject has been taken up by the National Research Council, the Engineering Foundation, and the General Electric Co., and extensive experiments were begun at the University of Illinois, under the direction of a Committee, with Professor H. F. Moore in direct charge of the work. Bulletins 124 (1921) 136 (1923) and 142 (1924) give the results thus far arrived at, but the experiments are still being carried on. The first of these gives a very complete bibliography of the subject.

With this introduction, the reader who desires to pursue the subject farther will be in a position to do so. A very good account of the matter, and particularly of Wöhler's and Bauschinger's results, will be found in Chap. XVI of Unwin's "Testing of Materials of Construction," third ed., Longmans, Green & Co., 1910.

2. Wöhler's Experiments.—Wöhler experimented on steel and iron under the following conditions: static tension, torsion, and bending; reversed bending in

revolving shafts; bending in one direction between variable limits; repeated tension between variable limits; repeated torsion.

In the machine for repeated torsion, a lever h_o (Fig. 436) was keyed on to the enlarged end of the test piece A, and by a revolving shaft the end of the lever was loaded at each revolution. To the other end of the shaft other levers h, h_1, were attached, the outer ends of which, d or d_1, bore on the short arm of a third lever

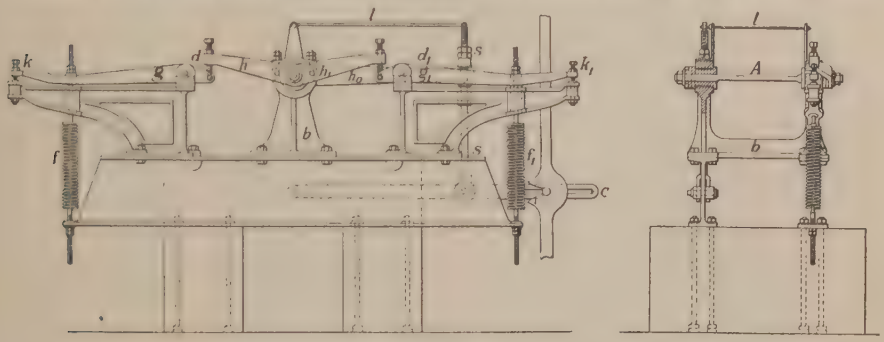


Fig. 436.

g or g_1 by means of an adjustable screw. At the further end of g or g_1 was a set screw k or k_1, and between k and the fulcrum was an adjustable spring, which was adjusted so that when the desired twisting moment was applied k would just rise from its seat. There were two levers g and g_1 so that the test piece could be twisted in opposite directions if desired. For twisting in one direction only, one lever may be omitted.

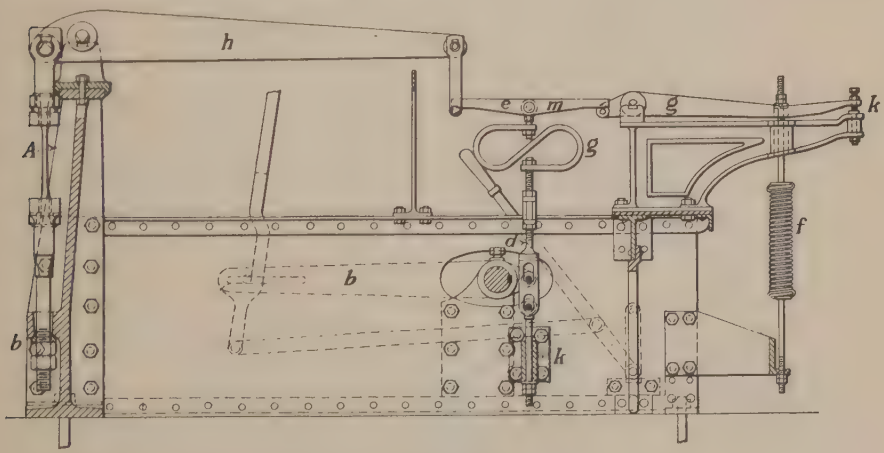


Fig. 437.

In the machine for repeated tension (Fig. 437), the test piece A was held at the bottom by the fixed frame, and at the top was attached to a lever h, which acted through the equal armed lever or beam em on the lever g, at the outer end of which was the set screw k, and on which acted the spring dynamometer shown. The maximum load just raises k, as adjusted by the spring. The load is applied

through a reciprocating rod which pulls the rod d downward. A desired minimum load is ensured by the regulating nut on the downward prolongation of d. Where d is attached to the lever b it is slotted.

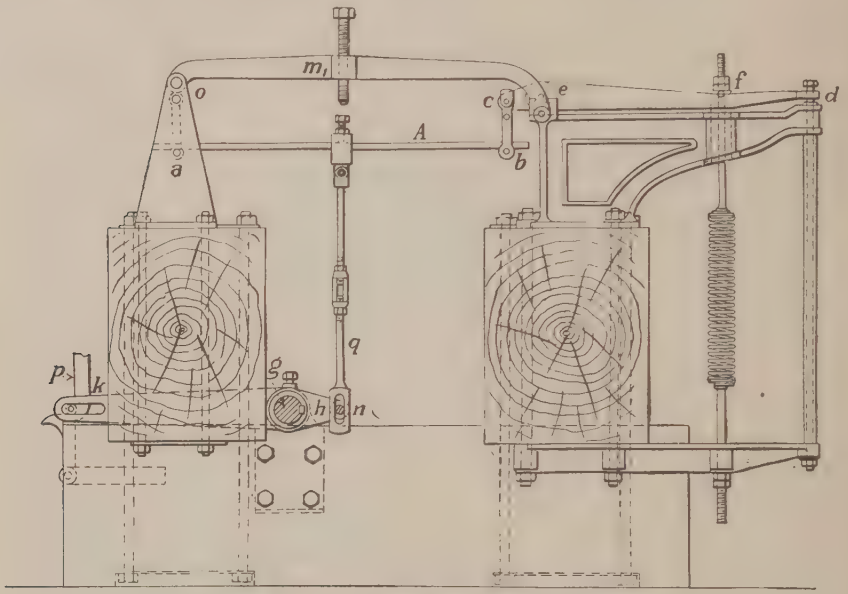


FIG. 438.

The machine for repeated bending in one direction (Fig. 438) was arranged to apply a load in the center of the test piece, which was supported at one end on a fixed support and at the other on one end of a lever cd. The maximum

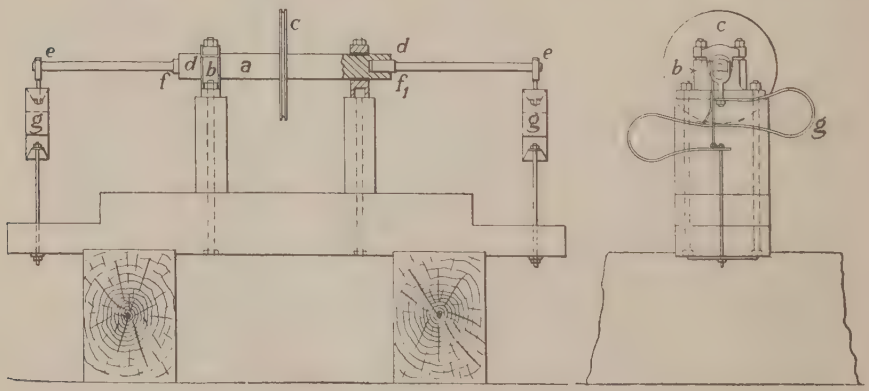


FIG. 439.

load was regulated by the calibrated spring so as to just raise the lever at d. The minimum was regulated by a set screw at m which prevented the loading rod from unloading the test piece below the minimum desired, which was itself first determined and adjusted by the calibrated spring.

In the machine for reversal of stress by continual revolution (Fig. 439) one end of the test piece was inserted in the revolving shaft driven by a pulley, and the projecting cantilever end was loaded and the load regulated by a spring dynamometer.

3. Wöhler's Laws.[1]—The laws discovered by Wöhler may be stated, in language different from his own, as follows:

(*a*) A smaller load than the static ultimate will cause fracture if repeated often enough.

(*b*) If the minimum stress is unvaried, the number of repetitions necessary to cause fracture is larger, the smaller the maximum stress.

An illustration is found in his tests on iron axles, repeatedly loaded in flexure in the same direction, the minimum stress being therefore zero, and the maximum f. Here the number of repetitions (n) necessary for fracture was

For f = 57,100	n = 169,750
51,900	n = 420,000
41,530	n = 1,320,000
31,150	$n >$ 48,200,000.

(*c*) If the maximum stress is unvaried, the number of repetitions necessary to cause fracture is larger, the larger the minimum stress.

As an example, the following are tests of cast spring steel, unhardened, f_o the minimum, f_1 the maximum, and n the number of repetitions;

f_o = 17,235;	f_1 = 103,830;	n = 62,000
34,575;	103,830;	n = 149,800
51,915;	103,830;	n = 400,050
60,533;	103,830;	n = 376,700
68,530;	103,830;	$n >$ 19,673,000.

(*d*) If the maximum is less than a certain limit, fracture will never occur, no matter how many times repeated. This maximum is called the *endurance limit.* It varies, obviously, according to the minimum.

(*e*) The endurance limit is greater, the greater the minimum stress. As an example, the following are tests with wrought iron, f_o the minimum, f the endurance limit:

f_o = −16,600;	f = +16,600
f_o = 0;	f = +31,150
f_o = +24,900;	f = +45,680
f_o = +46,720;	f = +46,720.

4. Wöhler concluded that in bending or tension there would be equal safety in stresses based upon the following ranges in ultimate stress per square inch:

Wrought iron.....	between +16,600 and −16,600
	between +31,150 and 0
	between +45,700 and +24,900

[1] The stresses herein given from WÖHLER's experiments differ from those given in some other books. The discrepancy seems to the writer to be due to different reductions of Wöhler's figures, which are given in "Centners" per square inch (Prussian). My own figures are from the original sources as given by Winkler in kilograms per square centimeter.

Axle cast steel	between +29,100 and −29,100	
	between +49,800 and 0	
	between +83,000 and +36,350	
Unhardened spring-cast steel	between +51,900 and 0	
	between +72,700 and +25,960	
	between +83,000 and +41,500	
	between +93,450 and +62,300	

and in shearing,

Axle cast steel	between +22,850 and −22,850	
	between +39,450 and 0.	

With $m = 4$, pure shear produces the same unit elongation as pure tension if $f_s = \frac{4}{5} f_t$ (see Art. 12, Chap. IV). Using the figure for shear when the minimum is zero, that is, 39,450, and multiplying this by $\frac{5}{4}$, the result is 49,310, which is very close to the figure for tension in the same material, 49,800. Similarly $22{,}850 \times \frac{5}{4} = 28{,}560$, which is close to the figure for tension, 29,100. Perhaps this indicates that the value $m = 4$ is very close to the truth.

5. Fatigue.—The term fatigue is used to denote the phenomenon that a material cannot sustain so large a stress if repeated or reversed as it can if continuously applied, and the term fatigue strength is sometimes used for the maximum stress that can be sustained under given conditions of repetition or range. Reversal of stress lowers the maximum much more than repeated stresses of the same kind, so that the phenomenon is of special importance in rotating parts, such as axles, in which the stress is completely reversed at each revolution. Fatigue strength is the same as endurance limit.

6. Dimensioning Based on Fatigue Strength.—There are several views regarding the applicability of the laws and facts of fatigue strength in proportioning the parts of structures or machines.

(*a*) The idea immediately suggests itself that the allowed or working stress should be based upon the fatigue strength, which depends upon the range of stress. This means that the allowed stress should be the maximum or fatigue strength divided by the proper factor of safety; and this allowed stress would thus be less for a repeated or live load than for a continuously applied or dead load, while for a reversed load it would be still less. This may be allowed for by multiplying a live stress by a factor greater than unity, to reduce it to an equivalent dead load, and using the allowed stress for a quiescent or static load; or by using a varying allowed stress, depending upon the range from maximum to minimum under the actual conditions.

(*b*) On the other hand, since rupture will never occur if the maximum stress is below a limit which—for stresses not reversed—is above the yield point, and since the actual stress allowed in many cases should not be above the yield point, in general, on account of the large deformation which takes place at that point, other engineers believe that the yield point, and not the maximum strength, should be the stress to which the factor of safety should be applied, and consequently that in dimensioning structures *in which the stress is not reversed* it is not necessary to pay any attention to fatigue strength.

This is the view generally held by structural engineers. The results of tests on repeated stress do not affect their method of dimensioning, if the stresses are

not reversed. Many formulae have been suggested and used, based on the experiments of Wöhler and later engineers, for proportioning by the first method; but none of them are now widely used unless the stresses are reversed.

For completely reversed stresses, the fatigue strength is often or generally less than the original yield point, and consequently must be considered in the design of parts subject to such stresses, and indeed of any parts in which there is stress reversal. Formulae for this purpose, or rules based on the results of tests of fatigue strength, are therefore still in common use.

(*c*) Still others have suggested that in the case of a structure or of machine part, it should first be decided how many repetitions or reversals will be allowable before the piece will need renewal in any case, due to wear or some other cause, and then that the allowed stress should be based on the maximum stress which will sustain that number of reversals or repetitions. This method is thus a modification of method (*a*), and would not be used where the latter is inapplicable. Indeed, the writer does not know that it has ever been actually used. It would only be used in the case of parts which would not be expected in any case to last indefinitely, but which would from other causes need replacement in a determinable time.

But in any case, whether the results are definitely used in designing or not, the study of fatigue and the experimental determination of the endurance limit, is of great interest and value, because it shows what materials have the greatest endurance, and so conduces to a selection of suitable materials for given purposes even if the results are not used in proportioning them.

7. Bauschinger's Experiments.—Professor J. Bauschinger published in 1886, in the thirteenth Heft of the Mittheilungen of his laboratory at the technical school of Munich, a communication entitled "On the change of the elastic limit and ultimate strength of iron and steel caused by high tensile andcompressive stresses, by heating and cooling, and by many repetitions of stress." This paper may be said to have been epoch-making, and it still remains the most complete and satisfactory treatment of the subject. Bauschinger drew many important conclusions from his tests, and although some of them relate to matters that are referred to in Chap. IV and XXIII, rather than to repetition, it may be well to state them all here.

Bauschinger measured deformations with his mirror apparatus (see Chap. XXI), which he considered accurate to $\frac{1}{250{,}000}$ inch; and he found that a stress above the elastic limit had the effect of raising the elastic limit, after a period of rest, to a point equal to, or even higher than, the stress applied. This was not new, for the fact had been stated by Wöhler, and by Thurston and by Beardslee in this country, but unaccompanied by any such conclusive evidence as Bauschinger offered.

He defined the elastic limit as the proportional limit. He said that the set limit was difficult to determine, requiring exceedingly (or impossibly) delicate instruments, while the proportional limit possessed three properties which made its determination relatively easy, *viz.:*

(*a*) While any permanent set within the proportional limit is very small, it suddenly increases greatly at or just beyond that limit;

(*b*) While repetition between zero and any load within the proportional limit always gives the same total and permanent elongation, both increase with every repetition if the upper limit of load is above the proportional limit;

(*c*) While the elongation for any fixed load does not increase with time, for any load within the proportional limit, it does increase, above that limit, for quite a time after the load is applied.[1]

From this definition of the elastic limit he drew the conclusion that if a repeated load above the elastic limit is not able to raise the elastic limit up to itself, the piece will surely fail if the number of repetitions of the load is great enough; for each application will increase slightly the permanent set. If, however, a repeated load does, after some repetitions, raise the elastic limit up to or above itself, the piece will never fail, no matter how great the number of repetitions.

He distinguished elastic limit from yield point, and stated that at the latter point, under a constant load, the deformation increases for several hours or even days; while after unloading it decreases for a long time, often for several days.

His definite conclusions may be stated as follows:

(1) A load above the yield point always immediately raises the yield point to the load; after the load is removed, if the bar is allowed to rest, the yield point rises to above the maximum load applied, and this rise continues for days, weeks, months, and perhaps years.

(2) A load above the yield point, if the bar is immediately re-tested after its removal, lowers the elastic limit, often to zero; in a period of rest the elastic limit will rise, reaching after several days the applied load, and after a sufficient time, certainly after several years, rising above the maximum load.[2]

The rise of the yield point and elastic limit may be delayed, but not prevented, by blows of a hammer. Repeated stressing between zero and a load below the original elastic limit does not affect the rise of yield point or elastic limit.

Repeated stresses between zero and a load below the original elastic limit do not change this limit. After a very high overstress, up to near the ultimate, the subsequent rise of the elastic limit during rest is very slow but not prevented.

(3) As a rule, a load above the yield point lowers the modulus of elasticity as well as the elastic limit; the former rises in a period of rest, like the latter, but probably more slowly; but after several years it is always considerably above its original value.

(4) A load above the elastic limit but below the yield point raises the elastic limit immediately, the more, the higher the load, up to a maximum when it is just below the yield point.

(5) Severe jars, as by cold hammering, lower an elastic limit which has been raised by overstress and rest. The yield point is also lowered by such treatment, but not much, and remains far above its original value.

(6) Heating and subsequent cooling, even if several times repeated, do not affect the elastic limit and yield point of steel if the heating is below 350° with rapid cooling, or below 450° with slow cooling, or below 400° for wrought iron with either rapid or slow cooling.

[1] The first two of these phenomena may be masked or counteracted by elastic recovery ("elastische Nachwirkung") which may be taking place, as in pieces which had been previously overstressed; but the third phenomenon always manifested itself.

[2] Muir has shown that the elasticity may be quickly restored by immersing in boiling water for a few minutes.

(7) Above these limits, heating and cooling always lower the elastic limit and the yield point, the more, the higher the heat. The effect on the elastic limit is much greater than on the yield point. (There may be exceptions to this rule, as, for instance, if the elastic limit is still rising due to previous overstress.)

(8) Rapid cooling after heating lowers the elastic limit and yield point, particularly the elastic limit, far more than slow cooling. Rapid cooling from 500°, certainly from a cherry-red heat, reduces the elastic limit of wrought iron or steel to zero, or nearly so; while slow cooling, even from cherry-red, cannot lower it so much. Elastic limit and yield point so lowered, do not rise in a subsequent period of rest.[1]

(9) By a stress over the elastic limit in tension (or compression), the elastic limit for the opposite kind of stress is much lowered, the more the higher the overstress. A small overstress (*i.e.* over the elastic limit) lowers the elastic limit for the opposite stress to zero or nearly so. It can be raised by repeated loads of the same kind. Such a lowered elastic limit does not rise in a period of rest up to three or four days, and but little, if at all, in weeks.

(10) Gradually increasing reversals of equal stress in tension and compression will not reduce the elastic limit for the opposite stress unless the original elastic limit is exceeded.

(11) When the elastic limit has been lowered by an overstress of the opposite kind, it can be raised by gradually increasing equal reversals, but only up to a point considerably below the original elastic limit.

Bauschinger obtained one of Wöhler's tension machines (Fig. 437), and made many tests with the lower limit zero. His conclusions regarding strength under many repetitions may be stated thus:

(12) Repeated stresses from zero to an upper limit below the original elastic limit will not cause fracture after five to sixteen million repetitions, if the piece has no defects. The smallest defect, even if scarcely noticeable, may reduce the number of repetitions before fracture to a small number, particularly in homogeneous materials like steel. (Wrought iron is less affected, being fibrous.) Surface defects, particularly, cause low endurance.

In using this principle, it is also essential to remember that the "original elastic limit" is very variable, depending on previous treatment, and may be artificially raised by overstress or cold working, as shown above; or may be lowered by overstress of the opposite kind. The principle applies to an original elastic limit not so artificially raised or lowered.

(13) Repeated stresses from zero to a maximum near or a little above the original elastic limit raises the latter to a point above, sometimes much above, the maximum load, the higher the greater the number of repetitions, but not above a certain limiting value.

(14) Repeated stresses from zero to a maximum which can raise the original elastic limit to a point above this maximum, will never cause failure; but if the maximum is so high that the original elastic limit cannot by this range of stress be raised above this maximum, failure will ultimately ensue.

A material free from defects and without artificially raised elastic limit is assumed.

[1] This principle explains the weakness of wrought-iron columns in a conflagration, when one side is suddenly cooled by a hose stream.

(15) The ultimate tensile strength for a static load is not lowered, but rather is raised, by millions of repetitions of stress.

(16) Repetition of stress makes no change in molecular or crystalline structure.

Remembering now (1) that if the elastic limit is lowered by overstress, it gradually rises in time; (2) that repeated overstress may in time raise the elastic limit to above itself; (3) that overstress of one kind lowers the elastic limit for the opposite kind without recovery by rest; (4) that a so lowered elastic limit may by gradually increasing reversals be raised to a certain limit; Bauschinger calls this last limit (see No. 11) the *natural elastic limit.* He concludes:

(17) The limits of completely reversed stresses must not exceed the natural elastic limit if the piece is to sustain an unlimited number of repetitions. Bauschinger put this forward as a hypothesis which he intended to investigate as soon as he obtained a machine for reversals of stress.

Bauschinger continued his experiments from 1886 to 1893, and the later results were published in the twenty-fifth Heft, as already stated; but no new conclusions were formulated.

8. Application of Wöhler's Results.—If we lay off as abscissas (Fig. 440) the minimum stress, and as ordinates the maximum stress which the material will

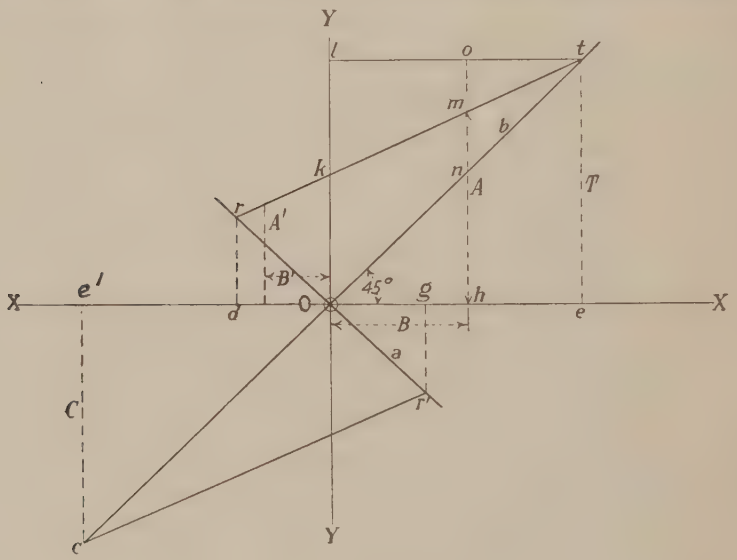


Fig. 440.

carry without ever breaking, no matter how many the repetitions, with the corresponding minimum—that is, the endurance limit with that minimum—the points so obtained will represent the result of the tests for any given material These points will necessarily be irregularly distributed. The problem is, to find the best smooth curve or line which will represent them. Tension may be represented to the right of O and above OX and compression to the left and below. Thus points in the first quadrant will represent stresses between an upper limit A and a lower limit B, both in tension; while points in the second quadrant

will represent stresses between an upper limit A' in tension and a lower limit B' (less numerically than A') in compression; and similarly points in the third quadrant will represent ranges entirely in compression and in the fourth quadrant between a maximum compression and a minimum tension. If two lines a and b are drawn at 45° with OX, only points in the areas rOt and $r'Oc$ will be plotted; for any tension below Or would be conjoined with a compression greater than the tension, and hence would be predominantly compression and should be plotted below Or'. The ordinate et will represent the static tensile strength T for which the maximum and minimum are equal; while $e'c$ will represent the static compressive strength. Ok will represent the greatest tension U which could be repeated indefinitely without fracture, if completely removed each time; while $rd = r'g$ will be the greatest stress which could be reversed an infinite number of times, without removal, or what is usually termed the *endurance limit*, in a restricted sense. (Whenever the term endurance limit is used, it should be made clear whether it refers to complete reversal or not.)

Various suggestions, based on Wöhler's results, were made soon after their publication, as to the shape of the line (or lines, one for predominant tension and the other for predominant compression) which would best represent the law.

Winkler, in 1877, suggested two straight lines rt and $r'c$, and he deduced as their equations:

For rt:

$$\left.\begin{array}{ll} \text{Wrought iron;} & A = 0.55T + 0.45B \\ \text{Cast steel;} & A = 0.44T + 0.56B. \end{array}\right\} \quad (1)$$

For $r'c$:

$$\left.\begin{array}{ll} \text{Wrought iron;} & A = 0.60C + 0.40B \\ \text{Steel;} & A = 0.37C + 0.63B. \end{array}\right\} \quad (2)$$

In these equations A is the maximum and B the minimum stress intensity, T the ultimate tension, C the ultimate compression. For wrought iron, dr, or the endurance limit for complete reversal, would have $A = -B$, hence $A = 0.38T$; while for gr', $A = -B$, hence $A = 0.43C$; and since these must be equal, $0.38T = 0.43C$, or $C = 0.88T$. This of course, is not the ultimate compression of a long column, but the local ultimate.

Launhardt, in 1873, suggested the relation

$$A - B = \frac{T - A}{T - U} \cdot A \quad (3)$$

which gives $A = U$ when $B = 0$, and $A = T$ when $B = A$, as must be the case. Using the data of the tests, namely, for wrought iron $U = \frac{6}{11}T$; and for steel, $U = \frac{5}{11}T$, he deduced from this

$$A = U\left(1 + \frac{B}{A} \cdot \frac{T - U}{U}\right)$$

For wrought iron; $A = \frac{6}{11}T\left(1 + \frac{5}{6}\frac{B}{A}\right)$.

For cast steel; $A = \frac{5}{11}T\left(1 + \frac{6}{5}\frac{B}{A}\right)$.

The value $\frac{D}{A}$ is $\frac{\text{minimum stress}}{\text{maximum stress}}$. Allowing that the maximum may be increased by impact, he modified these formulae to

For wrought iron; $A = \frac{6}{11} T \left(1 + \frac{1}{2} \cdot \frac{\text{min.}}{\text{max.}}\right)$.

For cast steel; $A = \frac{5}{11} T \left(1 + \frac{3}{4} \cdot \frac{\text{min.}}{\text{max.}}\right)$.

Weyrauch, in 1876, modified Launhardt's formulae, making them different for stresses of the same kind from what they are when the stresses are reversed. He gave:

For only tension or compression:

Wrought iron; $A = \frac{6}{11} T \left(1 + \frac{1}{2} \frac{\text{min.}}{\text{max.}}\right)$.

Cast steel; $A = \frac{5}{11} T \left(1 + \frac{9}{11} \frac{\text{min.}}{\text{max.}}\right)$.

For compression, T is replaced by C.

For reversed stresses:

Wrought iron; $A = \frac{6}{11} T \left(1 + \frac{1}{2} \frac{\text{min.}}{\text{max.}}\right)$.

Cast steel; $A = \frac{5}{11} T \left(1 + \frac{5}{11} \cdot \frac{\text{min.}}{\text{max.}}\right)$.

here T is ultimate tension or compression, according to the kind of the greater stress; and the minimum is to be called negative, so that for reversed stresses between a maximum tension and a smaller compression, the formula for wrought iron would be

$$A = \frac{6}{11} T \left(1 - \frac{1}{2} \frac{\text{max. compression}}{\text{max. tension}}\right).$$

Gerber, Director of the South German Bridge Co. in Munich, in 1874, assumed the curve to be a parabola, but he made several assumptions which seem to the writer inadmissible and Bauschinger's results did not fit his formula. Unwin gives his formula as follows, if Δ is the range of stress:

$$f_{\text{max.}} = A = \frac{\Delta}{2} + \sqrt{T^2 - n\Delta T}$$

where n is a constant which Unwin gives, from the experiments, as 1.5 for ductile steel or iron, and 2 for hard steel. Three cases may be taken,

(*a*) A single load; here $\Delta = 0$, $A = T$.

(*b*) A repeated load from zero to A; here $\Delta = A$, and

$$A = 2T(\sqrt{1 + n^2} - n),$$

which, for $n = 1.5$, gives $A = 0.6T$.

(*c*) Completely reversed stress; here $\Delta = 2A$; $A = \frac{T}{2n}$.

Hence for this curve $Ok = 0.6T$, $dr = \frac{T}{3}$.

The Illinois tests indicated, for iron or steel, an endurance limit (for complete reversal), greater than $\frac{T}{3}$, varying from $0.38T$ to $0.6T$, and averaging about $0.5T$. Professor Moore recommends $0.45T$.

In the Illinois tests, the endurance limit for complete reversals was in every case but one below the original tensile yield point; and, excluding one or two cases where the proportional limit had been depressed by cold work, was below the proportional limit in all but two cases. One of these cases was for 0.02*C* steel, or nearly pure ferrite; in which case the homogeneity of the crystals may explain the fact. The other case, in which the endurance limit was above the proportional limit, and but little below the yield point, was for 0.93*C* steel, or almost exactly the eutectoid composition (see Vol. 2, in which case, again, homogeneity may be the explanation. Also, it must be remembered that the endurance limit is *calculated*, and may be erroneous (see Art. 22).

In none of the ferrous metals tested did the endurance limit for complete reversals fall below 36 per cent of the ultimate tension, for only one did it fall below 40 per cent, while for several it was over 50 per cent.

Annealing raised the ratio of endurance limit to proportional limit; though it reduces both, it apparently reduces the latter more, probably by relieving internal stresses.

9. Of the methods which have been given, that of Winkler, representing the maximum by a straight line, is the simplest.[1] It is to be noticed that since the line *Ot* is inclined at 45°, at any point *h*, where *hm* is the maximum tensile stress, *hn* = *Oh* is the minimum, so that *nm* is the range of stress. The lines *rO* and *Ot* may therefore be called lines of minimum stress, and *rt* the line of maximum stress, distances between them representing the range; and similarly for compression. According to Winkler's formulae, *Ok* is 0.55*T* for wrought iron and 0.44*T* for cast steel.

Professor Goodman, of Leeds, in his book on "Mechanics Applied to Engineering," 1903, gives a diagram similar to Fig. 440, but in which *Ok* is 0.5*T*, so that *kt* is 0.5*T*. This is often referred to as "Goodman's diagram." According to this diagram, *dr* is $\frac{T}{3}$, instead of 0.38*T* for wrought iron and 0.43*T* for steel as by Winkler's equations. Goodman justified taking $Ok = \frac{T}{2}$ by the fact that if this is done *mo* = *mn*. The minimum stress *hn*, being unvarying, is like a dead stress; while the range or added stress *mn* is like a live stress; and if the effect of a live stress, due to sudden application, is doubled, *hn* + 2*mn* should be *T*, which is the case if $Ok = \frac{T}{2}$. It is rather singular that this point of view is so closely borne out by the result of Wöhler's tests, for the constants in Winkler's formulae were determined from those tests by the method of least squares.

If $Ok = \frac{T}{2}$, the equation for maximum stress becomes

$$4 = \text{max. stress} = 0.5T + 0.5B$$
$$= \frac{\text{static ultimate}}{2} + \frac{\text{minimum stress}}{2}.$$

If the minimum is made negative when reversed, this equation holds for all stresses, from complete reversal to static ultimate.

[1] The writer believes that Winkler's discussion of this subject is not known in America or in England, for he has never seen a reference to it.

If S_r is the endurance limit for complete reversals ($=\frac{T}{3}$ according to Goodman's line), the above may be expressed thus:

$$\text{Maximum} = \frac{3S_r}{2 - \frac{\text{min.}}{\text{max.}}}.$$

This is plotted in Fig. 441, with the results of some tests.

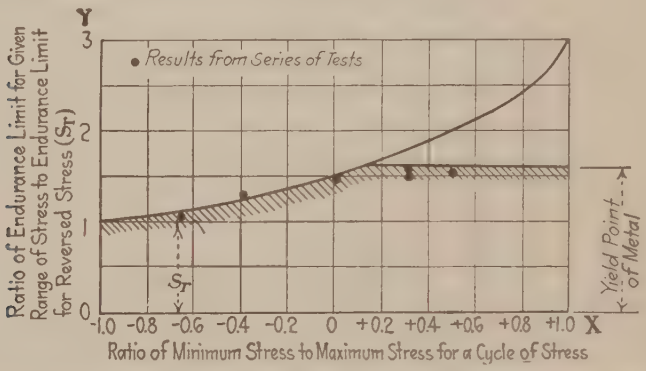

Y
3
2
1
0
• Results from Series of Tests
Ratio of Endurance Limit for Given Range of Stress to Endurance Limit for Reversed Stress (S_r)
S_r
Yield Point of Metal
-1.0 -0.8 -0.6 -0.4 -0.2 0 +0.2 +0.4 +0.6 +0.8 +1.0 X
Ratio of Minimum Stress to Maximum Stress for a Cycle of Stress

FIG. 441.—Effect of stress on endurance limit. (*Moore, Kommers and Jasper, A.S.T.M.* Vol. 22, 1922, pp. 286–290.)

10. Later Experiments and Machines.—Reference must now be made to the very extended experiments at the University of Illinois, embodied in *Bull.* 124, by H. F. Moore and J. B. Kommers, and in *Bulls.* 136 and 142, by H. F. Moore and T. M. Jasper. These tests are also described in reports of the Engineering Foundation. The principal tests were on rotating beams loaded with two equal

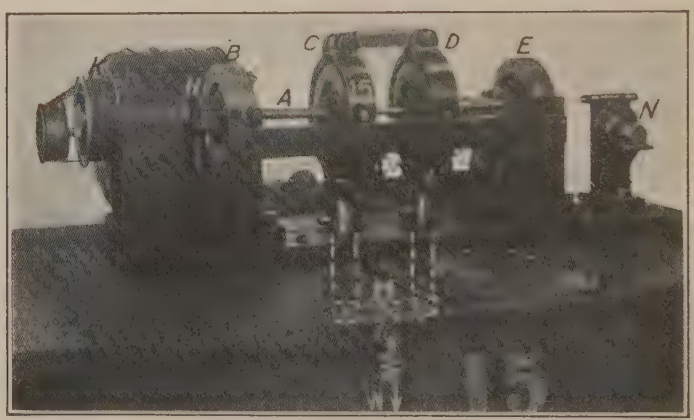


FIG. 442.—Rotating-beam testing machine. (*Farmer.*)

loads symmetrical on either side of the center, the magnitude of the stresses being computed by the usual flexure formulae. The speed was high, or generally 1,500 revolutions per minute. No tests were made for Bulletin 124 on repeated tension, but only on rotating beams, or on bending back and forth, which involved a complete reversal of stress at each revolution. The principal machine was the

Farmer machine, shown in Figs. 442 and 443. The shape of the test specimen is shown in Fig. 444. In another type of machine with which some tests were

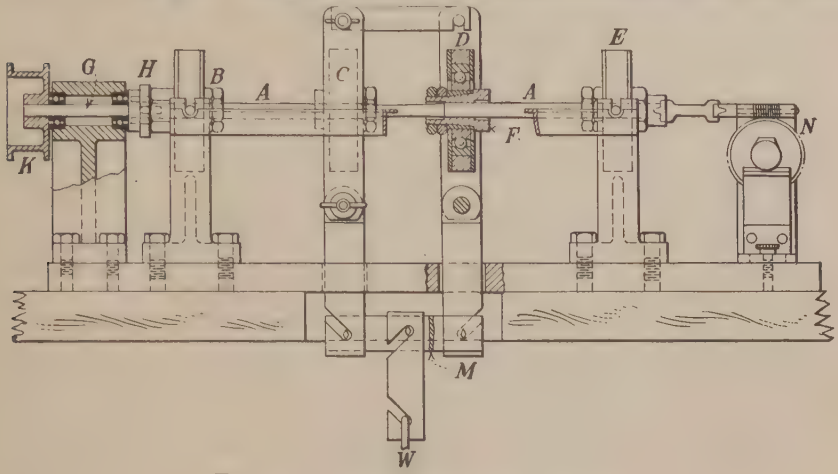


FIG. 443.—Farmer rotating beam machine.

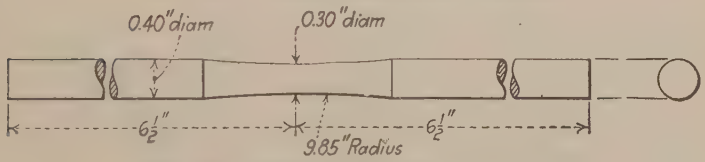


FIG. 444.—Test piece for Farmer machine.

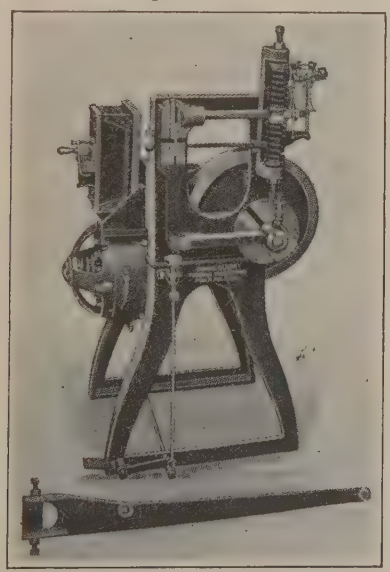


FIG. 445.—Upton-Lewis machine.

made, a projecting cantilever was loaded at the end and rotated (Wisconsin machine). Some tests were made with the Upton-Lewis machine, Fig. 445, in

which a small specimen was bent backward and forward, and the load measured by a calibrated spring; and some tests in torsion were made with the Olsen-Foster reversed torsion machine; also some with a machine for repeated impact, in which a grooved specimen held at the bottom as in the Izod machine is struck a series of blows, alternately on one side and on the other, by means of two swinging hammers operated by cams. In the later Bulletin, another machine for reversed shear is described.

In a revolving shaft there is a complete reversal of stress at each revolution, from tension to equal compression. There is no opportunity to test the effect of any other range. To allow this, a machine has been used in which the test piece is a projecting cantilever, which is encircled by a coiled spring pushing at one end against the bearing or a collar on the test piece, and at the outer end against an adjustable nut by which any desired compression could be produced in the spring, and equal tension in the test piece. When the test piece is loaded at the outer end through a ball bearing, and rotated, the top fiber is at any instant exposed to a tension equal to that due to the bending plus that due to the spring,

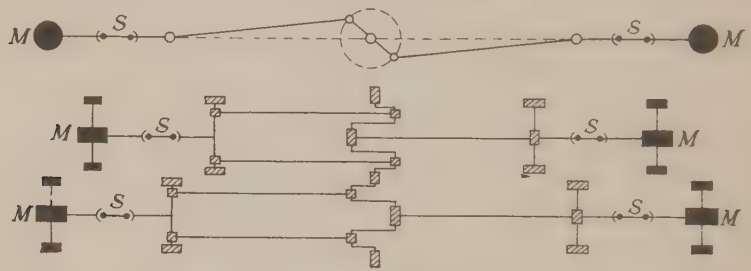


FIG. 446.—Osborne Reynolds inertia machine.

while the bottom fiber is exposed to the difference of these two stresses. By varying the load and the force in the spring, any desired range may be produced between a given tension and a smaller compression (but not between a tension and a larger compression). Such a machine was used in some of the later Illinois tests.

Another type of machine for reversed stresses is that of Osborne Reynolds, which depends upon the inertia of a mass placed in line with the test piece and connected to it, both being driven forward and backward in the direction of the test piece, by a crank and connecting rod, as shown in Fig. 446, which indicates four test pieces, *S*, simultaneously driven by one shaft. During one part of the revolution of the shaft the test piece will be in compression, and during the other part it will be in tension. The maximum tension and compression will not be equal, but the ratio of the two will depend upon the ratio of length of crank to length of connecting rod; the smaller that ratio, the more nearly equal the tension and compression. The maximum stresses will depend upon the reciprocating mass, and will vary with the square of the velocity; hence a careful speed regulation is necessary. Machines depending upon the inertia of rotating parts have been used in tests for reversed torsion.[1] Owing to the difficulty of speed regulation, inertia machines were not used in the Illinois tests, but they have been much used in England.

[1] D. J. McADAM, JR., "A High Speed Alternating Torsion Testing Machine," *A.S.T.M.*, p. 366, 1920.

11. Object of Tests.—The objects of tests of alternating or reversed stresses are two; (1) to find the maximum stress which, with a given minimum, may be applied an indefinite number of times—really infinite—without causing rupture; or, in other words, to determine the line *rt* in Fig. 440; (2) with a given minimum, to find the relation between a variable maximum and the number of applications which will cause rupture.

In seeking these objects, the first question which suggests itself is, how many applications of a load suffice to justify the conclusion that that load would never cause fracture. Because a piece sustains ten million repetitions—or any other number—is not proof that the next one may not break it. Here is one fundamental difficulty in attaining the first object stated above, and a source of possible

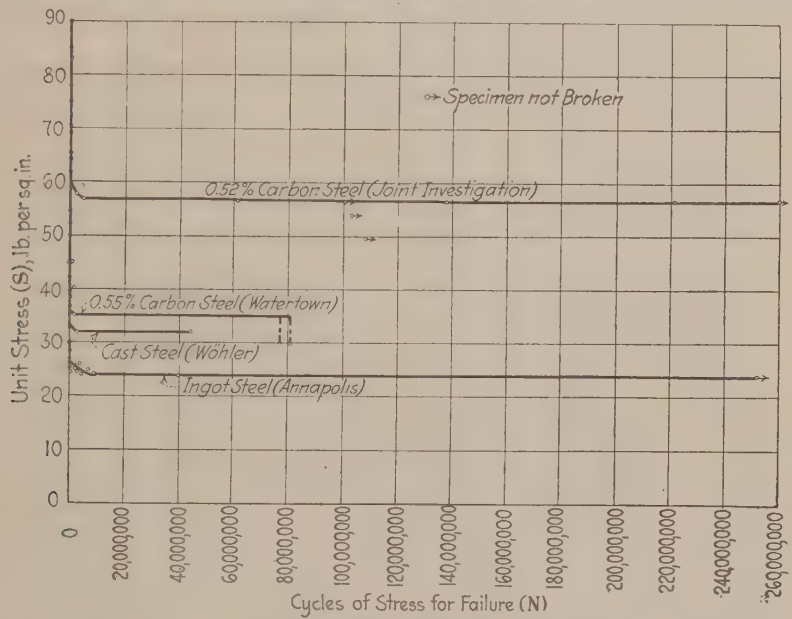


FIG. 447.—Fatique, or progressive failure of metals under repeated reversals of stress. (*Moore, Kommers and Jasper, Am. Soc. Testing Materials*, Vol. 22, 1922, p. 278.)

error in all of the published data. Bauschinger apparently thought that five to ten million repetitions, sustained without failure, indicated permanent ability to sustain the same load. Goodman thought four million sufficient. Bauschinger's maximum number of repetitions was 17,286,000, and Wöhler's was 132,250,000. With increased possibilities of high-speed machines, the necessary number has been considered greater, and in the Illinois tests was thought to be one hundred million for the Farmer machine; but even this, as stated, is no guarantee that the next application will not cause fracture. If so large a number is necessary, a test becomes very long. In Wöhler's machine the number of revolutions per minute is not stated, but it seems to be thought that it was not over 100; in which case, with an eight-hour day, there would be but fourteen million applications in one year of 300 days. Such a long test is only useful for research. But, by the same token, it may be that one hundred million applications, or even five million, may

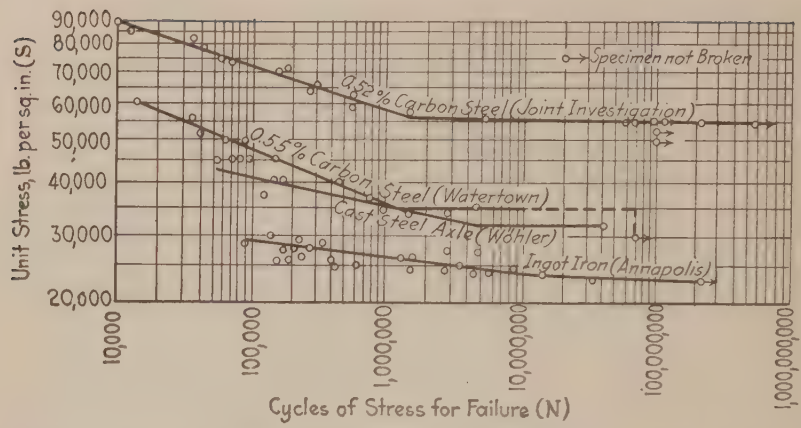


FIG. 448.—(*Moore. Kommers and Jasper, A.S.T.M.*, Vol. 22, 1922, pp. 280, 281.)

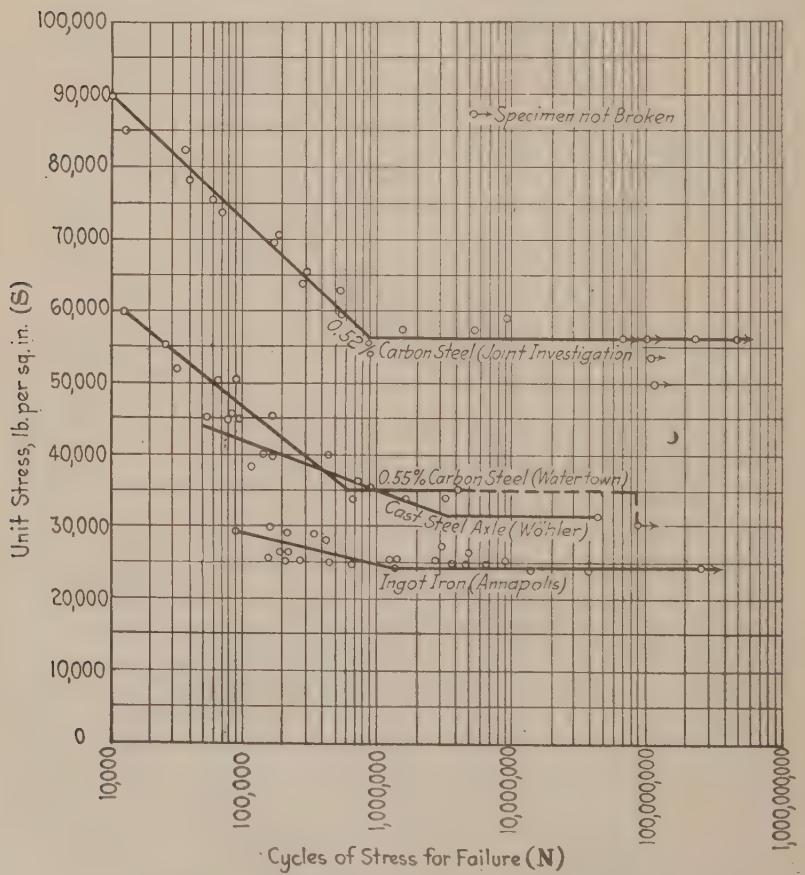


FIG. 449.—(*Moore, Kommers and Jasper, A.S.T.M.*, 1922, Vol. 22, p. 279.)

be well beyond the number which a given machine part could possibly sustain in its natural life; so that it may be justifiable to estimate that maximum number of actual applications and to proportion the piece for an endurance limit corresponding to that number.

The Illinois tests indicated that the endurance limit was developed at less than 10,000,000 repetitions for iron and steel, so that more than this number are unnecessary. On the other hand, it is stated that for some non-ferrous materials more than 100,000,000 repetitions are necessary. Some recent experimenters (*e.g.*, Gough) consider 3,000,000 repetitions sufficient.

12. S-N Curves.—These curves show, *for completely reversed stresses*, the relation between the maximum stress (in a revolving shaft), S, and the number of reversals

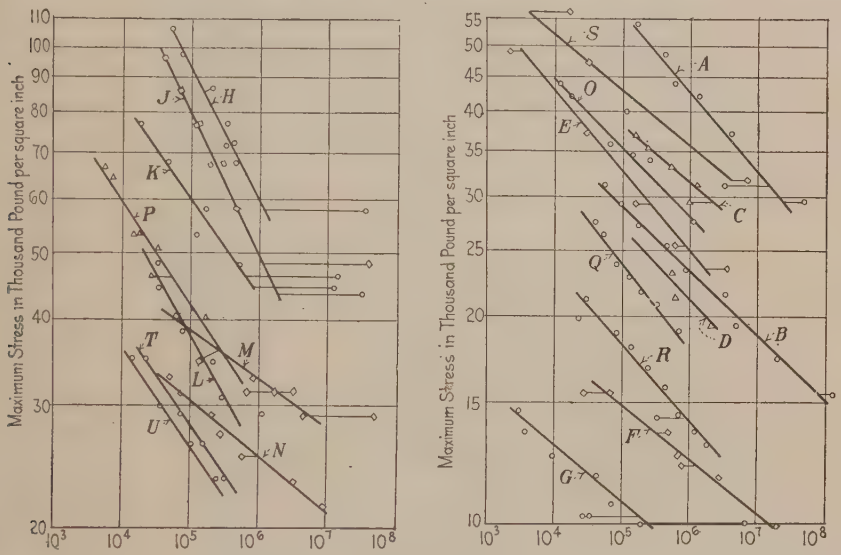


FIG. 450.—Number of repetitions before rupture. (*Basquin on Endurance Tests, A.S.T.M.*, Vol. 10, p. 628.)

required to break it, N. If these are plotted with the usual coordinates, as in Fig. 447,[1] they give no idea of the relations for small values of N. To avoid this difficulty, the logarithms of S and N may be plotted (logarithmic diagram), (Fig. 448) or the logarithm of N only (semi-logarithmic diagram), (Fig. 449).

For the relation between S and N, Professor O. H. Basquin[2] proposed, in 1910, an "exponential law," namely

$$S = CN^n$$

in which C is a coefficient and n a negative exponent to be found by experiment. He gave the results shown in Figs. 450 and 451. If these are plotted to logarithmic coordinates

$$\log S = \log C + n \log N$$

[1] See paper by MOORE, KOMMERS and JASPER on "Fatigue of Metals," in *A.S.T.M.*, p. 266, 1922; which should be read by every student of the subject.

[2] See *A.S.T.M.*, p. 625, 1910.

and the S-N curve is a straight line. If so, it follows that any load, however small, will break a piece if applied a sufficient number of times, and that there is no such thing as an endurance limit. Some engineers and metallurgists seem to hold this view, but the writer cannot subscribe to it. On the other hand, Professor Moore and his assistants believe that the tests indicate a decided

Curve Letter	Experimenter	Kind of Test	Material	Coefficient (Thousands)	Exponent
A	Wöhler	a	Wrought Iron Axles, Phoenix Co.	2.7	-0.12
B	Wöhler	b	Wrought Iron Axles, Phoenix Co.	103	-0.0
C×	Wöhler	b	Cast Steel, Borsig	103	-0.09
D×	Wöhler	b	Cast Steel, Borsig	94	-0.11
E	Wöhler	b	Homogeneous Iron, P.C. & Co.	130	-0.12
F	Wöhler	b	Bar Copper, Heckmann	36	-0.08
G	Wöhler	b	Cast Iron, Locomotive Cylinder	29	-0.09
H	Wöhler	a	Krupp's Spring Steel, hardened	1,000	-0.20
J	Wöhler	a	Krupp's Spring Steel, unhardened	920	-0.21
K†	Wöhler	c	Krupp's Axle Steel	320	-0.15
L†	Wöhler	c	Krupp's Axle Steel	310	-0.18
M	Wöhler	b	Krupp's Axle Steel	90	-0.07
N	Wöhler	b	Cast Steel, Vickers and Sons	97	-0.08
O	Benjamin Baker	d	Bars from Forth Bridge	45	-0.10
P	Benjamin Baker	b	Hard Steel	250	-0.15
Q‡	Reynolds & Smith	e	Annealed Cast Steel	92	-0.13
R‡	Reynolds & Smith	e	Annealed Mild Steel	66	-0.11
S§	F. Rogers	b	Unannealed Steel	110	-0.08
T	Föppl	b	Steel A, grooved	50	-0.15
U	Föppl	b	Steel B, grooved	135	-0.14

KIND OF TEST:

a - Bending in one direction only (+ and 0)
b - Rotating under bending load (+ and −)
c - Tension only (+ and 0)
d - Bending back and forth (+ and −)
e - Tension and smaller compression (+ and −)

× - *C* has round shoulders near grip; *D* has square shoulders
† - *K* has round shoulders near grip; *L* has square shoulders
‡ Ordinates are "Half Range of Stress" instead of maximum stress
§ Carbon, 0.32 per cent; Yield Point, 39,000 lbs. per square inch. Most points show the mean of more than one test.

$$S = CN^n$$

FIG. 451.—(*Basquin on Endurance Tests, A.S.T.M.*, Vol. 10, p. 629.)

angle in the S-N line, and a horizontal line to the right, the ordinate to which is the endurance limit, as shown in the figures. Others think that the line should be a curve, with no sudden break, though perhaps asymptotic to a horizontal line. (See McAdam in *A.S.T.M.*, p. 288, 1922.) It can never be conclusively proved which of these views is correct. It seems probable, however, that the exponential law holds nearly true over a considerable range of stress.

13. Stresses Not Completely Reversed.—As the exponent N in Basquin's formula is negative, the formula may be written

$$S = \frac{C}{N^m}.$$

Messrs. Moore and Seely[1] have proposed a modification to allow for incompletely reversed stresses, namely

$$S = \frac{B}{(1 - Q)N^{0.125}}$$

where $Q = \frac{\text{Minimum}}{\text{Maximum}}$, which is negative for reversed stresses (the minimum being taken algebraically), so that Q is zero when the minimum is zero, and -1 for completely reversed stresses. They give the following values for the constant B:

For wrought iron and structural steel	B = 250,000
For steel with 0.45 per cent C	350,000
For cold-rolled steel shafting	400,000
For tempered spring steel	600,000 to 800,000
For hard steel wire	600,000
For gray cast iron	100,000
For cast aluminum	80,000
For hard-drawn copper	140,000.

If R is the range of stress, the above formula is really

$$R = \frac{B}{N^{0.125}}.$$

In Illinois *Bull.* 134, the ratio of the endurance limit (or maximum stress) for any given algebraical ratio r of minimum to maximum, to the endurance limit for complete reversal, is studied, and it is concluded that the relation is

$$\text{endurance limit for ratio } r = \text{endurance limit for complete reversal} \times \frac{r + 3}{2}.$$

Probably our knowledge of the subject is as yet too incomplete to allow of the formulation of a reliable formula.

14. Correlation of Endurance Limit for Complete Reversal with Other Properties.—The Illinois tests indicate that there is a close proportionality between the endurance limit and the Brinell hardness number, until that number reaches about 400, when the steel ball deforms too much; that there is somewhat less close proportionality between endurance limit and ultimate tensile strength, and still less close with yield point and with proportional limit. A material with a lower elastic limit than another may have a higher endurance limit. There is no relation between endurance limit and the Charpy strength, or with resistance to repeated impact. In the British National Physical Laboratory the Brinell hardness is measured first, in order to estimate from it the approximate endurance limit.

15. Speed of Repetition.—The Illinois reports conclude that speed of repetition has little effect upon the endurance limit for complete reversals, below 2,000 revolutions per minute. At 200 revolutions it averaged about 2 per cent lower than at 1,500, and at 5,000 it was practically no higher than at 1,500. Stanton,

[1] See *A.S.T.M.*, p. 437, 1915; and *Trans. Am. Soc. C. E.*, p. 1474, 1916; *A.S.T.M.*, p. 470, 1916 (with diagrams for computation).

also, in his paper on "Resistance of Materials to Alternate Stress," at the 1912 International Congress, stated that, at the National Physical Laboratory investigations of the effect of speed, between 200 and 2,200 revolutions per minute, had "shown conclusively that in no case is there any difference in fatigue strength due to a variation in the rapidity of alternation between these limits."[1] Bairstow, in his paper of 1909, makes practically the same statement.

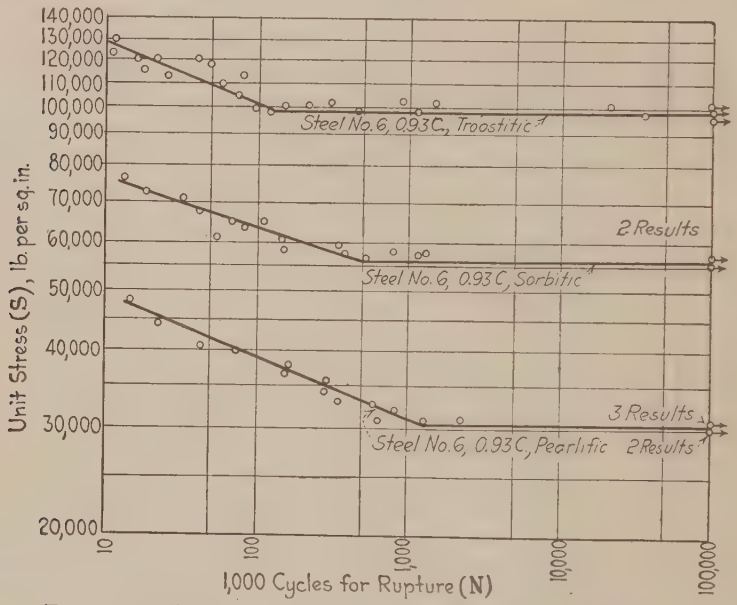


FIG. 452.—Effect of heat treatment. (*Bulletin, Univ. of Illinois.*)

16. Effect of Surface Finish, and of Defects.—An exceedingly important point, and one that vitally affects the value and use of the results of tests of repeated stress, is the fact that the endurance limits of iron and steel are greatly affected by minute flaws or surface scratches, even if they are so small as to be unnoticeable to the eye. In many of Bauschinger's test pieces which broke, thanks to the

[1] On the other hand, STANTON and BAIRSTOW, in *Proc. Inst. C. E.*, vol. 166, p. 94, give the following comparison:

Experimenter	Wrought iron		Steel	
	Reversals per minute	Max. range of stress for one million reversals, tons per square inch	Reversals per minute	Max. range of stress for one million reversals, tons per square inch
Wöhler	60	23.0	60	28.6 (cast steel)
Reynolds and Smith	1,890	11.7	1,892	11.9 (cast steel)
Stanton and Bairstow	800	19.3	800	28.4 (steel with 0.44 *C.*)

The apparent discrepancy here is probably due to the fact that the figures in the table are based on one million reversals, and do not give the real endurance limits.

In some machines, however, there is vibration, and the speed affects the results, so that high speeds may be impracticable.

careful record of the tests, it was noted that failure started at some minute defect. One minute imperfection may be fatal, and may lead to the failure, after a comparatively small number of repetitions, of a material having normally a high endurance limit. This is due to the fact that a fatigue failure starts with a minute crack or slip, which gradually extends till fracture occurs. The Illinois

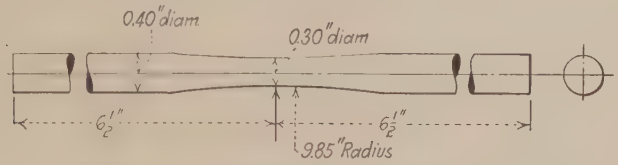


FIG. 453b.—Specimen for rotating beam test.

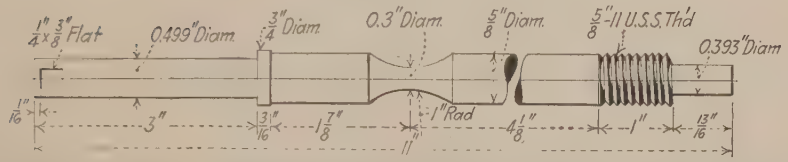


FIG. 453a.—Specimen for tests under steady tension combined with reversed bending.

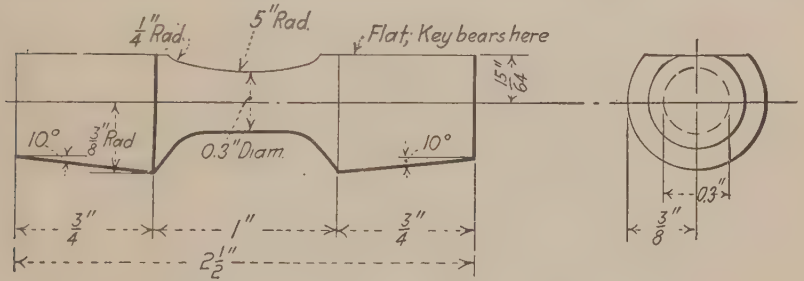


FIG. 453c.—Specimen for tests in reversed torsion.

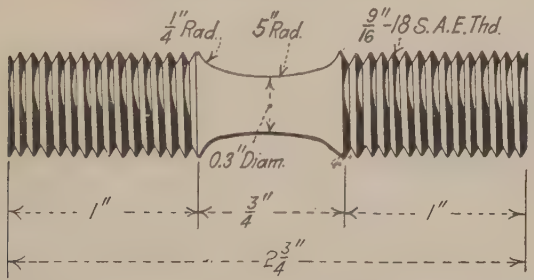


FIG. 453.—Specimen for tests under reversed axial stress.

tests showed that even differences in surface polish had a considerable effect. "Specimens of Bessemer steel with a turned surface showed a fatigue strength about 18 per cent lower than specimens of the same material which had been turned and polished." A surface defect which would not affect the static strength may greatly affect the fatigue strength.

17. Appearance of Fractures by Repetition.—The appearance of the fracture by repeated stress is characteristic. There is no necking-down, the fracture is sudden and without warning, apparently like that of a brittle material, with no general deformation. The fractured surface presents a general crystalline appearance, frequently with a dark spot near the edge, where the break may have started. Many slip bands show on the crystals (see Vol. 2). The crystals appear coarser than in a static break, because in the latter they have become elongated in the necking-down.

18. Other Influences.—Many other influences affect the endurance limit, most of which might be inferred qualitatively.

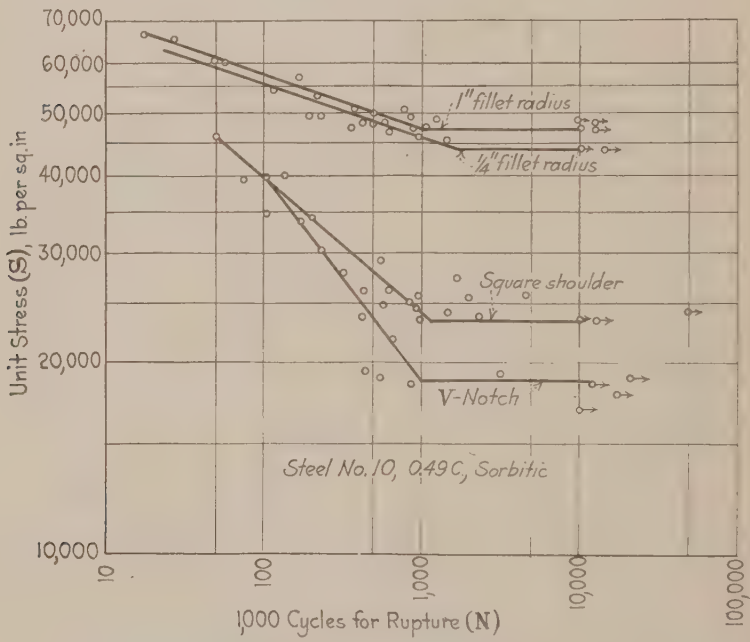


Fig. 454.—Effect of shape. (*Bulletin Univ. of Illinois.*)

Heat treatment has a great effect, as shown by Fig. 452 (See Vol. 2 for an explanation of the terms "troostitic," etc.). Suitable heat treatment may greatly raise the endurance limit as well as the static strength, with little or no sacrifice of ductility.

Abrupt changes of section, or even curves, by increasing the local stresses, have much influence. In the Illinois standard test piece for the Farmer machine, the piece had a radius, longitudinally, of 9.85 inches as shown in Fig. 453*b*. In other test pieces the radius is 1 inch (see Fig. 453*a*) and the results vary accordingly. For results of value, high localized stresses must be avoided. The effect is shown in Fig. 454. A groove 0.003 inch in depth may lower the fatigue strength of a soft steel shaft by 40 per cent. Lack of internal homogeneity may act like changes of section to increase local stresses. Professor Moore, in his book on "Materials," suggests the following multipliers, by which to multiply the

computed maximum fiber stress in the outer fiber of shafts, in order to obtain the actual maximum unit stress.

Ratio of radius of fillet or groove to diameter of shaft	Multiplier
0.1	2
0.5	1.6
1	1.2
2	1.1
Sharp V-thread	3
Whitworth V-thread	2
U. S. Standard V-thread	2.5

He also suggests, for parts finished with a lathe tool, to take local maximum stress 1.2 times the computed stress; for parts finished with a grinding wheel 1.06; and for parts carefully polished, 1.

All such rules are of course necessarily uncertain.

Cold working, as is shown in Chap. IV, raises the yield point, raises the ultimate much less, and the endurance limit less in proportion than the yield point, because of the effect of alternate stresses explained in Bauschinger's conclusions in Art. 7; hence it lowers the ratio of endurance limit to yield point. If the surface is roughened by cold working, the endurance limit may be lowered.

If a severe tensional overstress is applied before testing for endurance, the endurance limit is decreased, but only slightly if the overstress is not over 30 per cent above the original endurance limit. If the overstress is from 50 to 70 per cent above the original endurance limit, that limit is reduced, according to the Illinois tests, by 20 to 23 per cent. This means that cold-rolled shafting has less resistance to reversed stresses than hot-rolled shafting.

The Illinois tests are stated to indicate the interesting possibility that if a piece is subjected to many repetitions of a completely reversed stress just below the endurance limit, that limit may be raised. Thus, the endurance limit of normalized 0.37 C. steel was found to be 33,000 pounds per square inch; yet a piece of such steel was given over one-hundred and seven million reversals of 33,900, and then of over fifty-million reversals of each of five higher stresses up to 41,500 pounds. This may have been due to an individual peculiarity of the piece tested (since the pieces could not all be uniform), for the initial stress was above the stated endurance limit. Still it would not be surprising if reversals just under the real endurance limit should raise that limit, just as a static load just under the ultimate may raise that ultimate.

In the Illinois tests the endurance limit (for complete reversals) was found to be lower in tests of direct axial stress than in reversed bending tests, the ratio of the two being found as 0.64 (Bulletin 142, p. 42). The careful experimenters making the research consider that "for parts subjected to reversed axial stress the endurance limit should not be regarded as more than 60 per cent of that for parts subjected to reversed flexural stress." In the writer's view, however, this difference is not a difference of real endurance limit, but of the endurance limit to be used with the common (erroneous) formulae. In Chap. X, Art. 37,

it has been pointed out that in flexure the actual stress above the elastic limit is always less than that given by the flexure formulae. The writer can see no reason why the real endurance limit should be different in the two cases; but the writers of the Bulletin referred to believe it reasonable to assume that the probability of developing fatigue failure is greater where all the fibers are stressed equally than where they are not.

19. Endurance Limit in Shear.—This is found by torsion tests; and the Illinois tests are stated to show that the reversed shearing endurance limit averages about 0.53 of the bending endurance limit (for revolving shafts).

20. Bairstow's Tests.[1]—Bairstow states Bauschinger's theory as being that "specimens subjected to repetitions of stress begin to be fatigued when the stresses applied in each cycle are so great that the extension of the specimen is not wholly elastic," *i.e.*, there is an increase of permanent set at each cycle. Fatigue is failure due to repetition of stress. Bairstow concludes from his tests that "the truth of Bauschinger's theories would now appear to be established." His tests were of Swedish iron, hard Bessemer steel, and axle steel. He tested in direct tension and compression (not revolving shafts) using test pieces having diameters of 0.17 to 0.25 inch, and a length of ½ inch in the straight cylindrical part, with fillets ⅜-inch radius at each end. Elongations were measured by a Martens mirror apparatus reading to $\frac{1}{10{,}000}$ millimeters.

21. Rapid Methods of Finding Endurance Limit.—To find the endurance limit by the application of one hundred million repetitions is a long process, and inapplicable except in research. It is found however, that when the endurance limit is reached or slightly exceeded, there is a sudden increase in the temperature of the bar. The repetition of any load may produce a hysteresis loop (see Chap. IV, Art. 23), which means that at each application a certain amount of net work is done on the bar, which is changed into heat. A hysteresis loop does not necessarily mean ultimate failure. If it is completely closed, there is no increase of permanent set at each application. In time, with repeated applications, a hysteresis loop may flatten out and become a straight line. But if the loop is not closed, and the permanent set increases with each repetition, ultimate failure is certain.

In the Illinois tests, a close correspondence was found between the endurance limit for complete reversals and a sudden increase in temperature. This is shown in Fig. 455. There was some rise of temperature before the endurance limit was reached, but the sudden increase was easily discernible. The rise of temperature under the endurance limit means a hysteresis loop, and this shows that a hysteresis loop does not mean failure. The Farmer machine cannot be used for temperature tests, and the piece cannot be examined without stopping the machine. For temperature tests, a cantilever test piece was held at one end in a bearing, and deflected at the other by a revolving bearing through which the load is applied. This loaded end was made to move in a small circle concentric with the axis of the piece, the radius of the circle being the deflection corresponding to the stress desired. The temperature was measured at the most stressed section close to the bearing and at the unstressed section beyond the bearing.

[1] *Phil. Trans.*, series A, vol. 210. pp. 35–55, 1909.

The use of rise of temperature was first suggested by Strohmeyer.[1] It applies with some materials, but not with others. In some materials the rise of temperature test gives no reliable indications of the endurance limit.

Another quick method is suggested in the paper by H. J. Gough.[2] A mirror is attached to the end of a rotating cantilever. Normally the strain will be indicated, at high speeds, by a steady band of light, if a spot of light from a lantern is reflected from the mirror to a scale. When the endurance limit is reached there is a sudden increase in the width of the band.

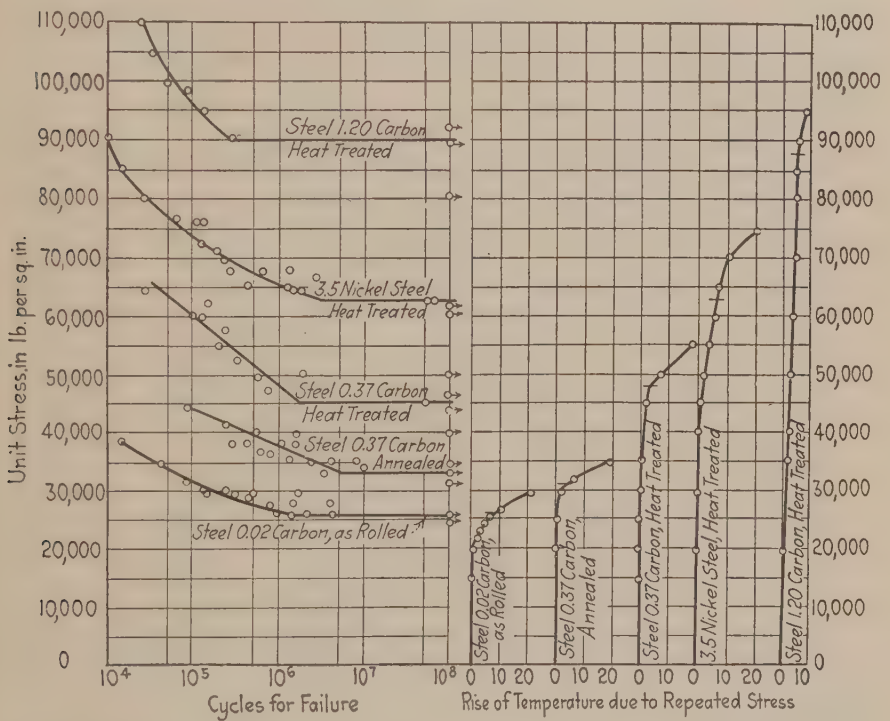


FIG. 455.—Comparison of reversed tests and temperature tests. (*Bulletin Univ. of Illinois.*)

22. Remarks and Conclusions Relative to Fatigue Tests.—The diagrams for fatigue strength, or *S-N* diagrams, show that the lines for different materials sometimes cross. This is well shown in Fig. 456, where the cold-drawn steel, though it showed better than the normalized steel under a small number of revolutions, had a considerably lower fatigue strength. Conclusions as to endurance limit cannot safely be drawn from tests with few revolutions, or different materials properly compared.

Tests of repeated or reversed stress by the tension machine, if it is accurate, give the true limits of stress. It is different with the rotating beam machine. Here the stress must be found by the formulae for flexure, and these are certainly

[1] *Proc. Royal Soc.*, 1914.

[2] *The Engineer*, Aug. 12, 1921.

incorrect above the elastic limit, hence in such tests, we cannot know the real stress if it is above the elastic limit. If the real stress, though at first above the elastic limit, can, by repetition, raise the elastic limit up to or above itself, then the formula will give the real stress when the elastic limit has been so raised; but, in such a case, by Bauschinger's principles, *the piece will never break*. Hence we see that *in any test of a rotating beam, in which the beam breaks, the real stress cannot be known*, and the error is greater, the smaller the necessary number of repetitions. The writer is therefore unable to agree with the statement in the *Illinois Bulletin* that "the magnitude of the stresses can be computed with a high degree of precision." He believes, on the contrary, that the real stresses are quite uncertain, and that this fact invalidates accurate conclusions from such tests. For accurate results, the stress should be directly weighed.

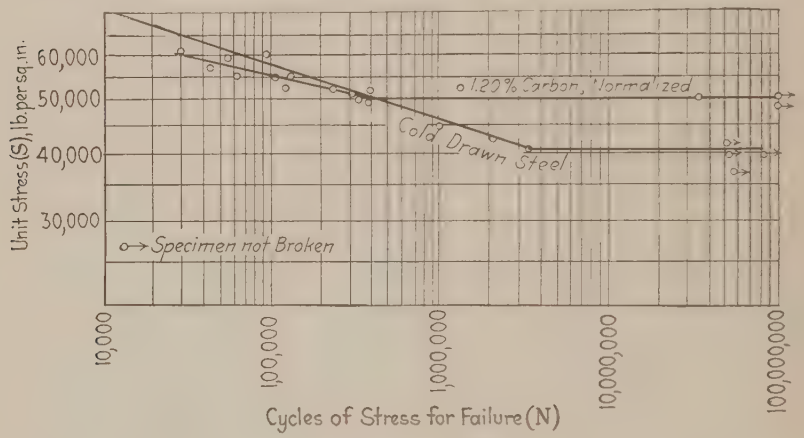


FIG. 456.—(*Moore, Kommers and Jasper, A.S.T.M.*, Vol. 22, 1922, p. 282.)

Still another fact discredits results of tests on rotating shafts, namely, that only one extreme fiber is exposed to the maximum stress. The results, therefore, cannot be applied to cases of direct stress in which the whole section is exposed to the maximum stress (see Chap. X, Art. 37 for discussion of this subject).

As a matter of fact, the test of a rotating shaft, in the writer's opinion, is a rather empirical test, and the results can only be properly applied to an identical or very similar case. If a rotating shaft requires a certain number of repetitions to break it, then another shaft of the same material and under the same computed stress, will probably carry about the same number; but it does not at all follow that a tension piece would do the same.

Another matter which may invalidate tests of rotating shafts is the possibility of an end bending moment due to the rigidity of bearings. Unless these are absolutely flexible in the plane of the loads, the computation of the bending moments and stresses may be seriously in error.

There are other sources of error in some machines, such as impact, suddenness of application, vibration, inertia of moving parts, etc.

For all of these reasons, and others which will be mentioned, the writer believes that the results of repeated tests on rotating bars must be regarded with doubt.

As a matter of fact, the results are extremely variable. Thus, when we read that one shaft, stressed to $\pm 59{,}700$ pounds, broke after 495,400 reversals, while another similar shaft of the same material and subjected to the same treatment, stressed to $\pm 59{,}900$ pounds, sustained 11,479,100 reversals before failure, what can we conclude as to endurance?

With reference to the use to be made of the results of fatigue tests, it is obvious that the *S*-*N* diagrams are of no value practically, but only useful as scientific research, unless in practice a part of a structure or machine is to be proportioned on the basis of the estimated number of repetitions which it will sustain during its life, by using a factor of safety on the endurance limit under the range of stress that it will sustain. Such a method of proportioning, as stated in Art. 6, has been suggested, and perhaps used. It may be applicable in the case of certain machine or automobile parts which will in any case be discarded after a determinable life. But for parts that are to last indefinitely, what is wanted is the endurance limit for an indefinite or infinite number of repetitions, not the maximum stress for a given number. And if this endurance limit is above the yield point, and the deformation at the yield point is objectionable, even the endurance limit becomes of no practical value. In the parts of structures, and in many parts of machines, the deformation at the yield point is so objectionable that the real stress should never be allowed to reach that point, but the factor of safety should be used on the yield point, not on the maximum. For such parts, then, tests of repeated stress, having demonstrated that the maximum for one kind of stress is above the original or primitive elastic limit, or at all events above the normalized elastic limit, and for reversed stress is above the "natural" elastic limit, no further tests of fatigue can have practical application.

There are parts of machinery in which the stress need not be kept below the yield point. A hoisting cable, for instance, is not made unserviceable by the elongation which occurs at the yield point. In such cases, and in others where the number of applications of stress during normal life can be determined with reasonable accuracy, the *S*-*N* diagrams may find a use.

But what shall be said about their reliability as a basis for design in view of the admitted fact that the minutest surface defect or scratch may be fatal to endurance? The parts of structures are not polished before use, though the parts of machines sometimes are made very smooth; but even in that case there may be minute surface flaws, which may entirely upset all endurance tests.

These considerations have led the writer to the view that while fatigue tests are of great value as indicating the relative endurance of different materials, their practical use in designing structures is very limited. For machinery parts, which are smooth and polished, they are much more applicable. Fatigue tests, however, are quite in vogue at present. Like all new things, they will probably be carried to excess and applied without due discrimination; just as, following Wöhler's tests, the profession was inundated by a flood of formulae for dimensioning, most of which have since been discarded, except for reversed stresses. The endurance limit for completely reversed stresses is an important and fundamental engineering constant for a given material, and should be determined.

CHAPTER XXIII

CAUSES AND CONDITIONS OF FAILURE OF A MATERIAL

1. It is generally, if not universally, assumed in designing that a material will fail—that is, break, or else deform so as to become unserviceable—when the stress of some kind (tension, compression, or shear) exceeds the ultimate strength for that kind of stress. If that is true, all that is necessary in designing is to compute the maximum stress of each kind, and to make sure that the safe limit—which is the ultimate (or the elastic limit or yield point) divided by a suitable factor of safety—is not exceeded.

But the different kinds of stress are inter-related, with each other, and with the deformations produced. These relations should be clear in the mind of the reader, and may be briefly repeated here.

A shear at a given point cannot exist on a single plane alone; there must be a shear of equal intensity on one plane at right angles to the first; and there must also be tension and compression of the same intensity on planes at 45°, and other shears and direct stresses on other planes.

Tension alone may exist on one plane, with no compression on any other plane through the point; and compression without tension; but in such cases there must be shear on all planes but the one on which the tension acts (or the compression) and the plane at right angles to it.

Tension may exist with no shear on any plane through the point, if there are three equal principal tensions at right angles; and similarly for compression. If the case is one of stress parallel to a plane, tension on a given plane must be accompanied by shear on some other planes; and the same for compression. Thus, if the stress is parallel to the plane of the paper (Fig. 457), a tension alone on plane *ab* necessarily means shear on oblique planes *cd*, all perpendicular to the paper; but if on two planes at right angles, *ab* and *a'b'*, there are equal tensions, then there is no shear on any plane passing through *O* perpendicular to the paper (see Chap. V, Art. 14). In this case, however, there *is* shear on planes through *O* not perpendicular to the paper such as *c'd'* in the side view. If, however, on three rectangular planes through *O* there are equal tensions at *O*, then there will be no shear at *O* or *any* plane through *O*. *Thus there may be tension or compression without shear, but not shear without both tension and compression.*

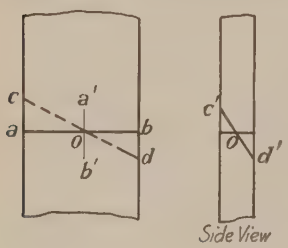


FIG. 457.

When there is normal stress alone (principal stress), on two or three planes at right angles, the case is said to be one of *biaxial* or *triaxial stress* (or loading).

The facts regarding deformation are similar. A tensile stress in one direction *a* causes elongation in that direction and contraction in a perpendicular direction

b; if there is combined with this stress a tension in direction b, the elongation in direction a is decreased. It is thus possible, in a homogeneous material, to have normal stress, either tension or compression, in a given direction, with no deformation in that direction, though this would require a larger normal stress (in steel, between three and four times) of the same kind in a single direction at right angles, and even a larger stress of the same kind in each of the two directions at right angles (in steel, one and five tenths to two times).

These considerations suggest two inquiries. The first is, whether failure is determined by stress or by deformation; will a material fail when the *stress* reaches a certain limit, or when the *strain* reaches a certain limit? The second is, since direct (normal) stress and shear are interdependent, whether a limit of one of these really determines failure; will a material fail when the shear reaches a certain limit, thus making the ultimate strength in tension or compression really dependent on the ultimate in shear; or will it fail when the direct stress reaches a certain limit, making the ultimate strength in shear depend on the ultimate in direct stress?

This leads to three theories of failure:

(a) The maximum stress theory.

(b) The maximum strain theory.

(c) The maximum shear theory.

Before considering these, some further explanation is desirable.

2. Failure May Not Be Fracture.—Failure of a material is not necessarily the same as fracture. Fracture means a rupture or breaking into separate parts. A material may fail or become unserviceable without fracture. Wood in compression, for instance, may fail by buckling or collapse of the cells or by crumpling of the fibers in shearing along some diagonal plane; an I-beam may fail by buckling of the top flange or by buckling of the web; a steel column may fail by buckling of one of its parts, such as an outstanding leg of an angle iron, or by buckling of lattice bars, causing buckling of one of the main parts of the column. In none of these cases need there be fracture or separation of the parts, and yet the piece may become entirely unserviceable.

The importance of this question is obvious when we remember that unless we know the conditions under which the material we are using will fail or become unserviceable, we cannot intelligently design it so that it will be safe. If the truth cannot be ascertained with certainty, we must at least have a reasonable hypothesis as to the conditions of failure, and must proceed in accordance with it.

Moreover, the structural engineer, who is studying a structural failure, must approach the problem with an entirely different attitude of mind from that in which he approaches a problem of design. A structure may have stresses far above those which it would be usual or proper to use in designing, and yet, under actual conditions, it may be perfectly safe, and if it fails, the failure may not be due to structural weakness.

The structural engineer who is charged with the maintenance of bridges, such as the railroad bridge engineer, has continually before him the question as to how far he may allow the actual stresses to exceed those used in designing, before he condemns a structure; that is, how far the factor of safety may safely be encroached on. Since a part of this factor is due to provision for overload due to increasing weight of rolling stock, some encroachment is reasonable. It is not so

easy to determine just how far it is safe to go (see Art. 15). The structural engineer may also be called upon to investigate the failure of a structure, and to determine its cause, which may not have been due to structural weakness at all, though stresses may have been much above those properly used in designing.

Failure, or fracture, can only occur in one of two ways: either the material is torn apart, separating the particles by tension, or one part must slide on another by shearing. This points to two of the theories of failure. The other theory involves the question, in either case, whether it is really stress or deformation which limits.

3. Furthermore, there are several criteria which may be assumed with reference to what constitutes failure, to which the proper factor of safety should be applied; as for instance, the limit of elasticity, the yield point, or the stress at rupture. Some engineers think that the yield point is the true criterion, because at that point a sudden and large increase in the deformation takes place, and a considerable permanent set is produced, which is objectionable. Some think that the elastic limit is the criterion, because above that point a slight permanent set is produced.

The elastic limit can scarcely be considered a criterion in general, because some substances appear to have no elastic limit; *i.e.*, limit of proportionality of stress to strain. For structural steel, the elastic limit and the yield point are so close together that for present purposes they may be considered as identical. The sudden and large increase in deformation which occurs at the yield point would be objectionable in the case of most structures, and the maximum stress allowed should be below this point. Some engineers, as already explained, think that the factor of safety should be the ratio of the stress at the yield point to the allowed stress rather than the ratio of the stress at fracture to the allowed stress, but this amounts to the same thing, for if the allowed stress is below the yield point there will be a factor of safety for both the yield point and the ultimate. The writer prefers to base the factor of safety on the ultimate, not only because there is in some materials no elastic limit, but also because sometimes a structure will be found to have stresses above the yield point, while under actual conditions it may be perfectly safe, and the deformation at the yield point may not be accompanied by any serious results.

4. There may be a difference in the criterion of failure between ductile and brittle materials, in view of the great difference in the stress-strain diagram in the two cases. It should also be remembered that any conclusions based on the theory of elasticity or Poisson's ratio are only valid up to the limit of elasticity, or proportional limit, within which E is constant, and that they require modification above that limit.

The conditions at actual rupture may have little relation to those existing at the yield point. The ultimate or breaking stress, however, does show what a piece is able to carry before it breaks, and in view of the fact that at the yield point an entire change takes place in the stress distribution, owing to the sudden tendency to deform the fibers which reach the yield point, it is obvious that conclusions based on the theory of elasticity afford no indication of what a piece will carry at the ultimate. Therefore, another reason why the writer considers that it is best to base conclusions as to safety on the ultimate strength, is because of the redistribution of stress at the yield point.

Failures of structures as a whole will be considered later. For the present we are considering the theories regarding the conditions of failure of a material itself.

5. Combined (or bi-axial) Stresses and Strains.—It will be convenient here to recapitulate the main conclusions shown in Chaps. IV and V regarding stresses and strains.

At any point in a body under stress parallel to a plane there are two planes (principal planes) at right angles, on which there is only normal stress. These so-called principal stresses may be of the same or of opposite kinds. If both are tension (Fig. 458), the unit strain in the direction of n_1 is, for $m = 4$ (Poisson's ratio $= \frac{1}{4}$)

$$e_1 = \frac{n_1}{E} - \frac{n_2}{4E} = \frac{1}{E}\left(n_1 - \frac{n_2}{4}\right).$$

If $n_1 = n_2$; $e_1 = \frac{3}{4}\frac{n_1}{E}$

If n_2 is compression; $e_1 = \frac{1}{E}\left(n_1 + \frac{n_2}{4}\right)$ or, if $n_1 = n_2$

$$e_1 = \frac{5}{4}\frac{n_1}{E}.$$

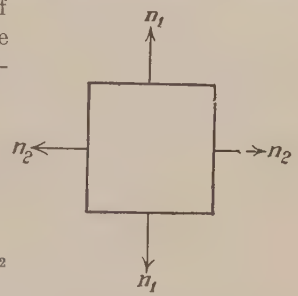


FIG. 458.

If the stresses are not parallel to a plane, there are three principal stresses, n_1, n_2, n_3, normal to three planes at right angles, and calling tension positive

$$e_1 = \frac{n_1}{E} - \frac{n_2 + n_3}{mE}. \qquad (1)$$

and similarly for e_2 and e_3

A stress in any direction produces a strain of the opposite kind in all directions at right angles to the first. Hence if n_2 is compression and $n_1 = 0$, and if the material can sustain but a small strain of elongation, n_2 may cause splitting along planes parallel to n_2.

For stress parallel to a plane the maximum shear intensity occurs on planes at 45° to n_1 or n_2, and is $\frac{n_1 - n_2}{2}$, in which n_1 and n_2 are taken algebraically; thus if both are tension or both compression, and equal, there is no shear on any plane. If $n_2 = 0$, the maximum shear intensity is $\frac{n_1}{2}$, as in a prismatic tension piece. If n_2 and n_1 are numerically equal, but one is tension and the other compression, the maximum shear intensity is n_1 or n_2.

For stress not parallel to a plane, maximum shearing stresses occur on six planes, in pairs, each pair being at right angles to each other and containing one principal axis, and bisecting the angle between the other two principal axes; the common stress intensity on these two planes of a pair being opposite in character and equal to half the algebraical difference of the two principal stresses whose directions they bisect.

6. Experiments.—The experimental study of this question is not easy, and it has not yet been solved satisfactorily. To do so, it would be necessary that a piece could be exposed to three principal stresses simultaneously, and that these principal stresses could be varied in any desired way. No way has been discovered of doing this. It is possible, however, to make tests with two principal

stresses which can be varied, by using as a test piece a hollow tube with variable internal hydrostatic pressure, and exposing it also to torsion and direct stress (either tension or compression). The internal pressure causes a circumferential principal stress (hoop tension) of any desired value f_n; the direct stress may be a tension f_t or a compression f_c; the torsion produces shear f_s on a section perpendicular to the axis, all of which may be varied. A thin tube is always used, in order that the stresses may be as certain as possible.

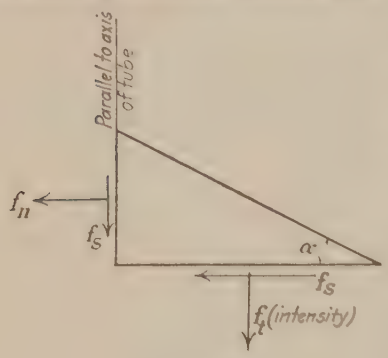


FIG. 459.

In an ordinary test in direct stress the principal stresses are 0 and $\pm p$. In a pure torsion test, they are $+p$ and $-p$, at 45° with the axis. In such tests it is impossible to have two principal stresses of the same kind; they must be either 0 and $+p$, 0 and $-p$, or $+p$ and $-p$. If, however, torsion is combined with direct stress and internal pressure, other combinations are possible. The case is as shown in Fig. 459, and the principal stresses n_1 and n_2 are

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{f_n + f_t}{2} \pm \frac{1}{2}\sqrt{(f_t - f_n)^2 + 4f_s^2}. \qquad (2)$$

By a combination of direct tension with torsion, and no internal pressure, any combination of principal stresses may be obtained, from $+p$ and 0 to $+p$ and $-p$. For in this case, in Fig. 459, $f_n = 0$, and f_t is the shear from torsion, and consequently

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{f_t}{2} \pm \frac{1}{2}\sqrt{f_t^2 + 4f_s^2}.$$

By making $f_s = 0$ (no torsion), $n_1 = f_t$ and $n_2 = 0$; while by making $f_t = 0$ (no direct stress), $n_1 = +f_s$ and $n_2 = -f_s$.

Similarly, by combining direct compression with torsion, with no internal pressure, any combination from $-p$ and 0 to $+p$ and $-p$ may be obtained.

By combining direct tension and internal pressure, without torsion, $f_s = 0$, and

$$\left.\begin{matrix} n_1 \\ n_2 \end{matrix}\right\} = \frac{f_n + f_t}{2} \pm \frac{1}{2}\sqrt{(f_t - f_n)^2}$$

or $n_1 = +f_t$ and $n_2 = +f_n$, and these may be made equal or one may be made zero.

It is thus obvious that any combination of two principal stresses may be obtained. The third principal stress is either zero or a radial compression which is generally small, and is neglected.

7. The Maximum Stress Theory.— The simplest and most widely used theory is that a material will fail under any given kind of stress when the maximum intensity of that stress reaches a definite limit, termed the ultimate; it will fail by tension when the ultimate tensile stress is reached, by compression when the ultimate compressive stress is reached, by shear when the ultimate shearing stress is reached. Inasmuch, however, as materials tend to become more rigid under

high compressive stress, and there are cases where no compression, however great, could produce failure, the theory is not applied to compression. This is of course subject to all relations between the stresses; thus, if a tensile stress f_t were always accompanied by a shearing stress of $\frac{1}{2}f_t$ (which is not true if there are two principal tensions at right angles), and if the ultimate shearing stress were less than half the ultimate tensile stress, there could be no tensile failure: it would always be by shear. This is the basis of the usual method of proportioning a structure, by keeping the maximum stress of each kind within what are considered safe limits for that kind of stress, the safe limit being determined by dividing the ultimate strength by an assumed factor of safety. The ultimate strength is found by testing a piece (or a number of pieces) under that kind of stress, until it breaks.

When a piece is tested to fracture under a given kind of stress, however, the fracture may not be due to that kind of stress, if other kinds exist at the same time in the piece. If a rod of steel is pulled in a testing machine, it will stretch as the load increases, and finally break when the load reaches a limit, which is called the ultimate strength in tension. This ultimate, however, may not be the ultimate strength in tension. The piece may break along a diagonal line, as in Fig. 460, and the failure may really be by shear; or, if the fracture is straight across the piece, the surface of the fracture may be cup-shaped, sloping from the edges toward the center at angles approximating 45°, and the failure may again be really by shear. If a piece under compression breaks by shearing along a diagonal plane, the friction on that plane, which in this case might be called internal friction, would modify the angle between the plane of fracture and the direction of the compression, as indicated in Art. 3, Chap. VI. The same may occur in tension.

FIG. 460.

We have seen that the strain or deformation in any direction depends not only upon the stress in that direction, but upon the stress at right angles thereto. The *maximum stress theory* assumes that the piece will fail under the kind of stress existing in it which first reaches the ultimate for that kind of stress; and that failure will occur under any given stress, independent of any stresses at right angles thereto. Of course, in non-homogeneous materials, the ultimate strength in one direction or on one plane may be very different from that in another direction or on another plane through the same point for the same kind of stress; as for instance in timber, where the tensile and shearing strength is very different along the grain from what it is across the grain. This, of course, is taken account of in the *maximum stress theory*.

When a tension bar is pulled, the *strain* increases with the load, and it may be strain and not stress that really determines fracture. It is obvious that there may be a tension in a given direction with no strain in that direction, if there is a tension in a direction at right angles thereto; or that there may be a much larger strain in the direction of the tension than would correspond to the tensile stress alone, if there is a compression in a direction at right angles thereto.

8. Professor Bridgman's Experiments.—Professor P. W. Bridgman of Harvard University has made a series of remarkable experiments at very high hydrostatic pressures, repeatedly reaching 30,000 atmospheres, or 441,000 pounds per square inch.[1] These experiments were of three kinds: (1) hydrostatic pressures

[1] "Breaking Tests under Hydrostatic Pressure and Conditions of Rupture," *Phil. Mag.*, July, 1912

on the circumference of solid bars, (2) hydrostatic pressures on the outer surface and closed ends of hollow cylinders, and (3) hydrostatic pressures on the interior of hollow cylinders.

In the first type, solid bars of various materials extended longitudinally through a hollow cylinder, passing through packing rings at each end. Pressure was applied to the outer surface by producing large hydrostatic pressures in the annular ring between the bar and the cylinder. The bars broke across a plane perpendicular to the axis, the two disconnected ends being expelled with violence through the packing rings. "On one occasion a specimen $\frac{5}{16}$ inch in diameter and 3 inches long penetrated 5 inches of wood, driven by a pressure of only 6,000 atmospheres." The pressure caused the bars to stretch longitudinally, and the friction of the packing rings caused a slight *compression* on the fractured area, in cases in which there was no drawing-down before fracture. Ductile materials like soft steel showed a drawing-down before fracture, and this resulted, of course, in tension on the fractured area; but this could not explain the *beginning* of the drawing-down. Hard materials like glass-hard tool steel or glass itself showed no necking-down. It may be, however, that the pressure, by reducing the diameter, caused slight annular shoulders where the rods passed through the ends of the cylinder, and so produced an axial tension. This would be the simplest

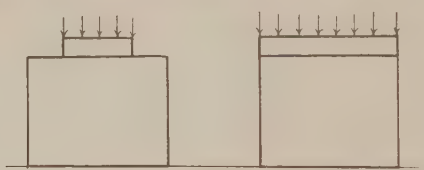


Fig. 461. Fig. 462.

explanation of the phenomenon. If it is not the true explanation, these experiments suggest that it may be possible for a bar of homogeneous material like steel or glass to be ruptured by *tearing apart* on a plane transverse section, just as it would be by a pure longitudinal tension, even though there may be *compression* on that section; in other words, that a tensile fracture or tearing apart may occur at a section where there is no tensile stress, or where there may even be a compressive stress. Also he found that in such cases the yield point and the breaking point coincided. He states, "No cases have ever been observed of a bar receiving set under this type of stress without rupture." This seems to conclusively disprove the maximum stress theory, at least as applied to tension, either for ultimate strength or yield point. A piece does not necessarily break in tension on a given plane when the tensile stress on that plane reaches a given intensity; nor does it always take a permanent elongation in a given direction when the tensile stress in that direction reaches a given limit. This agrees with what we have learned regarding the different action of local stress on one fibre of a beam, and a stress over the entire section (see Chap. X, Art. 37).

With reference to compression, it is also easy to see that this theory is untenable; for if a round steel ball be exposed to a uniform hydrostatic pressure over its entire surface, the intensity of the compressive stress may be raised very high without failure; indeed, it is hardly conceivable that such a pressure could

ever produce fracture. Also, the same compressive intensity of stress in a prism may produce very different results according to the area of surface to which it is applied. Thus, the same intensity applied to a block B over a part of its surface, as in Fig. 461, has very different effects from a compressive stress of the same intensity applied over the entire surface, as in Fig. 462.

These experiments, therefore, disprove the maximum stress theory, either for brittle or ductile materials, for tensile or compressive stress. Whether the theory fails for shearing will perhaps be clearer after discussing the next theory.

9. The Theory of Maximum Shear, or Maximum Difference of Principal Stresses. Considering stress whether parallel to a plane or not, this theory is that a material will fail or reach its yield point, when the algebraic *difference of two principal stresses* at that point reaches a certain limit, and not when any one reaches it. If the principal stresses are of the same kind, the difference is numerically smaller than the maximum; if they are of opposite kinds, the difference is the numerical sum of the principal stresses. It has been shown that the maximum shearing intensity is one-half the algebraic difference of the principal stress intensities, so that this theory is really a theory of maximum shear, and means that a piece will break when the shearing intensity on any plane reaches a limit which is the ultimate strength in shearing, and that it will not break by tension or compression until this ultimate shearing intensity is reached on some plane; that is, it means that all failures are due to shear, and occur when the shearing intensity reaches its ultimate.

We know that for stress parallel to a plane — say, the plane of the paper — if the two principal stresses are equal and of the same kind, there is no shear on any plane perpendicular to the paper through the point in question; but there will be shear on planes not perpendicular to the paper. If, however, there are three principal stresses (stress not parallel to a plane) and if all three are equal and of the same kind, there is no shear on any plane through the point. If the maximum shear theory is correct, then, a piece tested in pure tension should break along some diagonal plane inclined at about 45° to the stress, or the fracture should be cup-shaped with sides at about 45° with the stress. While tension breaks of a ductile material are often of this kind, others are clean tension breaks almost square across the piece. If the theory is correct, it must require that such a square break is really composed of very small conical or saw-tooth breaks, as indicated in Fig. 463. In such case, the maximum shearing intensity is one-half the maximum tensile intensity, and the theory must require that the ultimate shearing intensity cannot be greater than one-half of the ultimate tensile strength *on the actual surface of rupture* (not the original section).

Fig. 463.

This is inconsistent with the phenomena of fracture of a ductile material like structural steel. If the per cent of reduction of area at fracture is p, then the actual area is $A\left(1 - \frac{p}{100}\right)$, and if T is the total tension, what is called the tensile strength is $\frac{T}{A}$, but the real *average* breaking stress is

$$\frac{T}{A\left(1 - \frac{p}{100}\right)}$$

hence the actual shearing stress is $\frac{T}{2A\left(1 - \frac{p}{100}\right)}$. For structural steel it is generally considered that the ultimate shearing strength, f_s, is about 0.8 of the ultimate tensile strength $f_t = \frac{T}{A}$. But calling $\frac{f_s}{f_t} = r$, from the above

$$r = \frac{1}{2\left(1 - \frac{p}{100}\right)} \tag{3}$$

so that if $r = 0.8$, $p = 37.5$ per cent reduction. Frequently $p = 50$ per cent, in which case $r = 1$, or the shearing strength equals the tensile strength as usually computed.

Light would be shed on this point if the tensile and shearing strength of a material were determined, together with the reduction of area; but, unfortunately, it is impossible to determine the shearing strength accurately by direct tests, because more or less bending is invariably present, as explained in Chap. VII. Even such tests, however, would not be conclusive, because, when a ductile material necks down in breaking, the stress is not distributed absolutely uniformly over the fractured surface. The case becomes in some degree that of a grooved specimen (see Chap. VI), and the maximum stress is probably at the outside. The ratio r, of Eq. (3), should, therefore, strictly be

$$r = \frac{c}{2\left(1 - \frac{p}{100}\right)}. \tag{4}$$

in which c is an unknown constant greater than unity.

It is quite possible that the maximum shear theory, if modified to take account of "internal friction," might explain failure. Referring to Fig. 462*a*, we have seen in Chap. VI that in a piece exposed to *pure compression*, if A is the cross-section at right angles to P, and A_1 the section at an angle α with A,

$$\text{The shearing intensity on } A_1 = \frac{P}{2A} \sin 2\alpha = s,$$

$$\text{The normal intensity on } A_1 = \frac{P}{A} \cos^2\alpha = n;$$

but the value of s necessary to shear is not the real or usual shearing strength f_s, which is the value when there is no normal stress, but is

$$s = f_s + n \tan \varphi \tag{5}$$

if φ is the angle of internal friction.

The maximum value of s is for $\alpha = 45°$; but the maximum value of $s - n \tan \varphi$ is for

$$\tan \varphi = -\cot 2\alpha$$

or

$$\alpha = 45° + \frac{\varphi}{2}. \tag{6}$$

This value gives, substituting the values of s and n,

$$f_s = s - n \tan \varphi$$

$$= \frac{P(1 - \sin \varphi)}{2A \cos \varphi} = \frac{f_c(1 - \sin \varphi)}{2 \cos \varphi}. \tag{7}$$

Similarly, *for tension*, the value of s necessary to shear on A_1 may be less than f_s on account of the normal tension n, or

$$s = f_s - n \tan \varphi; \alpha = 45° - \frac{\varphi}{2};$$

from which is deduced, by the same process,

$$f_s = \frac{f_t(1 + \sin \varphi)}{2 \cos \varphi}. \tag{8}$$

In the general case of three principal stresses, we have seen that there will be three pairs of planes; each pair at right angles, containing one axis of principal stress, and bisecting the angle between the other two principal stresses; the shearing intensity on the two planes of each pair being the same, and greater than on any other plane containing the same principal stress axis, and being numerically equal to one-half the algebraic difference of the other two principal stresses.

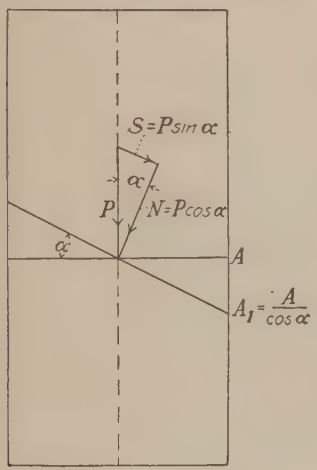


FIG. 462a.

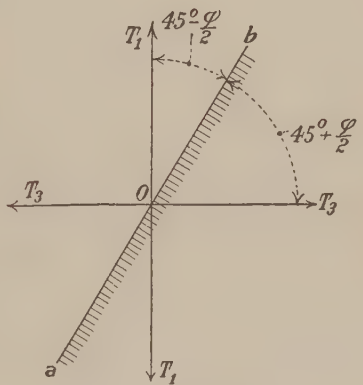


FIG. 462b.

The greatest shear will be on the pair of planes containing the middle principal stress if all are tension. That is, in Fig. 462*b*, if T_1 is the greatest principal stress intensity in tension, and T_3 is the least, the middle one being T_2, acting through O perpendicular to the paper, the numerical maximum shear will be $\frac{T_1 - T_3}{2}$ acting on planes making angles of 45° with T_1 and T_3. The stress acting on the surface ab will be T_1 acting upward and T_3 acting to the left; and the plane ab, which gives maximum shear necessary to shear, taking account of "internal friction," will make an angle $45° - \frac{\varphi}{2}$ with T_1 and $45° + \frac{\varphi}{2}$ with T_3. The real or usual f_s (for no normal stress) will be

$$f_s = \frac{T_1(1 + \sin \varphi)}{2 \cos \varphi} - \frac{T_3(1 - \sin \varphi)}{2 \cos \varphi}$$

$$= \frac{1}{2 \cos \varphi}[T_1 - T_3 + (T_1 + T_3) \sin \varphi]. \tag{9}$$

The first term of this equation is the same as Eq. 8, with T_1 instead of f_t: the second term is obtained from Eq. 8 by substituting $-\varphi$ for φ, since the angle with T_3 is not $45° - \frac{\varphi}{2}$, but $45° + \frac{\varphi}{2}$.

For pure torsion, in which there is a shearing intensity f'_s, and but two principal stresses, both equal and equal to f'_s, one tension and one compression, making $T_1 = f'_s$ and $T_3 = -f'_s$,

$$f_s = \frac{f'_s}{\cos \varphi}. \tag{10}$$

Note that f'_s, or the shearing stress in torsion, is not the same as what we have called the real or usual ultimate strength f_s when there is no normal stress, because the pure shearing is complicated by the diagonal tension and compression. According to this theory, a torsion break should not be square across, but should consist of surfaces making angles of $\frac{\varphi}{2}$ with the cross-section (which section is at 45° with the principal stresses).

Chido Sunatani[1] considers that his tests confirm this theory. He found, for mild steel, $\varphi = 20°$, very nearly. It seems also confirmed by the experiments of Gilbert Cook.[2] The value of φ probably differs for different materials, and it may also be different in compression from what it is in tension; though Sunatani believes it is the same in all materials.

If the maximum shear theory is correct, it should be impossible to cause fracture by three equal principal stresses. This is the case in compression, as where a cube or sphere is exposed to hydrostatic pressure on all sides; but the explanation is that fracture cannot occur unless the fractured parts have somewhere to go to, and here they have not. It is difficult if not impossible to conceive that it would be impossible to produce fracture by three equal tensions, no matter how large.

The study of torsion throws some light on this subject. In pure torsion there is a pure shear on a transverse section, and tension and compression on planes at 45°, of the same intensity as the shear. Cast iron, a brittle material, breaks with a helical fracture, in tension, hence shear does not govern. Soft steel, a ductile material, breaks square across, by shearing; this may be, however, because the ultimate shearing strength is less than the ultimate tensile strength. These facts show that the maximum shear theory does not hold for brittle materials. Professor Bridgman's experiments also show that it does not hold for brittle materials; for the maximum stress difference in pure tension is the ultimate tensile strength, and since, in the case of a bar extending through a cylinder and exposed to hydrostatic pressure, there is compression on all planes containing the axis, and only a slight longitudinal compression due to the friction in the packing (there being no drawing-down and if there is no shoulder at the end), the greatest stress difference is practically the transverse stress, and the bars should have failed when the hydrostatic pressure equalled the ultimate tensile strength, if the theory were true; while in fact they carried considerably more.

This theory, however, seems to many engineers more reasonable than the theory of maximum stress. Present opinion seems to be that for ductile materials

[1] Reports of the Tohuku Imperial University, Vol. III, No. 1, 1922.
[2] Engineering, Dec. 15, 1911.

the maximum shear theory holds, while for brittle materials the maximum stress theory holds; though Bridgman's tests seem to show this view to be incorrect. But Professor Boyd states "there is no question as to the truth of the theory" (of maximum shear);[1] and J. J. Guest[2] considers it proved by his tests, for a uniform ductile material, as the condition of initial yielding. Metallurgists seem to believe that failure of steel occurs by slipping on some plane or planes in the crystals.

10. The Maximum Strain Theory.—This theory is that fracture occurs when the maximum *strain* reaches a certain ultimate value, no matter how the strain is produced. The ultimate value is, of course, very different for different materials, or in different directions in non-homogeneous materials. For a homogeneous material the maximum unit strain in the direction of the maximum principal stress n_1 will be $\frac{n_1}{E} - \frac{n_2}{mE}$ algebraically, for stress parallel to a plane, and $\frac{n_1}{E} - \frac{n_2}{mE} - \frac{n_3}{mE}$ for stress in general.

A boiler is exposed to circumferential or ring tensile stresses tending to rupture it longitudinally, and also to longitudinal tensile stresses half as great. It is usually proportioned for the former stresses without reference to the latter, by the maximum stress theory. If the maximum strain theory is true, the axial stress should be considered, and the boiler is stronger than if it is neglected.

We have seen that in Bridgman's experiments tearing occurred on a section on which there may have been compression; but the tensile *strain* here may have been great. It is possible, however, for a material to tear when under a compressive strain. Bridgman has exposed a steel cylindrical rod, on which was a short close-fitting ring of hard rubber, to hydrostatic compression. Here everything was in compression, but the rubber ring tore at the inside, where there must have been compression. The explanation is that, if the steel had not been there, the rubber would have compressed more than the steel allowed it to do; or it may be said that it tore because the steel did not allow it to contract *as much as it wanted to*. If the theory is true, it would therefore seem that the tensile strain must be measured from the condition that would exist, under the action of compressive forces, if no tensile tendency were present. As we shall see, there are anomalies and conditions regarding compression that conflict with any theory. In the above case, the failure was not by shear, so that the shear theory failed also.

Professor Bridgman considers that the maximum strain theory, as well as the other two theories, is disproved by his experiments on thick hollow cylinders exposed to internal pressure. In such cases the principal stress, principal strain, stress difference and strain difference, all have their maximum values at the inner surface, *as computed by the usual formulae*, which, however, are founded on assumptions that may not be true (see Chap. XVIII), and which do not hold above the elastic limit anyway. But in these cylinders *the break began at the outside* and progressed inward, though in one case "it was not possible to ascertain whether rupture really started from the outside or not," and in another place he says, "There might still be some question as to whether the rupture really did begin at the outside." However, his tests of nickel-steel cylinders convinced

[1] "Strength of Materials," 3d. ed., p. 326.

[2] *Phil. Mag.*, pp. 69–132, 1900.

him that rupture really did begin at the outside, in ductile materials. For glass tubes, however, he says that "there can be little doubt that the rupture does begin at the inside." This matter will be further discussed in Art. 13.

Notwithstanding certain objections to the maximum strain theory, the writer believes it to be the best—and in general a valid theory, at all events for a ductile material like structural steel, under ordinary conditions, and it is widely held. The weighty authority in favor of the maximum shear theory, however, should prevent dogmatism on this matter.

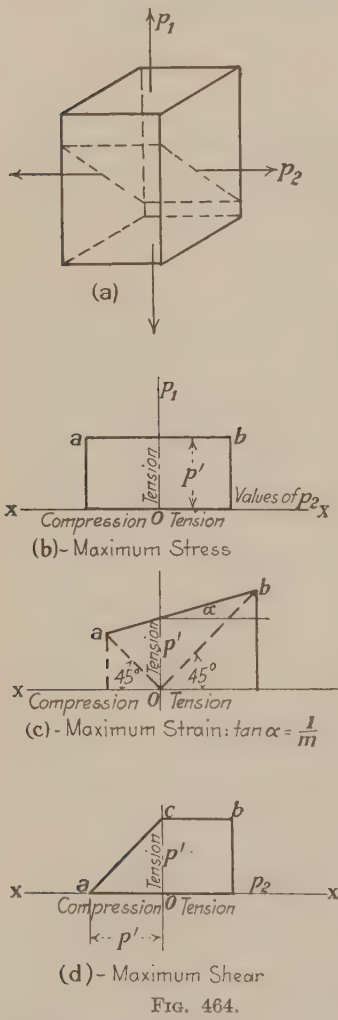


FIG. 464.

11. Comparison of Theories for Stress Parallel to a Plane.—Assuming that there are two principal stresses p_1 and p_2 (Fig. 464*a*), the third, perpendicular to the paper, being zero, p_1 being tension and p_2 smaller numerically than p_1, but of the same or opposite kind, the three theories require failure to occur when p_1 has the value shown by ordinates from xx to the line ab, for different values of p_2 less than p_1 numerically. The ultimate for pure tension is p'.

By the maximum stress theory, the value of p_1 will not depend on p_2, so that it will be equal to p' at all values of p_2 from $+ p'$ to $- p'$. The figure is thus two squares (Fig. 464*b*).

By the maximum strain theory the ultimate strain per unit length in pure tension is $\frac{p'}{E}$ (p' being stress intensity), and for any other condition the strain along p_1 must equal this. Hence, if $\frac{1}{m}$ is Poisson's ratio,

$$\frac{p'}{E} = \frac{p_1}{E} - \frac{1}{m}\frac{p_2}{E}$$

$$p_1 = p' + \frac{p_2}{m}. \qquad (11)$$

When $p_2 = 0$, max. $p_1 = p'$. As p_2 increases, being tension, the ultimate p_1 increases along the line ab, inclined at an angle whose tangent is $\frac{1}{m}$, to the point where $p_1 = p_2$. If p_2 is compression (negative), p_1 decreases as p_2 increases numerically, along the same line, to a point where $p_1 = p_2$ numerically, (Fig. 464*c*).

To use the maximum strain theory in practice the value of p_1 must be found from Eq. (11), and used as the ultimate. It is thus necessary to find both principal stresses. If both are equal, $p_1 = p_2$, and if $\frac{1}{m} = 0.3$, which is nearly its value for steel,

$$p_1 = p' \cdot \frac{m}{m-1} = 1.43p'$$

so that the gain is 43 per cent. If p_2 is compression and equal to p_1,

$$p_1 = p' \cdot \frac{m}{m+1} = 0.77p'$$

so that the loss is 23 per cent. All this assumes E to be constant, which is not true for brittle materials.

By the maximum shear theory, the maximum shear will be independent of p_2, so long as p_2 is tension, and will occur on a plane, shown dotted in Fig. 464*a*, containing the axis of p_2 and bisecting the angle between p_1 and p_3; and its intensity will be $\frac{p_1}{2}$; hence the value of ultimate p_1 will be the same, or p', no matter what the value of p_2, and the line *ab* will be the same as for the maximum stress theory if p_2 is tension. If p_2 is compression, the maximum shear will not occur on the dotted plane in Fig. 464*a*, but on a plane at right angles to the paper (containing p_3) and bisecting the right angle between p_1 and p_2, and its intensity will be *numerically* $\frac{p_1+p_2}{2}$. This must equal $\frac{p'}{2}$. Hence the line *ac*, to the left of 0, will be as shown, and p_1 will be zero when $p_2 = -p'$. For in simple tension, with $p_2 = 0$ and $p_1 = p'$ the ultimate shear is $\frac{p'}{2}$, and this must be its value whatever the value of p_2 according to the theory; if p_2 is negative, then numerically

$$\text{Maximum shear} = \frac{p_1 + p_2}{2}$$

which must equal $\frac{p'}{2}$, or $p' = p_1 + p_2$, or $p_1 = p' - p_2$, which represents the line *ac*, and $p_1 = 0$ when $p' = p_2$ numerically.

It is thus obvious that the maximum strain theory is the only one which makes the ultimate in tension in any given direction depend upon the tensile stress in directions at right angles. If it could be once demonstrated that the ultimate tension in any direction does increase as the *tension* at right angles increases, the maximum strain theory would be established as the one most nearly in accord with fact, at least for both principal stresses tension. This would be a crucial test.

12. Experiments.—Bridgman's tests have been already referred to. Other experimenters have tried to solve the problem,[1] but not always with success.

Becker experimented with thin tubes, 5.56 inches in internal diameter, and $\frac{3}{32}$-inch thick, exposed to internal hydrostatic pressure combined with axial tension or compression. The criterion of strength was taken as the "apparent elastic limit, where the slope of the stress-strain curve is 50 per cent greater than at the origin." The axial strains were measured on a length of 4 inches. The results, as estimated from the published diagrams, are shown in the following table. It appears that for axial tension there was a distinct increase in the yield point along p_1 as tension p_2 increased, until the ratio of p_2 to p_1 was about 0.5, after which the yield point p_1 remained nearly constant.

[1] Becker, "The Strength and Stiffness of Steel under Biaxial Loading," *Bull. Univ. of Ill.*, No. 85, 1916.

Guest, J. J., "On the Strength of Ductile Materials under Combined Stress," *Phil. Mag.*, pp. 69–132, 1900.

Ratio of circumferential to axial stress	Apparent elastic limit, pounds per square inch	By Fig. 464c the apparent elastic limit should be	By Fig. 464d the apparent elastic limit should be
Series 1			
Axial tension			
0	45,000		
0.24	48,000		
0.475	50,000		
0.69	50,000		
0.92	50,000		
Series 2			
Axial tension			
0	22,000		
0.475	26,500		
0.92	26,500		
0.92	27,000		
Series 3			
Axial compression			
0.00	25,500		
0.2	24,000	24,300	21,250
0.3	22,500	23,700	19,600
0.6	19,000	22,200	16,000
0.9	16,500	20,800	13,400

In the third series the axial force (p_2) was compression, and p_1 was tension. According to Fig. 464c the values of p_2 would be as shown in the third column if $m = 4$; this would be by the maximum strain theory. According to Fig. 464d, the maximum shear theory, the values would be as in the fourth column.

Becker concludes from the fact that above $\frac{p_2}{p_1} = 0.5$ the yield point remained the same, that two laws govern this matter, the maximum strain theory holding up to $\frac{p_2}{p_1} = 0.5$, and the maximum shear theory for greater ratios. This does not seem a necessary conclusion, for with two tensions the maximum stress theory agrees with the maximum shear theory, so that it might equally well be concluded that the maximum stress theory holds for values of $\frac{p_2}{p_1}$ above 0.5. For opposite principal stresses, the maximum strain theory (column 3 of table) is more in agreement with the tests than the maximum shear theory, and with a different value of m would be closer still. But the agreement is poor for either.

These tests of Becker, therefore, afford no satisfactory confirmation of the maximum shear theory.

One of the most important investigations of this subject was that of J. J. Guest, who, as a result, propounded the theory of maximum shear, which has

since been termed "Guest's law." He used small tubes of steel, brass, and copper, about 1¼-inch outside diameter and from 0.025 to 0.036 inch in thickness, exposed, like Becker's, to internal pressure and to axial stress, and also to torsion. The tubes were one foot long, the axial twist and strain were measured on a length of 8 inches, by a specially devised apparatus, partly optical. The specimens tested were small, the measuring apparatus new, and, as Becker says "Criticism may be made of the repeated use of the same specimen, since the yield point stress is raised by repeated loading beyond the yield point stress of the first test." In most of the tests the stress ratio was between 0.5 and 1., within which limits Becker believes "the shear theory undoubtedly holds." But it has been shown above that this is not the case, since it may be the stress theory that holds. The place to test a theory is under conditions where it conflicts, if true, with every other theory, that is, along the line *ac* in Fig. 464*d*, that is for principal stresses of opposite kinds. The writer knows of no tests under these crucial conditions which justify the maximum shear theory.

13. Other Criteria.—To pursue the subject of failure farther, it is clear that there are other criteria that must govern in some cases. *A piece cannot fail*, (no matter how large the stress), *unless the parts into which it tends to be broken have some place to go to.* Thus, a cannon ball, if homogeneous and exposed to uniform normal pressure all over, cannot possibly collapse. In the same way, a hollow tube, if absolutely homogeneous and of perfect geometrical form, could not collapse under external pressure. If of ductile material it could close up tight, but not break. In practice, such tubes may fail by collapsing, folding or creasing but it is because they are not perfectly homogeneous, or because the pressure is not perfectly uniform. If the thickness is so great that it will not fold or crease, the effect of pressure is to close up the hole uniformly, eventually closing up perfectly tight if the pressure is great enough, without rupture. Glass cylinders of heavy tubing sealed at the ends, and thick hollow spheres, were subjected to pressures reaching 360,000 pounds per square inch "with neither crushing nor alterations in the dimensions of measurable amount." This disproves the theories of maximum stress and of maximum shear.

Another obvious criterion must be that *the failure or yielding must tend to relieve the stress*, otherwise it would not take place or could not spread.

In each case, therefore, a physical conception must be formed as to the result of failure, as in the case of the cannon ball above. Consider, for instance, a thick tube of ductile material exposed to external or internal pressure. A portion of a curved bar (Fig. 465) exposed to pressure from the outside tending to break it at a section SS_1 by tension beginning at S, would tend to rotate as shown by the curved arrow, and could do so. But if it were part of a hollow cylinder the rotation would be prevented by the reaction on the surface *ab*, and there would be compression on the whole of every section; it could not fail because the parts could not go where they tend to go. If the pressure were from the inside, the part A would tend to go in the opposite direction, beginning to tear at S_1, and could do so; but if the piece were part of a hollow cylinder (Fig. 466) exposed to internal pressure, how could the parts A and B rotate as shown by the arrows so as to cause tearing at S_1, unless there has been a previous break from the outside as indicated by the dotted lines—in other words, an initial fracture beginning at the outside?

Professor Bridgman conceives that, after the yield point is passed at the inside, the metal there becomes plastic, transmitting pressures in all directions hydrostatically, so that after this stage is reached there is circumferential *compression* at the inside of the cylinder. But how could the large yielding take place which occurs at the yield point, tending to rotate A and B as shown in Fig. 466 unless there were previous fracture from the outside to the right and left? It seems simpler and clearer to the writer to consider that notwithstanding the large circumferential tension at the inside, strain does not occur as assumed, and that when a fracture starts at the outside, the tensile *strain* there is really greater than the limit, and greater than at the inside. In other words, this leads to the conclusion that *the strain in the direction of a given stress does not depend upon that stress alone, or upon it together with the stresses at right angles, but also upon whether physical conditions will allow the strain (which tends to exist) to really exist.* These considerations will show why the writer does not consider that Bridgman's experiments, or any others that he knows of, disprove the maximum strain theory,

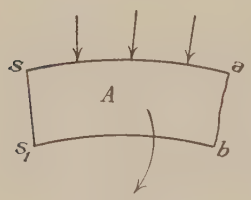


Fig. 465.

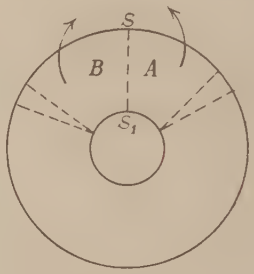


Fig. 466.

while they do clearly disprove the maximum stress and the maximum shear theories.

14. Practically, it makes not a particle of difference to the engineer whether the maximum shear theory is true or not. Engineers determine the ultimate strength or the yield point in simple tension or compression and in pure shear, they find the real maximum stress of each kind, and they allow what they consider a safe stress intensity, which is a fraction of the ultimate or of the yield point (factor of safety). They do not care whether the failure is primarily by shear or not. The only theory that concerns them, as possibly modifying their procedure, is the maximum strain theory. If this theory is correct, then, if two principal stresses are of the same kind, their usual procedure is a little on the safe side, for the real ultimate maximum principal stress is larger than it would be if—as they assume—there is no principal stress at right angles; the maximum gain being 43 per cent if $\frac{1}{m}$ is 0.3, and 33⅓ per cent if $\frac{1}{m}$ is 0.25 (see Art. 11). If, however, the two principal stresses are of opposite kinds, then their usual procedure is not on the safe side, or their real factor of safety is not as great as they assume, the maximum loss being 23 per cent if $\frac{1}{m}$ is 0.3, and 20 per cent if $\frac{1}{m} = 0.25$. This may be of importance.

As tests appear to indicate the correctness of the maximum strain theory quite as much as they do the correctness of any other—or more—it is clear that *it is wise to adopt the maximum strain theory whether it is strictly true or not.* The maximum stress allowable in tension should therefore be less when there is a compression at right angles than when there is none, according to formulae that have been given; and more when there is tension at right angles.

15. The theory of failure has not only the bearing on structural design that has been indicated in Art. 14, but it has an even greater bearing in the investigation of the cause of failure of a structure, which the engineer is often called upon to make. The attitude of mind in dealing with a question of failure is entirely different from the attitude in making a design. In the latter case the unit stresses used are those common in conservative practice, and they cover a multitude of items included in the factor of safety, one of which is a possible increase in the loads. It must be recognized that the stresses in an actual structure may greatly exceed those used in design, and yet the structure may be perfectly safe. A committee of the A.R.E.A. recently recommended allowing, in railroad bridges, a tensile stress of 26,000 pounds per square inch before considering that a bridge required strengthening, instead of 16,000 prescribed for design by the specifications of the society; and other stresses in proportion. Even a stress above 26,000 pounds if the material is good, would not be really dangerous. Steel will not be injured at all by stresses of one kind up to the elastic limit or yield point, even if repeated indefinitely, with good material, and even a much higher stress would not cause failure, as will be shown.

A tensile stress up to the elastic limit or yield point will cause no appreciable permanent set, and no injury. If above the yield point, a permanent set will result, and if such a stress occurs in a main tension member, it may cause a considerable total permanent stretch of the bar, which will seriously affect the secondary stresses, and for this and other reasons would be objectionable; but there are many instances in structures where even such a permanent set would have no ill effects. If the overstrain occurs in a detail, the resulting permanent set will be small, and may be harmless even if the overstrain is considerably above the elastic limit.

These questions are likely to arise, and to be very important, in ascertaining the cause of a failure, and they may figure in law suits involving large sums of money.

The differing effects of tensile, compressive, and bending stresses must be constantly borne in mind. Tension in a bar tends to straighten it if bent either initially or by transverse loads, and a considerable stretch may occur without injury to the bar itself. Compression produces further bending if any exists, and stress beyond the yield point may produce a deformation or buckling leading to failure; the yield point is really the ultimate. In flexure, the true maximum stress, if the usual formulae indicate it to be above the elastic limit, can never be known, because those formulae do not apply to stresses above the elastic limit, and the outside fibers cannot be stressed above that limit without throwing stress in the adjoining fibers and so causing a radical redistribution of the stress. In flexure the maximum stress will therefore not be above—or very little above—the elastic limit, if under the redistribution which takes place the bending moment can be resisted without exceeding that limit.

The effect of repetition (see Chap. XXII) must also be borne in mind. The stress which will ultimately break a piece depends upon the range of stress from maximum to minimum, and upon the number of repetitions, as already fully explained in the chapter referred to.

16. A most important point governing failure is the fact that a stress above the elastic limit raises the elastic limit up to the previous stress and sometimes above it (see Chap. IV). This fact, coupled with the maximum strain theory, affords a reasonably certain foundation for a satisfactory explanation of failure. In Fig. 467 the elastic limit and yield point are near e, and up to that point there is no permanent set and but a small elastic deformation (about 0.001 of the length). If the piece is stressed to b and then unloaded, the stress and strain pass nearly down the line bc, parallel to Oe, and the permanent set is Oc. If the same stress is again applied, it will be found that the elastic limit is raised nearly or quite to b, and if not raised quite to b, it may be so raised by a few applications. Sometimes, after a period of rest, or by a few repetitions of the load, it may be

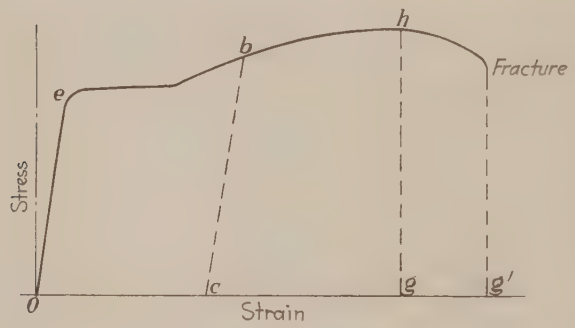


Fig. 467.

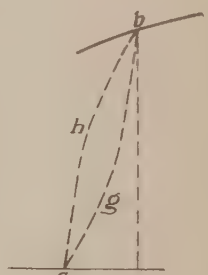


Fig. 468.

found to be raised above b. This means that no matter how many times that load is applied; there will be no increase of strain. The strain necessary to break the piece by a single application is Og, before necking-down begins, and Og' at fracture.

This theory, which applies to stresses of the same kind, was first suggested by Bauschinger, in his paper of 1885. There seems at present little question of its correctness, though it is doubted by some engineers. Professor Unwin in his discussion of the paper by Stanton and Bairstow[1] said that it has been "completely made out," and that is the present writer's belief. There seems no doubt that a stress which, if repeated, can never raise the elastic limit up to itself, so that each repetition would cause an increase of strain, would ultimately cause failure. No experiments are needed to prove this; it seems evident. It is not so evident that a load which could raise the elastic limit up to itself would never cause failure, though this would follow if the maximum strain theory is correct. It is true that often in removing a load from a point such as b in Fig. 467, and applying the same load again, the diagram does not follow down an inclined straight line bc and back again, but encloses a loop as shown in Fig. 468, the

[1] *Min. Proc. Inst. of C. E.*, vol. 166, p. 115, 1905–6.

unloading along *bgc* and the reloading along *chb*. This is called a *hyteresis* loop. In such a case the work done on loading exceeds the work given up in unloading by the area of the loop, which means a consumption of energy at each application of the load, and this must be transformed into heat. The shape and size of this loop depend upon circumstances; the slower the loading and unloading, the smaller the area; and also, after a number of repetitions the loop may get smaller and disappear, or it is conceivable that it may become larger. Whether the loop exists when the load is repeated very rapidly, as in the machines for testing by repeated stresses, cannot of course be known. It is reasonably certain that there is no hysteresis loop if the material is perfectly elastic, that is, up to the elastic limit;[1] consequently, if the elastic limit is raised to the applied stress there is no loop and no absorption of work at each repetition.

There is, however, no reason why, even if for a given maximum stress there is a hysteresis loop and a consequent absorption of heat at each repetition, that stress, if repeated continuously, should ultimately cause fracture. The experiments of Professor Moore[2] result in the determination of an "endurance limit," that is, the greatest stress which if repeated an infinite number of times, would never break a piece. It is found that, as the stress is increased, when the endurance limit is reached there is a sudden and considerable increase in the rise of temperature, so that the endurance limit may be quite accurately determined by finding when this sudden increase of temperature occurs. But even at stresses very much below this endurance limit there is a considerable and measurable increase of temperature. If these observations are correct, therefore, they prove conclusively that the presence of a hysteresis loop if closed, does not indicate a stress that will ultimately cause fracture.

If, however, b is higher, or nearer h, it may be found that in the second application the elastic limit will not be raised to b, and it may take a considerable number of applications to so raise it. It may even be found impossible to raise the elastic limit to the maximum stress applied, no matter by how many repetitions; such a load will ultimately break the piece, because each application will cause a small increase of strain, and, if there are enough repetitions, fracture will surely ensue. The stress for square inch, f, which will break the bar after an infinite number of repetitions, is the stress which, if applied an infinite number of times, would never be able to raise the elastic limit up to itself. Any stress f which, after repetitions, could raise the elastic limit up to itself, would never cause fracture, no matter how many times applied, because successive applications would not increase the strain.[3]

This is the theory for simple tension. The same thing would apply in compression or shear. It may be modified if there are any physical conditions that require it, as suggested in Art. 13.

For tension alternating with compression, the same theory is applicable, but here the important fact must be remembered, that stress above the elastic limit in tension lowers the elastic limit in compression, sometimes to zero; and *vice versa*. The maximum value of two equal alternating stresses which will just suffice to

[1] Bairstow, "The Elastic Limits of Iron and Steel under Cyclical Variations of Stress," *Phil. Trans. Royal Soc. of London*, series A., vol. 210, pp. 35–55.

[2] *Bull.* 124 *and* 136 *of the University of Illinois.*

[3] See the Chapter on Fatigue, for further discussion of this subject.

cause fracture after a very large number of repetitions is in this case not above the original elastic limit, and is in fact generally considerably below it.

The foregoing principles will prove frequently useful and illuminating. They will frequently help in deciding on the cause of a failure of a structure. There are few subjects in which an erroneous conclusion may be reached more easily than in the study of failures, one reason being that it is often or generally impossible to distinguish cause from effect. A broken piece found in the wreck may not have been the cause but may be the effect of the collapse.

CHAPTER XXIV

MATHEMATICS, TESTING AND COMMON SENSE

1. In the investigation of problems of the strength of materials and the theory of structures, there are two indispensable agencies or instruments, mathematics and experiment. It may be well to examine the character and limits of these instruments.

No reader of these pages can fairly charge the writer with a prejudice against mathematics. It has been used, freely and constantly, as a tool that is necessary to the solution of the problems discussed. By most people this book would be considered quite a mathematical treatise. But mathematics has been used as a tool or a servant, not as a master.

Just as there is no vice worse than excessive or misapplied virtue, so there is no engineering mistake worse than excessive or misapplied mathematics, though many people seem to think that there cannot be an excess of it. Mathematical analysis is a rigid machine, which works with absolute accuracy upon the data supplied to it, and gives results dependent on the data. If the data are correct, the results will be correct; but a small error in the data may make a large or a small error in the results. The danger in the use of mathematics is that the machine itself is so interesting that attention will be concentrated upon that machine and its operation, and diverted from the data, which are more important. Mathematical results always depend on certain assumptions or data, and in engineering problems these data are often vague or uncertain. Oftentimes, if the data are correct, common sense without mathematics, will indicate the result, at least qualitatively if not quantitatively. Mathematical results, being of absolute accuracy if the data are accurate, many people who do not know are led to think that they must be correct, even though they may be based on false data. The *habit of mind* cultivated by the continued use of the mathematical machine discourages the *habit* of carefully analyzing the data, and often leads to the acceptance of mathematical results which a little common sense would show to be wrong. It also leads to the idea that if an answer is the result of a long and complicated mathematical process it gains in reliability thereby. It is the same with statistics. Many people are impressed with a multitude of statistical tables, while the fact is that statistics are notoriously unreliable because they so often fail to include all the facts. Almost anything may be proved by statistics, if the statistics used may be selected; and similarly, almost anything may be proved by mathematics if we are allowed to choose the data. There is in all this an example of the awe felt by the average person for something that is abstruse and complicated and a little beyond him, and of the lack of mental initiative which leads a man to believe what seems supported by a great mass of testimony which he has not the ability or knowledge to disprove.

2. Sir William Hamilton[1], the logician, said that he had never met with a mere mathematician who was not "credulous to a fault." This was because

[1] *Edinburgh Review*, 1837.

they were so accustomed to thinking about the abstract and abstruse problems of mathematics and the use of the machine, that they had lost the power of dealing with physical realities. He also said "Mathematics affords us no assistance, either in conquering the difficulties or in avoiding the dangers which we encounter in the great field of probabilities wherein we live and move." He further maintains that of all the subjects in the usual curriculum there is none which cultivates a smaller number of intellectual faculties, or cultivates them in a more inadequate and partial manner. While this is probably going too far, it seems to the writer clear that the study of the mathematical machine does tend to obscure the realization that the problems of life are rarely mathematically exact, and the habit of considering the necessary errors of the data; so that it tends to obscure the common sense, and to replace it with mathematical hair-splitting. Even John Stuart Mill, who rather weakly disagrees with Hamilton, says that the applications of mathematics should be studied "in such a manner that the intellect *is aware of what it is about, and does not go to sleep over algebraical symbols.*" (Italics ours.) There is precisely the danger: the intellect is generally so concentrated on the machine that it is *not* conscious of what it is about, and it *does* go to sleep over algebraical symbols. Karl Pearson the historian of the "Theory of Elasticity," says,[1] (to repeat the quotation on page 143):

"We find even in mathematicians of the standing of Lamé not infrequently an omission to state clearly the physical principle upon which they base their calculations of a physical phenomenon. The history of mathematical elasticity gives many examples of this divorce between theory and physical fact; the mathematician has too often identified elasticity with the solution of certain differential equations, the constants of which are to be determined by a purely fanciful and often practically idle, if not impossible distribution of load."

And so we sometimes see complicated mathematical demonstrations based on untrue or impossible assumptions, or even on the assumption that a thing is and is not at the same time, as in the case referred to in Art. 5, of Chap. VI. The results may or may not be true; we do not know; the theory is certainly wrong; the writer has no time to waste in speculating on theories based on such conflicting data. The mathematician revels in it.

3. The principal modern developments of mathematics have been in the field of *analysis*, as contrasted with that of *geometry*. The geometry of today is essentially the geometry of Euclid, though some fields, such as projective geometry, are modern, and seldom taught. Analysis is the machine. Once the equations are set up—and this requires some thought—the rest is largely machinery, and very interesting machinery. The tendency of today is to do everything possible by machinery; and so, in mathematics, geometry and its applications are neglected. Geometry requires a good deal of mental effort, to visualize and really *see* what is being done, and the meaning of the results—too much effort for the usual student. Descriptive geometry is shunned and disliked, because it is difficult, and so it is made as easy as possible for the feeble minds so common now. The machine is the thing that we learn to use. Even mental arithmetic is neglected, and superseded by the slide rule. The college graduates and engineering graduates of today cannot solve "in their heads" the problems of

[1] TODHUNTER, "A History of the Theory of Elasticity and of the Strength of Materials," edited and completed by Karl Pearson, Cambridge Univ. Press, 1886, vol. 1, p. 626.

Colburn's mental arithmetic, that was used in preparatory schools two generations ago. Let the reader who doubts this try it and see. The graduate today would be surprised if told that he ought to be able, instantly, to find the square of 20.25.

The writer is of course perfectly well aware just how much of analysis is machine work and how much is not; the statement that it is largely machine work is believed justified.

4. In studying mathematics or its applications, it is essential to be conscious of the meaning of it all, to see clearly the assumptions, to interpret the results consciously, not to "let the mind go to sleep over algebraical symbols." A formula is an intelligible sentence, which tells us something; too often, to the student, it is merely a result expressed by letters for which numbers are to be substituted in any special case: he does not *see* what it means, what inferences may be drawn from it, or when it is applicable. He looks upon it, too often, as merely something that may be manipulated by the methods of analysis. Most students, for instance, after completing the study of the calculus, have no clear understanding of the meaning of $\frac{d^2y}{dx^2}$. Such an attitude is fatal, but it is largely encouraged by our methods of education, and by the fact that many teachers of mathematics and of engineering have had no practical experience in applying mathematics, and do not themselves fully realize the approximations, probabilities and uncertainties of the world in which we live and move. A professor of mathematics once told a former student of the writer that he should not bother to know the *meaning* of a mathematical expression, that all he needed was to know "what to do with it;" in other words, to know how to manipulate the mathematical machine, not to know what it was all about.

5. Many students are lost without their books. If they would perceive clearly the fundamental principles and the assumptions made, and remember these, they could, without books, work out fundamental formulae at once. The student should, for instance, be able to sit down with only pencil and paper, and work out the formula for a circular shaft, in less than half a minute, in some such way as this: the principle is that the stress intensity varies as the distance from the axis; very well; then the circumference of any little ring distant r from the axis is $2\pi r$, its area is $2\pi r dr$, the total stress on it, if c is the stress intensity at a unit distance from the axis, is $2\pi c r^2 dr$, its moment is $2\pi c r^3 dr$, and the total moment or torque is $M = \frac{1}{2}\pi c R^4$, or if f is the maximum stress intensity $M, = \frac{1}{2}\pi f R^3$, as in Art. 2, Chap. VIII. The formula need not be remembered. The same holds for much that is in this book.

If this were practiced, there would be no need for the student to work out many examples. Our schools, in the opinion of the writer, give too many examples, in which the students merely substitute numbers in formulae; and they do not sufficiently impress principles or train common sense.

All this is no criticism of mathematics, which is, as already stated, an indispensable tool for the engineer. It is merely a caution against its misuse, and a criticism of the way in which it is frequently taught and studied.

6. The other indispensable tool is experiment, or testing. There is no other way to discover the properties of materials, or to penetrate into the laws of nature, than to get the facts by experiment and to verify conclusions by tests.

Yet it is possible to go too far in this direction, as in any other. This is the age of testing, of laboratories, of weighing and measuring.

7. Fifty or sixty years ago there was not a testing laboratory in any of our colleges, and few if any physical or chemical laboratories. The student of chemistry was taught by listening to lectures, and by seeing the professor perform experiments, and by reasoning from his text books. Soon it was realized that this was wrong, that the student should test and experiment for himself, not only to understand the laws better, but to learn the use and manipulation of apparatus. Now the pendulum has swung to the other extreme, great laboratories of physics, chemistry, and the branches of engineering have sprung up, and the attitude is largely that the only way to learn is by testing. We test every conceivable thing. We often resort to testing to avoid the necessity for thinking. In too many cases, if we do not understand a thing, instead of trying to reason it out, we test it. Testing, and the laboratory method, are—in the writer's opinion—being carried too far, and this, also, is being encouraged by our methods of instruction. The revolution which has properly taken place in teaching physics, chemistry, and engineering, meaning by this the introduction of the laboratory method, has led teachers of other subjects to believe that they are not up to date unless they too have revolutionized their subjects. Even English, History, and Mathematics have had to be so revolutionized. The writer found his children, studying a text-book on Geometry by a university professor, required to prove that the sum of the three angles of a triangle was 180°, by drawing the triangle, cutting it out, and measuring the angles with a protractor. This is the laboratory method in mathematics—absurd and mentally demoralizing because by drawing a single line on a free hand sketch any child can easily *see* that the sum of the angles is 180°. But even this small amount of mental exertion is discouraged, and a "test" is made. The test proves nothing, for the sum will never, except by chance, add up to *exactly* 180°, and so the child is left with the idea, perhaps, that the thing is approximate anyway. Similarly, the metric system is taught by requiring the children to buy a metric scale, and to measure a distance by it and by the foot rule, comparing the results. The simple statement that a meter is 39.37 inches, and the comparison of some lengths and areas based on that fact would probably involve too much mental effort and visualizing, and so measurements must be made.

So, too, in science, a boy or girl is now apparently not supposed to be able to learn that the heart beats and sends the blood through the body, by studying diagrams and by feeling his own pulse and his own heart-beats; but he must dissect a poor little frog, take out its heart, and *see* it beat—not for scientific purposes (for the writer does not oppose legitimate vivisection for proper scientific research), but purely to use the laboratory method for a quite unnecessary purpose.

8. If a test involves the use of simple and accurate instruments, as in hydraulic tests for the determination of constants in theoretically correct formulae, and in many tests of materials, as described in Chap. XXI, an experiment may be fairly accurate, and only subject to the usual errors of observation. Where they involve the use of extensometers, an instrument is involved which is very delicate, and which must be exceedingly accurate. A measurement to $\frac{1}{10{,}000}$ of an inch,

or, as sometimes attempted, to $\frac{1}{50{,}000}$ or $\frac{1}{100{,}000}$ of an inch, if such accuracy is required in order to obtain correct results, is difficult to accomplish. There are many possible errors, such as slipping of the points of contact, inertia of parts, initial bends of the specimen, etc., as discussed in Chap. XXI, which may vitiate the results. When such measurements are attempted, not in the laboratory, but in the field on actual structures, the chances of error are greatly multiplied.

9. The writer must confess to a profound distrust of field extensometer measurements, except, perhaps, to gain some idea of the amount of impact. If a train is run very slowly over a structure and then at speed, extensometer measurements may afford an approximate idea of the amount of impact, though not of the actual stress. Even in the laboratory, where every precaution may be taken to ensure accuracy, extensometer measurements are often to be distrusted, especially where made on short gauge lengths, unless made with the greatest care, with almost miscroscopic accuracy, and by competent observers, of experience and judgment. Testing engineers will no doubt maintain that these statements are made because the writer is not a testing engineer, which is a fact; but he has perhaps had occasion to study the results of tests as thoroughly as most testing engineers themselves. But it is a fact that some published results of elaborate extensometer measurements can be shown to be wrong, because they do not conform to necessary relations. The writer knows of some measurements of this kind in which the same deformation was shown under quite different loads; hence the strain and stress do not depend directly upon the load, which is of course absurd. The writer has also heard an experienced and capable testing engineer testify in an important case as to the results of his laboratory tests, and his conclusions therefrom; when the simplest arithmetic sufficed to show conclusively that the results could not possibly be correct. Some of these extensometer measurements had been made on a gauge length of one inch, and the writer would place no reliance on such measurements. Some extensometer measurements show tension on one side and compression on the other side of a piece in tension, due perhaps to the straightening of a slight bend.

10. Just as the mathematician is under the temptation to become so interested in his machine that he loses sight of the big things, so the testing engineer is under the temptation to become so interested in his machines and measuring devices that he, too, loses sight of fundamental relations, and propounds results that are doubtful or even impossible.

11. Another thing that testing engineers, or mathematicians, are very apt to do, is this; they will take a small piece from a structure, test it, and assume that the test is a fair indication of the strength or action of the part when in the structure, which may be entirely untrue. For instance, a large oil tank is composed of a number of circular rings, each made up of many plates riveted together. If one of these fails, the testing engineer will cut a small piece from one of the lap joints of a ring, and fail to see that the action of that joint in the testing machine is very different from its action in the structure. According to the discussion in Arts. 30–35, of Chap. X, the action of a fiber of a steel beam, under flexure, is very different from its action if cut out and tested in tension; the stress-strain diagram in the latter case shows little as to the action in the beam. So in testing columns, the action in the machine may be quite different from the action in a

structure. If the law of similarity is not observed, even careful laboratory tests may be useless in practice. If more than one variable is involved, a laboratory test may apply only to a case identical in all respects with the case tested.

12. The point is to remember that in testing, as in the use of mathematics, common sense should always be in command, that instrumental errors should be avoided wherever possible, that results should be calmly scrutinized to see if they are reasonable, and that the scientific method should be employed. The writer has often found himself in a position where he has been obliged, regretfully, to disbelieve results produced by mathematicians or by testing, because they did not "seem reasonable." This may appear a very flimsy criterion, but it will be found a good one in the hands of a man with experience, judgment, and common sense. The reader will do well to cultivate the habit of distrusting what does not seem to him reasonable. His "seeming" may be wrong; the result may be correct; in that case his senses need cultivating, to the end that he may finally come to be able to "sense" a thing correctly. If a thing "seems wrong," when we are sure, by mathematics or testing, that it is right, we must cultivate our senses till they are in conformity with truth; in other words, until we can see straight.

Finally, it may be appropriate to refer to a remark of Aristotle, who observed that it is a mark of an instructed mind to rest satisfied with that degree of precision which the nature of the subject admits, and not to seek for exactness where only an approximation to the truth is possible.

The subject of the practical Strength of Materials is one in which mathematical accuracy is rarely, if ever, possible.

INDEX

A

Abbott, on hardness, 498
Abrams, Professor D. A., 76
Allowable stresses, 64
Alternation of stress, 508
American Bridge Co., standard screw threads, 129, 130
Institute of Steel Construction, 430, 455, 456
Soc. C. E. column committee, 441, 442, 445, 449, 451
Soc. for Testing Materials, 86, 492, 498, 501, 502, 505, 507, 522, 523, 524, 525, 526
Amorphous Cement theory, 85
Arc, circular, center of gravity of, 29
length of, 24
Areas, 25
Aristotle, 562
Aston, James, 485
Axis, 178
discontinuous, 246

B

Bairstow, Dr. Leonard, 528, 532, 555
Barba, Law of Similarity, 68
Basquin, Professor O. H., 435, 525, 526
Bauschinger, Professor J., 79, 496, 508, 513, 554
Beam, continuous, 181, 358
definition, 161, 179
deflection, 206
fixed at one end, shears and moments, 254, 257
at one end, slope and deflection, 295
at both ends, shears and moments, 287
at both ends, slope and deflection, 322
at one end, and supported at the other, 284, 317
flitched, 243, 244
general case of deflection, 332
of different materials, 243
of uniform strength, 29, 331
Beam, overhanging, 314
reinforced concrete, 244, 368
simply supported, 180
supported at both ends, 258, 272, 300
unsymmetrical, 242
Bearing, on rivets, 390
Becker, A. J., 549
Beilby, 83
Bending moment, 184
Berry strain gauge, 494
Betti, 217
Biaxial loads, 57, 358
Boiler Code, A. S. M. E., 398, 400, 401
Bond in reinforced concrete, 375
Bouscaren, column tests, 448
Bouton, Chas., 119
Boyd, Professor J. E., 458, 547
Brewster, Sir David, 124
Bridgman, Professor P. W., 541, 546
Brinell, test of hardness, 497
Brittleness, 54, 82
Building Laws, formulae for steel columns, 455
cast iron columns, 459
Bussey Bridge, accident, 353

C

Cantilevers, 365
Cast iron, beams, 242
columns, 458
effect of time in tests, 76
modulus of elasticity, 55
Center of gravity, 27
Central ellipse, 42
Centroid, 27
Chain riveting, 387
Change of cross-section, 68
Change of volume, by stress, 62
Charpy impact machine, 503
specimens, 502
Chemical tests, 506
Church, Professor I. P., 22
Circle of inertia, 43
of stress, 98
Circular arc, length, 24

Circular sector, center of gravity, 32
segment, area, 26
center of gravity, 32
Clapeyron, Theorem of Three Moments, 364
Clebsch, 220
Coker, Professor F. G., photoelasticity, 121, 124, 125
Cast iron, 458
Columns, 411
comparison of formulae, 429
Euler's formula, 420, 425
forms of, 440
Gordon's formula, 418
ideal, never exists, 432
Parabolic formula, 424
practical formulae, 456
Rankine's formula, 418
reinforced concrete, 376
straight line formula, 422
tests, 445
of L. Bouscaren, 448
of Jas. Christie, 445, 448, 454
of T. C. Clarke, 447, 448
of L. Tetmajer, 449
ultimate strength, 440
with bend, 433
with eccentric load, 434
with fixed ends, 415
with one end fixed, 417
with pin ends, 415
with round ends, 417
wooden, 457
Combined stresses, 139, 335
work of, 355
Common sense, 557–562
Compression, combined with flexure, 344
mode of fracture, 117
pure, 116
piece, shear in, 118
Concrete, effect of time in tests, 76
modulus of elasticity, 55
reinforced beams, 368
columns, 376
Conjugate stresses, 88, 101, 108
Continuous beams, 338
Cooper, Theodore, 419
Corrosion cracking of brass, 473
Coulomb, C. A., 177
Couple, 6
Couples, composition of, 11
Cross-section, deformation in flexure, 201
Curved bars, flexure, 476
Cyclical variations of stress, 79
Cylinders hollow, 461

D

Dalby, Professor W. E., extensometers, 496, 497
d'Alembert's principle, 15, 83
Dead loads, 3
Deflection of beams, by method of work, 217
by moment-area method, 209
due to shearing, 219, 221
example, 221
general case of, 332
general principles, 206, 294
in terms of unit stress, 225
example, 327
of reinforced concrete, 376
of uniform strength, 331
principle of reciprocity, 217
summary of methods, 218
variable moment of inertia, 328
Deformation, 52, 111
of circular shaft, 137
of cross-section, 201
Devries, R. P., on hardness, 501
Dimensioning, 512
Disc, rotating, 409
Distribution of normal stress, 50, 150, 232
of torsional stress, 236
Drift pin, 381
Drilled holes, 380
Drop test, 502
DuBois, Professor A. J., 22
Durand's rule, 26
Ductility, 53, 80, 82, 497
Dynamic tests, 502

E

Efficiency, of a riveted joint, 391
Elastic limit, 65, 72
line, 210
determined by tests, 493
"original" and "natural," 79
raised by overstress, 77
Elastic line, in flexure, 178
weights, in flexure, 213
illustrations, 214
Elasticity, 52, 82, 493
modulus of, 54

Ellipse of inertia, 39
of stress, 94
Ellipsoid of stress, 109
Elliptical section, torsion on, 146
Elongation, relative, 68
Endurance limit, 511, 512
Euler, formula for columns, 420
Ewing, Professor, 79
Extensometers, 494
tests, 561
Eye-bars, stress in, 126

F

Factor of safety, 64, 538
Failure, theories of, 536
comparison of theories, 548
maximum shear theory, 543
strain theory, 547
stress theory, 540
Fairbairn, formula for tubes, 463
Farmer rotating testing machine, 520
Fatigue, 508
tests, 506
Fixed points, of central ellipse, 42
Flexure, pure, 158, 178
assumptions in theory, 183
neutral axis in, 183
stresses in, 181
Flexure, formula, limits of use, 237
of curved bars, 476
with compression, 344
with tension, 336
with torsion, 349
work in, 203
Flitched beams, 243
Floor beams, 251
effect on shear and moment, 253
Föppl, Professor Aug., 121, 508
Forces, classification, 3
equilibrium, 9
not in a plane, 11
resolution, 7
resultant, 5
sense, 4
Fränkel, Professor W., 217
Fremont impact machine, 506
Friction, 17
coefficient of, 19
on pin, 412
rolling, 20
Fuller and Johnson, 22

G

Galileo, 176
Gerber, H., on repeated stress, 518
Girder, 181
plate, 181
Goodman's diagram, 519
Gordon's formula for columns, 418
Gough, H. J., repeated stress, 533
Grashof, Professor F., 110
Grips, for testing, 490
Grooved section, 70, 406
Guest, J. J., 547, 549, 550
Gyration, radius of, 37

H

Hamilton, Sir William, 507
Hancock, Professor E. L., 78
Hardening, of steel by overstress, 79
Hardness, 53, 82, 497
Hartmann, L., lines caused by stress, 84
Helical spring, 140
Herring-bone lines, 86
Heymans, Professor Paul, 124
Hinge, 20
Hodgkinson, Eaton, 76, 427, 458
Hole in tension piece, 120
Hollow cylinders, principal stresses in, 464
thick, 465
thin, 461
with torsion, 464
Hook, 353
Hooke's law, 55
Hoop tension, 461
Howard, J. E., 471, 472, 473
Hutton's formula, 464
Hysteresis loop, 77

I

Ideal stress intensity, 58
Impact tests, 502
Inertia, circle of, 43
ellipse of, 39
moment of, 36
product of, 38
variable moment of, 328
Indirect transmission of stress, 409
Influence lines, 250, 291
Inglis, C. E., effect of cracks on stress, 125

Initial stresses, 470
due to cold straightening, 471
measurement of, 471
Inner forces, 3
Izod impact machine, 505

J

Johnson, J. B., 424
materials of construction, 119, 120, 127, 485
Johnson, Professor L. J., 22, 177
Johnson, R. S., 85
Johnson, T. H., column formula, 423
Joints, butt, 386, 400
efficiency, 391
lap, 386, 396
riveted, 377

K

Kernel of section, 168
examples, 168
in curved bars, 482
use of in finding stress, 170
Keystone column, 440
Kinetics, 21

L

Lamć, 465
Lanza, Professor G., 450
Lap joint, 386
Lateral strain, 56
Launhardt, on repeated stress, 517
Lines, curved, 24
Load line, 184, 223
Loads, dead, 3
live, 3
suddenly applied, 82
Longitudinal shear, 188
Loose filler, 410
Lüders' lines, 84
Ludwik cone, 498

M

Machines *vs.* structures, 1
Malleability, 53
Mariotte, 177
Martens, 485
extensometer, 496
tests of screw threads, 127
Mathematics, 557–559
Maurer, Professor E. R., 22
Maxwell, J. Clerke, 217
McAdam, D. J., Jr., 522
Merica and Woodward, on initial stresses, 471
Merriman, Professor M., 422, 428
Mesnager, M., 124
Microscopic tests, 506
Mill, John Stuart, 558
Miller, L. H., column diagrams, 455, 456
Modulus of elasticity, 54
of resilience, 80, 203
Moh's scale of hardness, 499
Mohr, Professor O., 214
Moment, 4
on a beam, 248
relation to shear in flexure, 185
absolute maximum on a beam, 281
on riveted connections, 394
Moment diagram, 265
Moment line, 185, 209, 223
Moment of inertia, 36
for structural shapes, 44
polar, 39
Moment-area method, 208, 209
Moncrieff, J. M., 435, 449
Moore, Professor H. F., 74, 508, 518, 520, 523–527
Moore and Seely, 74
Morley, Arthur, 465
Morris, Professor C. T., 435
Mörsch, Professor E., 135
Movable loads, shear and moment due to, 266
Müller-Breslau, H., 177

N

Navier, L., 177
Necking down, 66
Neumann bands, 85
Neutral axis, 162
construction of, 167
position of, 239
relation to central ellipse, 165
Neutral point in a panel, 273
Nickerson, Louis, 124
Non-homogeneous section, 175
Notched section, 70

O

Obliquity of stress, 95
Osborne-Reynolds, testing machine, 522
Outer forces, 3

P

Panel, of a bridge, 252
Parabola, length of, 25
Parabolic column formula, 424
Parabolic segment, area of, 26
  center of gravity of, 33
Persy, 177
Phoenix column, 440
Photo-elasticity, 124
Pearson, Karl, quoted, 143, 558
Pin, moment of friction on, 412
Pin connections, 378
Pitch, of rivets, 386
Planar distribution of stress, 156
Plasticity, 52, 82
Poisson's ratio, 56
Polar moment of inertia, 39
Position of loads for max. moment, 269
Preuss, E., 120, 121
Principal strains, 114
  stresses, 92, 104
  stresses and strains, relation between, 115
Pritchard, H. S., 475
Product of inertia, 38
Proportional limit, 65
Punched holes, 380
Punched sections, 119

Q

Quadrilateral, center of gravity of, 31

R

Radius of gyration, 37
Rankine's formula for columns, 418
Reamed holes, 380
Reciprocity of deflections, 217
Reinforced concrete beams, 368
  columns, 376
  T-beams, 372
Repetition of stress, 508
Resilience, 54, 79, 82
  modulus of, 80, 81, 203
Reversal of stress, 508
Rigidity, 53, 82
  torsional, 138
Ritter, Professor W., 177
Riveted joints, 377
  butt, 386, 400
Riveted joints, efficiency of, 391
  lap, 386, 396
  lap *vs.* butt, 402
  tests, 406
  three methods of considering 395
Riveting, chain, 387
  staggered, 387
Riveting machines, 378
Rivets, 376
  size and shape, 384–386
  stresses on a group, 392
  tension in, 388, 394
  unequal distribution of stress on, 407–409
Rockwell hardness tester, 500
Rosenhain, W., 85
Rotating discs, 469

S

St. Venant, 143, 220
Scleroscope, 499
Scoble, stress in piece with hole, 121
Scratch test for hardness, 499
Screw threads, 127
Section modulus, 188
Shaft, deformation, 137
  elliptical, 146
  hollow, 138
  non circular, 142
  power transmitted by, 137
  rectangular, 147
  relative economy, 149
Shear, 16, 148
  action of, 131
  deformation due to, 61, 134
  distribution in a beam, 188
  double, 132, 388
  in reinforced concrete, 375
  involves tension and compression, 60, 132
  on planes at right angles, 58
  relation to moment in flexure, 185
  single, 132
  work done by, 134, 204
Shear, on a beam, 248
  tests of, 490
Shear line, 185, 223
Shearing stress, distribution, 188
  examples, 197
  in web of I-beam, 192
  longitudinal, 188, 196
Shells, hollow spherical, 463

Shore scleroscope, 499
Similarity, Barba's law, 68
Simpson's rule, 26
Sine curve, 34
  area of, 35
  center of gravity of, 35
Skelton, R. A. & Co., 437
Slip bands, 84
Slenderness ratio, 411
Slocum, Professor S. E., 22
Slope, by method of work, 217
  by moment-area method, 209
  due to shear, 219
  of elastic line, 206, 298
  summary of methods, 218
S-N. curves, 525
Spangenberg, Professor L., 508
Specifications for riveting, 382
Speed of repetition of stress, 527
Spring, helical, 140
Stanton, T. E., 527, 528
Steel columns, 456
Steel hardening by overstress, 79
  modulus of elasticity, 55
Stiffness, 53, 82
Strain, 52, 111
  gauges, 494
  general case of, 113
  principal, 114
Strain-hardening, 501
Strength, 54, 82
Stress, 3, 16, 48
  actual distribution, 232
  at the surface of a body, 89
  bending, 50
  circle of, 98
  conjugate, 88, 95, 101, 108
  distribution, 16, 50
  ellipse of, 94
  ellipsoid of, 109
  indirect transmission, 409
  intensity, 4
  laws of variation, 110
  max. and min. on a section, 172
  method of studying, 87, 151
  normal, 16, 48
  obliquity of, 95
  on a rivet, 387
  parallel to a plane, 60, 90
  planar distribution, 156
  principal, 92, 104
  real, in stress-strain diagram, 67
Stress, repetition and reversal, 508
  tangential, 16, 48
  Wöhler's experiments, 508
    laws, 511
Stress-strain diagram, 64
Stringers, 251
Strohmeyer, 533
Structural shapes, moment of inertia, 44
Structures *vs.* machines, 1
Sudden application of load, 82
Summation of effects, 250
Sunatani, Chido, 119, 546
Surface finish, effect of, 528
Suyehiro, K., plates with holes, 121

T

Tanks, 462
Temperature, rise of in repeated stress. 532
Tension, mode of fracture, 117
  piece, shear in, 118
    with hole, 120
    with notch, 119
  pure, 116
Test pieces, standard, 492, 502, 529
Tests of materials, 485
  chemical, 506
  compression, 490
  elasticity, 493
  fatigue, 506
  hardness, 497
  impact, 502
  limitations of, 559–562
  microscopic, 506
  shearing, 490
  specimens, 529
  standard pieces, 492
  tension, 488
  torsion, 491
Testing machines, 487
  Farmer, 520
  for fatigue, 509–511, 520
  Olsen, 488, 491
  Osborne-Reynolds, 522
  Upton-Lewis, 521
  Wisconsin, 521
Tetmajer, L. v., 242, 410, 427, 449
Theorem of Three Moments, 362
Thurston, Professor R. H., 495
Timber, modulus of elasticity, 55
Time, effect on stress-strain curve, 75
Todhunter, I., 143

Tohuku Imperial University, 119
Torque, 136
Torsion, 136
  behavior in, 141
  deformation, 137
  general theory, 144
  non-circular sections, 142
  power transmitted by, 137
  with compression, 352
  with flexure, 349
  with tension, 351
Torsional rigidity, 138
Toughness, 54, 82, 502
Trapezoid, center of gravity, 30
Triangle, center of gravity, 30

U

U. S. Bureau of Standards, 437, 451, 471, 487
University of Illinois Bulletins, 520, 531, 533, 549
Unsymmetrical beams, 242
Unwin, Professor W. C., 68, 74, 75, 402, 485, 490, 508
Upton, Professor G. B., 69, 234, 475, 491
Upton-Lewis machine, 521

V

Vierendeel, Professor A., 427
Volume, change of due to stress, 62
  modulus, 63

W

Watertown Arsenal tests, 141, 406, 450, 451, 487
Weyrauch, Professor J., 518
Winkler, Professor E., 517, 519
Withey, Professor M. O., 485
Wöhler, A., 508
Wooden columns, 457
Worcester, J. R., 472
Work, 21, 79, 203
  done in compression, 134, 204
    in flexure, 205
    in shearing, 134, 204
    in tension, 204
    in torsion, 137
    of combined stresses, 355
Wrought iron, modulus of elasticity, 55

Y

Yield point, 66, 72
Youngs modulus, 54